

ON NEARLY SEMIFREE CIRCLE ACTIONS

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ABSTRACT. Recall that an effective circle action is semifree if the stabilizer subgroup of each point is connected. We show that if (M, ω) is a coadjoint orbit of a compact Lie group G then every element of $\pi_1(G)$ may be represented by a semifree S^1 -action. A theorem of McDuff–Slimowitz then implies that $\pi_1(G)$ injects into $\pi_1(\text{Ham}(M, \omega))$, which answers a question raised by Weinstein. We also show that a circle action on a manifold M which is semifree near a fixed point x cannot contract in a compact Lie subgroup G of the diffeomorphism group unless the action is reversed by an element of G that fixes the point x . Similarly, if a circle acts in a Hamiltonian fashion on a manifold (M, ω) and the stabilizer of every point has at most two components, then the circle cannot contract in a compact Lie subgroup of the group of Hamiltonian symplectomorphism unless the circle is reversed by an element of G .

1. INTRODUCTION

This paper is an attempt to understand topological properties of Lie group actions. Its starting point was the following theorem of McDuff–Slimowitz [1] concerning circle subgroups of $\text{Symp}(M, \omega)$, the group of symplectomorphisms of a symplectic manifold (M, ω) . Recall that an effective circle action is **semifree** if the stabilizer subgroup of each point in M is either the circle itself or the trivial group. Also, we say that a circle subgroup Λ of a topological group \mathcal{H} is **essential in \mathcal{H}** if it represents a nonzero element in $\pi_1(\mathcal{H})$ and **inessential in \mathcal{H}** otherwise.

Theorem 1.1. *Any semifree circle action on a closed symplectic manifold (M, ω) is essential in $\text{Symp}(M, \omega)$.*

This is obvious if the action is not Hamiltonian since in this case the flux homomorphism

$$\text{Flux} : \pi_1(\text{Symp}(M, \omega)) \longrightarrow H^1(M, \mathbb{R})$$

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does not vanish on Λ . However, if the action is Hamiltonian with generating Hamiltonian $K : M \rightarrow \mathbb{R}$ then the result is not so easy: the proof in [1] involved studying the Hofer length of the corresponding paths $\phi_t^K, t \in [0, T]$, in $\text{Ham}(M, \omega)$.

The first result in this paper uses the theorem above to answer a question posed by Alan Weinstein in [4]. Let G be a semisimple Lie group with Lie algebra \mathfrak{g} . Let $M \subset \mathfrak{g}^*$ be a coadjoint orbit, together with the Kostant–Kirillov symplectic form ω . If the coadjoint action of G on M is effective, then G is naturally a subgroup of $\text{Ham}(M, \omega)$, the group of Hamiltonian symplectomorphisms of (M, ω) . This inclusion induces a natural map from the fundamental group of G to the fundamental group of $\text{Ham}(M, \omega)$. Weinstein asks when this map is injective. We prove that this map is injective for all compact semisimple Lie groups. In [3] Vina established a special case of this result by quite different methods.

Theorem 1.2. *Let a compact semisimple Lie group G act effectively on a coadjoint orbit (M, ω) . Then the inclusion $G \rightarrow \text{Ham}(M, \omega)$ induces an injection from $\pi_1(G)$ to $\pi_1(\text{Ham}(M, \omega))$.*

In view of Theorem 1.1, this is an immediate consequence of the following result, which we prove in Section 2.

Proposition 1.3. *Let a compact semisimple Lie group G act effectively on a coadjoint orbit (M, ω) . Then every nontrivial element in $\pi_1(G)$ may be represented by a circle that acts semifreely on M .*

Theorem 1.1 immediately implies that if a compact Lie group G acts effectively on a closed symplectic manifold (M, ω) , then any semifree circle subgroup $\Lambda \subset G$ is essential in G . The other results in the paper generalize this claim.

