Math 211.01 Review for Midterm 2

• The exam will be based on Ch 4 (everything except for Facts 4.3.4 and 4.3.5) and Ch 5, secs 1 through 3 (everything except for Facts 5.1.9, 5.1.10, 5.1.11, correlations (cf Def 5.1.13), the QR factorization, Facts 5.3.9 and 5.3.10.).

• You are expected to be able to work with the definitions of **linear space, subspace, basis and dimension of a linear space, coordinates; linear transformations (image, kernel, rank, nullity); isomorphisms; the $\mathcal{B}$-matrix of a linear transformation, and change of basis matrix. Orthogonality and length, orthonormal bases, orthogonal projection, angle, orthogonal transformation and matrix, transpose of matrix.**

• You are expected to memorize and be able to reproduce the easier definitions: **Def 4.1.3, 4.2.1, 4.2.2, Def 5.1.1, 5.1.2, 5.1.12, 5.3.1.**

• You are expected to know how to:

- decide whether a given set is a linear space;
- decide whether a given subset is a subspace;
- find a basis for a linear space;
- find the coordinates of an element $\vec{x} \in V$ in terms of a given basis $\mathcal{B}$ for V;
- decide whether a given function $T : V \to W$ is linear, and whether it is an isomorphism;
- find the matrix of a linear transformation in terms of a given basis $\mathcal{B}$;
- find the change of basis matrix;
- find an orthonormal basis for a given subspace of $\mathbb{R}^n$;
- find the orthogonal projection of an element $f \in \mathbb{R}^n$ onto a given subspace $V \subset \mathbb{R}^n$;
- find the angle between two given vectors;
- decide if a given linear transformation is orthogonal;
- find the transpose of a matrix A and the inverse of an orthogonal matrix.‘

Sample questions Note: you must EXPLAIN all your answers. The exam will have about 5 questions similar to those given below.

1: Let $V = \mathbb{R}^{2\times 2}$, the linear space consisting of all 2 by 2 matrices. Which of the following are subspaces of V?

(i) The set of matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = 1$;

(ii) The set of matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = 0$.

2: Let $V = U^{2\times 2}$ the space of upper triangular 2 by 2 matrices with basis

$$\mathcal{B} = \left\{ A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(i) What does it mean to say that (x, y, z) are the coordinates of a matrix C with respect to $\mathcal{B}$?

(ii) Find the coordinates of the matrix $C = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ with respect to $\mathcal{B}$.

3: (i) Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}$. Find A^T.

(ii) Let B be the permutation matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Show that B is orthogonal.

What is B^{-1}?

4 (i) Find an orthonormal basis for the subspace $x + 2y - z = 0$ of $\mathbb{R}^3$.

(ii) Find the orthogonal projection $\vec{y}$ of the vector $\vec{x} := (1, 1, 2)$ onto this subspace V.
(iii) Check that $\vec{x} - \vec{y}$ is perpendicular to V.

5 Let $V = P_2$ the space of polynomials of degree ≤ 2. Define $T : P_2 \rightarrow P_2$ by $T(f) = f' - f(3)$, where f' is the derivative of f.
(i) Show that T is linear.
(ii) What is the kernel of T? What is its image?
(iii) What are the rank and nullity of T? Why is their sum equal to 3?
(iv) Find the matrix of T with respect to the standard basis for P_2, i.e. $\mathcal{B} = (1, t, t^2)$.

6: Consider the linear space $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ with the two bases:

$$\mathcal{B} = \left\{ f_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}, \quad \mathcal{A} = \left\{ v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

(i) Find the $\mathcal{A}$-coordinates of f_1.

(ii) Find the $\mathcal{A}$-coordinates of f_2.

(iii) Find the $\mathcal{A}$-coordinates of $f = af_1 + bf_2$.

(iv) Find the $\mathcal{B}$-coordinates of $f = af_1 + bf_2$.

(v) Find the change of coordinates matrix $S_{\mathcal{B}\rightarrow\mathcal{A}}$.