Math 211.01 Notes on Vectors

Linear combinations of vectors (cf p 31 in Bretscher: This is useful background for sec 1.3 ex 6.)

A vector $\vec{b}$ is said to be a **linear combination** of the vectors $\vec{v_1}, \vec{v_2}$ if there are scalars x_1, x_2 such that $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2$. For example,

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

If we write $\vec{e}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then any vector in $\mathbb{R}^2$ can be written as an linear combination of $\vec{e}_1, \vec{e}_2$ as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

See Fig 1.

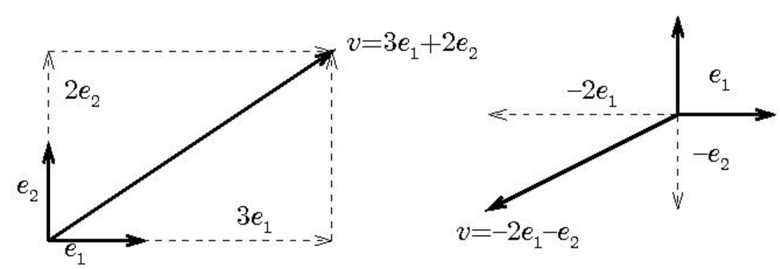

FIGURE 1. Writing $\vec{v}$ as a linear combination of $\vec{e}_1, \vec{e}_2$

IMPORTANT FACT Given *any* pair of noncollinear vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$ one can write an *arbitrary* vector $\vec{b}$ in $\mathbb{R}^2$ as a linear combination of $\vec{v}_1, \vec{v}_2$. This is obvious geometrically: see Fig 2.

An explicit example is worked out on the lower half of page 33 in Bretscher. Exactly the same argument works in general. Here are the details.

Notice that to say that $\vec{v}_1, \vec{v}_2$ are noncollinear means that $\vec{v}_1 \neq k\vec{v}_2$ for any k and also that $\vec{v}_2 \neq k\vec{v}_1$ for any k, i.e. neither of the vectors is a multiple of the other. (You need to write both these inequalities in case one of the vectors is zero.) Hence if you row reduce the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with row vectors $\vec{v}_1 = [a, b], \vec{v}_2 = [c, d]$ you get a matrix with two nonzero rows, i.e. rank $(A) = 2$. We saw in class that this is equivalent to the condition $ad - bc \neq 0$.

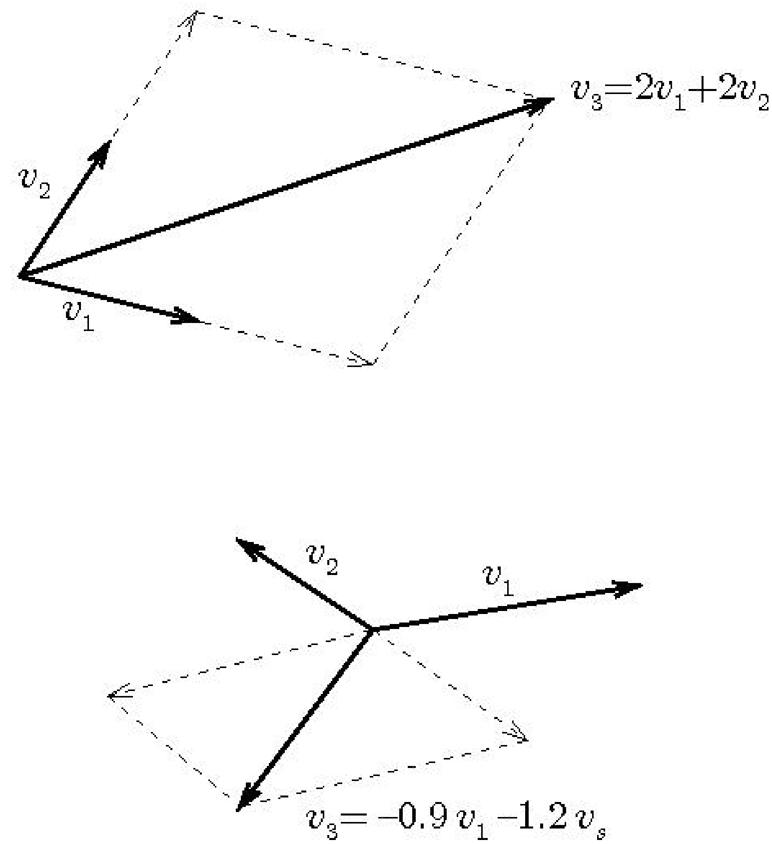

FIGURE 2. Writing $\vec{v}_3$ as a linear combination of $\vec{v}_1, \vec{v}_2$

You can also check this directly. For example, if $ad - bc = 0$ and $a \neq 0$ then $d = bc/a$. So

$$\vec{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \frac{c}{a}a \\ \frac{c}{a}b \end{bmatrix} = \frac{c}{a} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{c}{a}\,\vec{v}_1.$$

Now notice that if $\vec{v}_1, \vec{v}_2$ are not collinear (and so satisfy the above condition), the equation $x\vec{v_1} + y\vec{v_2} = \vec{v_3}$ (with unknowns x, y) that expresses $\vec{v_3} = [p, q]$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$ can be written out as

$$x \begin{bmatrix} a \\ b \end{bmatrix} + y \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

which is the same as the system

$$\begin{aligned} ax + cy &= p \\ bx + dy &= q. \end{aligned}$$

The matrix of coefficients $B = \begin{bmatrix} a & c \\ b & c \end{bmatrix}$ is called the **transpose** of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and has the same rank, i.e. 2. Hence these equations have a unique solution.

To be continued.