Observe that Theorem 1.1 does not immediately extend to the smooth (non-symplectic) category. For example, Claude LeBrun pointed out to us that the circle action on S^4 induced by the diagonal action of S^1 on $\mathbb{C}^2 = \mathbb{R}^4 \subset \mathbb{R}^5$ is semifree but gives a nullhomotopic loop since $\pi_1(SO(5)) = \mathbb{Z}/2\mathbb{Z}$. Nevertheless, the semifree condition does have consequences in the smooth category, even if the action is only semifree on a neighborhood of a component of the fixed point set; we shall say that such components are **semifree**. Further, given a circle subgroup $\Lambda \subset G$ we say that $g \in G$ **reverses Λ in G** if $gtg^{-1} = t^{-1}$ for all $t \in \Lambda$. Finally, a component F of the fixed point set M^Λ of Λ is **symmetric in G** if there is an element $g \in G$ whose action on M fixes F pointwise and which reverses Λ .

Theorem 1.4. *Let Λ be a circle subgroup of a compact Lie group G which acts effectively on a connected manifold M . If there is a semifree component of the fixed point set M^Λ which is not symmetric in G , then Λ is essential in G .*

Example 1.5. First, let $G = SU(2)$ act on $\mathbb{C}\mathbb{P}^2$ by the defining representation on the first two copies of \mathbb{C} , and let $\Lambda \subset G$ be the circle subgroup

given by $\lambda \cdot [z_0 : z_1 : z_2] \mapsto [\lambda z_0 : \lambda^{-1} z_1 : z_2]$. This action has a semifree fixed point, namely $[0, 0, 1]$. Moreover, this circle subgroup is inessential in G . Therefore, by the theorem above, there exists $g \in G$ which reverses the circle action and fixes $[0, 0, 1]$. In fact, we can take any g which lies in the normalizer $N(\Lambda)$ but not in Λ itself. Note that $g^2 = -\mathbb{1}$ for any such g .

In contrast, consider the natural action of $G = PU(3)$ on $\mathbb{C}\mathbb{P}^2$, and let $\Lambda \subset G$ be the circle subgroup given by $\lambda \cdot [z_0 : z_1 : z_2] \mapsto [\lambda^2 z_0 : z_1 : z_2]$. This action is semifree and essential, but is not reversed by any $g \in G$. To see this, note that the circle has order 3 in $\pi_1(G)$, whereas every circle that can be reversed has order 1 or 2.

One can weaken the semifree hypothesis in the above theorem, at the cost of adding a global isotropy assumption and working once more in the symplectic category. We say that a circle action has **at most twofold isotropy** if every point which is not either fixed or free has stabilizer $\mathbb{Z}/(2)$. Recall, also, that a symplectic action of G on (M, ω) is Hamiltonian if it is given by an equivariant moment map $\Phi : M \rightarrow \mathfrak{g}^*$.

Theorem 1.6. *Let Λ be a circle subgroup of a compact Lie group G which acts effectively on a connected symplectic manifold (M, ω) . If Λ has at most twofold isotropy and if there is no $g \in G$ which reverses Λ , then Λ is essential in G .*

Example 1.7. This theorem does not extend to circle actions which have at most threefold isotropy. For example, the action of S^1 on $\mathbb{C}\mathbb{P}^3$ given by $\lambda \cdot [x, y, z, w] = [\lambda^2 x, \lambda^{-1} y, \lambda^{-1} z, w]$ is inessential in $PU(4)$. However, since F_{\max} and F_{\min} are not diffeomorphic, this action has no reversor.

We also need the symplectic hypothesis. To see this, consider the obvious action of $SU(3)$ on $S^6 := \mathbb{C}^3 \cup \{\infty\}$. The subgroup $\Lambda := \text{diag}(\lambda^2, \lambda^{-1}, \lambda^{-1})$ acts with at most twofold isotropy but has no reversor.

