Math 211.01 Review for Final

The final will be cumulative, with a slight extra emphasis on the last two chapters. You are expected to know (almost – see below) all the topics that I listed on the other two review sheets. In addition:

• Chapter 6.1 through Fact 6.1.6; Ch 6.2 through Fact 6.2.8; much of Chapter 6.3 but not Fact 6.3.4, 6.3.6, 6.3.7, 6.3.8.

• Chapter 7.1 (except for p 300, 301); all of Ch 7.2, 7.3 and 7.4 (except for the infinite dimensional examples).

• We omitted Ch 3.4, which on retrospect was a mistake since we have used the ideas a lot. Some good exercises from this section: 25, 27, 37, 41, 43, 59, 60, 63. There will not be an explicit question from this section on the exam.

• Some topics the final exam will NOT contain: Sec 5.5; the QR factorization; row operations expressed as multiplication by elementary matrices (as in ex 50-52 in Ch 2.4); the change of basis formula in Fact 4.3.5.

• You should be able to calculate dot products, cross products, area of triangles, determinants, products of matrices. You should also know what orthogonal matrices are and about similar matrices.

Review problems:

Note: The actual exam will mostly contain simple questions like 4, 6 and 8. Only a few will ask for explanations as in 3 and 9. But I think these are good review problems.

1. Show that the matrices $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ are similar.

2 Is $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ diagonalizable? If so, find matrices S, D, where D is diagonal, such that $AS = SD$.

3. (i) Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix}$.

(ii) Notice that 0 is a root. What does this tell you about A?

(iii) What do the other roots tell you about A?

(iv) Can you tell from this polynomial that A is diagonalizable, or would you have to do more work to find this out?

4. (i) Use a determinant to find the area of the triangle in $\mathbb{R}^2$ with vertices at $A = (2, 0), B = (1, 2)$ and $C = (-4, 1)$. Draw a diagram.

(ii) Calculate $\det(A)$ where $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 3 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$. Is A invertible?

5. (This question is a bit too hard for the exam, but is a good review question, I think.) Suppose that A and B are similar, invertible $n \times n$ matrices. Are the following statements true or false? Give a reason for your answer or a counterexample. (If you are unsure of an answer, try some 2×2 examples.)

(i) A^{-1} is similar to B^{-1}.

(ii) It is impossible for A to be upper triangular (with some nonzero entries above the diagonal) while B is diagonal.

(iii) A and B have the same trace and the same determinant.

(iv) If A is diagonalizable so is B.

(v) It is possible for A to have zeros on the diagonal while B is diagonal.

6. Let $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$. Find A^{10}. (**Hint**: diagonalize A.)

7. Let $V = P_2$ the space of polynomials of degree ≤ 2. Define $T : V \to V$ by $(Tf)(t) = (t+1)f'(t)$.

(i) Find all eigenvalues and eigenvectors of T by finding all solutions to the equation $Tf = \lambda f$, for $f \in P_2$.

(ii) Does T have an eigenbasis?

(iii) Find the matrix of T with respect to the standard basis $\mathcal{B} = (1, t, t^2)$, and use it to check your answers to (i) and (ii) above.

8. (i) Find a basis for the subspace

$$W = \{(x, y, z, t) \in \mathbb{R}^4 : 2x + 3y + z = 0, x - y + z + t = 0.\}$$

(**Hint**: Express this as the kernel of a certain matrix A, and then find this kernel.)

(ii) What is $\dim W$?

9. Define $T : \mathbb{R}^3 \to \mathbb{R}^4$ by $T\vec{x} = A\vec{x}$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

(i) Find a basis for the image of T. What is its dimension?

(ii) Is it possible to write $\mathrm{im}(T)$ as the solution set of a single linear equation $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$? Explain your answer.

10. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ 2 & 1 & 4 \end{bmatrix}$.

(i) Calculate A^{-1} using row reduction.

(ii) Use your answer to (i) to find the inverse of A^T. Check your solution.

11. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$. Calculate $A^T A$. Interpret your result in terms of the columns of A.

12. (i) Find the line of intersection of the planes $x_1+x_2+2x_3 = 1$ and $x_1+x_2+x_3 = 2$.
(ii) What is its direction vector?

(iii) Find the cross product of the normals to these planes.

(iv) Compare your answers to (ii) and (iii) and explain what you find.

13. Let $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and let $T_D : \mathbb{R}^2 \to \mathbb{R}^2, \vec{x} \mapsto D(\vec{x})$ be the corresponding transformation.
(i) Describe T_D geometrically.
(ii) Let $S = \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}$. Describe the transformation $T_{SDS^{-1}}$ geometrically.

14. (i) Are the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

linearly dependent or independent?

(ii) What is the dimension of the space they span.?

(iii) Find a basis for the space they span.

15. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the line

$$L = \{(x, y) \in \mathbb{R}^2 : x - 2y = 0\}.$$

(i) Let

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

How are these vectors related to L? Draw a diagram.
(ii) Find the matrix P of T with respect to the basis $\mathcal{B} := (\vec{v}_1, \vec{v}_2)$.

(iii) Find the matrix A of T with respect to the standard basis of $\mathbb{R}^2$.

16. (i) Find an orthonormal basis for the subspace

$$V = \{(x, y, z) \in \mathbb{R}^3 : 2x + z = 0\}.$$

(ii) How many vectors do you need to add to get an orthonormal basis for $\mathbb{R}^3$?

(iii) Add vectors to the basis found in (i) to get an orthonormal basis $\mathcal{B}$ for $\mathbb{R}^3$.

(iv) You should find that the coordinates of the vector $\vec{x} = (2, 0, 1)$ with respect to $\mathcal{B}$ are:

$$[\vec{x}]_{\mathcal{B}} == \begin{bmatrix} 0 \\ 0 \\ \sqrt{5} \end{bmatrix}.$$

Why?