Remark 1.8. If G is a simple group, we do not need to assume that (M, ω) is symplectic in Theorem 1.6; we only need to assume that there exists a point p which is fixed by a maximal torus containing Λ but is not fixed by all of G . Note that, in contrast, in the example above, the only points fixed by Λ are fixed by all of G .

Remark 1.9. If Λ is *any* circle subgroup of $SO(3)$ – or indeed a subgroup of any simple group of type B_n , C_n , or F_4 – then there exists a $g \in G$ which reverses Λ . In this case, Theorem 1.6 is trivial and the force of Theorem 1.4 is that we can choose g so that it also fixes p .

Remark 1.10. In the proof of the above theorems, we pick a maximal torus T which contains Λ . The reversor g that we construct lies in the normalizer $N(T)$ and has the property that g^2 lies in T . However, as we saw in Example 1.5, g^2 may not be equal to the identity.

Theorems 1.4 and 1.6 have the following easy corollaries:

Corollary 1.11. *Consider a Hamiltonian circle action Λ on a closed symplectic manifold (M, ω) with moment map $K : M \rightarrow \mathbb{R}$, normalized so that $\int_M K \omega^n = 0$. If F is a semifree fixed component, then Λ is essential in every compact subgroup $G \subset \text{Symp}(M, \omega)$ that contains it, unless there is a symplectomorphism g of M that fixes F and reverses Λ . In this case, all the following hold:*

- (1) $K(g(p)) = -K(p)$ for all $p \in M^\Lambda$.
- (2) There is a one-to-one correspondence between the positive weights at p and the negative weights at $g(p)$, and vice versa.
- (3) g induces an isomorphism on the image of the restriction map in equivariant cohomology $H_{S^1}^*(M) \rightarrow H_{S^1}^*(M^\Lambda)$.
- (4) $g(F) = F$. In particular,
 - (a) $K(F) = 0$
 - (b) The sum of the weights at F is zero.

Corollary 1.12. *Consider a Hamiltonian circle action Λ on a closed symplectic manifold (M, ω) with moment map $K : M \rightarrow \mathbb{R}$, normalized so that $\int_M K \omega^n = 0$. If the action has at most twofold isotropy, then Λ is essential in every compact subgroup $G \subset \text{Symp}(M, \omega)$ that contains it, unless there is a symplectomorphism g of M that reverses Λ . In this case, all the following hold:*

- (1) $K(g(p)) = -K(p)$ for all $p \in M^\Lambda$.
- (2) There is a one-to-one correspondence between the positive weights at p and the negative weights at $g(p)$, and vice versa.
- (3) g induces an isomorphism on the image of the restriction map in equivariant cohomology $H_{S^1}^*(M) \rightarrow H_{S^1}^*(M^\Lambda)$.

It is unknown whether the existence of such g is necessary for Λ to be inessential in $\text{Symp}(M, \omega)$. We make partial progress towards answering this question in [2].

All the results in this paper are proved by a case by case study of the structure of semisimple Lie algebras.

2. COADJOINT ORBITS

In this section, we prove Proposition 1.3. We begin with a brief review of a few facts about Lie groups.

Each simply connected compact semisimple Lie group is a product of simple factors, and its center is the product of the centers of its simple factors. Moreover, since its Lie algebra splits into a corresponding sum, the coadjoint orbits also are products of coadjoint orbits of simple groups. Therefore, we may assume that G is simple.

Let G be a compact simple Lie group. Let \tilde{G} denote the universal cover of G , and \widehat{G} denote the quotient of G by its center. Let \mathfrak{g} denote the Lie algebra of G , and let \mathfrak{t} denote the Lie algebra of a maximal torus $T \subset G$. Let $\ell \subset \mathfrak{t}$, $\tilde{\ell} \subset \mathfrak{t}$, and $\widehat{\ell} \subset \mathfrak{t}$ be the lattices consisting of vectors $\xi \in \mathfrak{t}$

whose exponential is the identity in G , \tilde{G} , and \widehat{G} , respectively. There is a one-to-one correspondence between ℓ and circle subgroup of G , $\tilde{\ell}$ and circle subgroups of \tilde{G} , and $\widehat{\ell}$ and circle subgroups of \widehat{G} , given by sending λ to $t \rightarrow \exp(t\lambda)$. Note that $\tilde{\ell} \subseteq \ell \subseteq \widehat{\ell}$. Because \tilde{G} is simply connected,

$$\pi_1(G) \cong \ell/\tilde{\ell} \subseteq \widehat{\ell}/\tilde{\ell} \cong \pi_1(\widehat{G}).$$

Let \mathfrak{t}^* denote the dual to \mathfrak{t} , and let $\Delta \subset \mathfrak{t}^*$ denote the set of **roots** of G , i.e. the nonzero weights of the adjoint action T on $\mathfrak{g}_{\mathbb{C}}$, where $\mathfrak{g}_{\mathbb{C}}$ is the complexification of \mathfrak{g} . The lattice $\widehat{\ell}$ is dual to the lattice in \mathfrak{t}^* generated by the roots, i.e. $\lambda \in \widehat{\ell}$ precisely when $\eta(\lambda) \in \mathbb{Z}$ for all $\eta \in \Delta$. If we use the Killing form (\cdot, \cdot) to identify \mathfrak{t} and \mathfrak{t}^* , then $\tilde{\ell}$ is generated by the set

$$\left\{ \frac{2\eta}{(\eta, \eta)} \mid \eta \in \Delta \right\}.$$

Further the set of weights at any fixed point p for the action of T on M is a nonempty subset of the set of roots. Therefore the result will follow if we find a representative λ for each nontrivial class in $\widehat{\ell}/\tilde{\ell}$ such that $|\eta(\lambda)| \leq 1$ for every $\eta \in \Delta$.

We will check this on a case by case basis; in each case we will use the Killing form to identify \mathfrak{t} and \mathfrak{t}^* . Let (\cdot, \cdot) be the standard metric on \mathbb{R}^k with the standard basis e_1, \dots, e_k , and define

$$\epsilon_i = e_i - \frac{1}{k} \sum_{j=1}^k e_j.$$

(I) For the group A_n , where $n \geq 1$, $\mathfrak{t} = \mathfrak{t}^* = \{\lambda \in \mathbb{R}^{n+1} \mid \sum \lambda_i = 0\}$ and the roots are $\epsilon_i - \epsilon_j = e_i - e_j$ for $i \neq j$. Hence

$$\widehat{\ell} = \{\lambda \in \mathfrak{t} \mid \lambda_i - \lambda_j \in \mathbb{Z} \forall i, j\}, \text{ and } \tilde{\ell} = \mathfrak{t} \cap \mathbb{Z}^{n+1}.$$

As representatives for the quotient $\widehat{\ell}/\tilde{\ell} \cong \mathbb{Z}/(n+1)$, we take $\lambda = \sum_{i=1}^k \epsilon_i$ for $0 \leq k \leq n$.

(II) For the group B_n , where $n \geq 2$, $\mathfrak{t}^* = \mathbb{R}^n$ and the roots are $\pm e_i$ and $\pm e_i \pm e_j$ for $i \neq j$. Hence

$$\widehat{\ell} = \mathbb{Z}^n, \text{ and } \tilde{\ell} = \left\{ \lambda \in \mathbb{Z}^n \mid \sum \lambda_i \in 2\mathbb{Z} \right\}.$$

As representatives of the quotient $\widehat{\ell}/\tilde{\ell} \cong \mathbb{Z}/(2)$, we take 0 and e_1 .

(III) For the group C_n , where $n \geq 3$, $\mathfrak{t}^* = \mathbb{R}^n$ and the roots are $\pm 2e_i$ and $\pm e_i \pm e_j$ for $i \neq j$. Hence

$$\widehat{\ell} = \{\lambda \in \mathbb{R}^n \mid \lambda_i \pm \lambda_j \in \mathbb{Z}, \forall i, j\}, \text{ and } \tilde{\ell} = \mathbb{Z}^n.$$

As representatives of the quotient $\widehat{\ell}/\tilde{\ell} \cong \mathbb{Z}/(2)$, we take 0 and $\frac{1}{2} \sum_{i=1}^n e_i$.

(IV) For the group D_n , where $n \geq 4$, $\mathfrak{t}^* = \mathbb{R}^n$ and the roots are $\pm e_i \pm e_j$ for $i \neq j$. Hence

$$\widehat{\ell} = \{\lambda \in \mathbb{R}^n \mid \lambda_i \pm \lambda_j \in \mathbb{Z}, \forall i, j\}, \text{ and } \widetilde{\ell} = \left\{ \lambda \in \mathbb{Z}^n \mid \sum \lambda_i \in 2\mathbb{Z} \right\}.$$

The quotient $\widehat{\ell}/\widetilde{\ell}$ is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ if n is even, and to $\mathbb{Z}/(4)$ if n is odd. Either way, as representatives of $\widehat{\ell}/\widetilde{\ell}$, we take 0 , e_1 , $\frac{1}{2} \sum_{i=1}^n e_i$ and $\frac{1}{2} \sum_{i=1}^n e_i - e_n$.

(V, a) For the group E_6 , $\mathfrak{t}^* = \mathbb{R}^6$ and the roots are 2ϵ , $\epsilon_i - \epsilon_j$, and $\epsilon_i + \epsilon_j + \epsilon_k \pm \epsilon$ for i, j , and k distinct, where $\epsilon = \frac{1}{2\sqrt{3}}(1, 1, 1, 1, 1, 1)$. Hence,

$$\widehat{\ell} = \left\{ n\epsilon + (\xi_1, \dots, \xi_6) \in \mathbb{R}^6 \mid \sum_{i=1}^6 \xi_i = 0, n \in \mathbb{Z}, \frac{n}{2} + 3\xi_i \in \mathbb{Z} \text{ and } \xi_i - \xi_j \in \mathbb{Z} \forall i, j \right\}, \text{ and}$$

$$\widetilde{\ell} = \left\{ n\epsilon + (\xi_1, \dots, \xi_6) \in \mathbb{R}^6 \mid \sum_{i=1}^6 \xi_i = 0, n \in \mathbb{Z} \text{ and } \frac{n}{2} + \xi_i \in \mathbb{Z} \forall i \right\}.$$

As representatives of the quotient $\widehat{\ell}/\widetilde{\ell} \cong \mathbb{Z}/(3)$, we take 0 , $\epsilon_1 + \epsilon_2$, and $-\epsilon_1 - \epsilon_2$.

(V, b) For the group E_7 , $\mathfrak{t} = \mathfrak{t}^* = \{\lambda \in \mathbb{R}^8 \mid \sum \lambda_i = 0\}$, and the roots are $\epsilon_i - \epsilon_j$, and $\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l$ for i, j, k , and l distinct. Hence

$$\widehat{\ell} = \{\lambda \in \mathfrak{t} \mid 4\lambda_i \in \mathbb{Z} \text{ and } \lambda_i - \lambda_j \in \mathbb{Z} \forall i, j\}, \text{ and } \widetilde{\ell} = \{\lambda \in \mathfrak{t} \mid \lambda_i \pm \lambda_j \in \mathbb{Z} \forall i, j\}.$$

As representatives for the quotient $\widehat{\ell}/\widetilde{\ell} \cong \mathbb{Z}/2\mathbb{Z}$, we take 0 and $\epsilon_1 + \epsilon_2$.

Every group of type E_8 , F_4 , and G_2 is simply connected, so no further argument is necessary. \square

3. LIE GROUP ACTIONS

This section contains proofs of Theorems 1.4 and 1.6. We begin by stating a lemma about root systems, that is proved at the end. We shall always assume that the positive Weyl chamber is closed.

Lemma 3.1. *Let G be a simply connected compact simple Lie group. Let \mathfrak{t} be the Lie algebra of a maximal torus $T \subset G$. Let $\widetilde{\ell}$ be the integral lattice, let Δ denote the set of roots, and let W denote the Weyl group. Use the Killing form to identify \mathfrak{t} and \mathfrak{t}^* . Fix $\lambda \in \widetilde{\ell}$. Choose a positive Weyl chamber which contains λ . Let $\delta \in \Delta$ denote the highest root. Then the following claims hold:*

- (a) *If $(\lambda, \delta) \leq 2$, then there exist orthogonal roots $\eta_1, \dots, \eta_k \in \Delta$ so that $\lambda = \sum a_i \eta_i$ and so that $(\lambda, \eta_i) = a_i(\eta_i, \eta_i) = 2$ for all i .*
- (b) *Let $L \subset \Delta$ be a set of roots which contains every root $\eta \in \Delta$ such that $\delta + \eta$ or $\delta - \eta$ is also a root. Assume also that L is closed under addition, that is, it contains every root which can be written as the sum of roots in L . Then L contains all roots.*

- (c) If $(\lambda, \delta) > 2$ and $-\text{id} : \mathfrak{t} \rightarrow \mathfrak{t}$ is not an element of the Weyl group, then for every nonzero weight $\alpha \in \tilde{\ell}^*$ there exists $\sigma \in W$ so that $|(\sigma \cdot \alpha, \lambda)| > 1$.
- (d) If $-\text{id} : \mathfrak{t} \rightarrow \mathfrak{t}$ is not an element of the Weyl group, then δ is the only root which lies in the positive Weyl chamber.

Using this result, we can find elements which reverse certain circle subgroups of simply connected compact simple Lie groups. Note that because G is simply connected, every circle subgroup of G is inessential in G .

Lemma 3.2. *Let Λ be a circle subgroup of a simply connected compact simple Lie group G .*

- (i) *Let $\rho : G \rightarrow \text{GL}(V)$ be a nontrivial representation of G . If Λ acts semifreely on V then there exists $g \in G$ that reverses Λ .*
- (ii) *Let $H \subsetneq G$ be a proper subgroup containing Λ . If the adjoint action of Λ on $\mathfrak{g}/\mathfrak{h}$ is semifree, then there exists $h \in H$ that reverses Λ .*
- (iii) *Let $H \subsetneq G$ be a proper subgroup containing a maximal torus which contains Λ . If the natural action of Λ on G/H has at most twofold isotropy, then there exists $g \in G$ that reverses Λ .*

The assumption in (ii) above is a special case of (i) since the representation V is restricted; however, the conclusion is stronger since it asserts that the reversor lies in H . Statement (ii) and (iii) are also related: the former makes a strong assumption about the action induced by Λ on the tangent space to G/H at the fixed point eH , the latter makes a weaker assumption about the action at all the fixed points on G/H .

We will now use the claims in Lemma 3.1 to prove Lemma 3.2. Let T be a maximal torus which contains Λ . Let $\tilde{\ell} \subset \mathfrak{t}$ denote the integral lattice. Let $\lambda \in \tilde{\ell}$ be the vector corresponding to Λ . Choose a positive Weyl chamber which contains λ . Let $\delta \in \Delta$ denote the highest root.

Recall that the Weyl group W is the quotient $N(T)/T$, where $N(T)$ is the normalizer of T in G . Every root η gives rise to an element $w_\eta \in W$ whose action on \mathfrak{t}^* is given by $w_\eta(\beta) = \beta - \frac{2(\eta, \beta)}{(\eta, \eta)}\eta$.

Proof of Lemma 3.2 (i). Let $\rho : G \rightarrow \text{GL}(V)$ be a nontrivial representation of G . Assume that Λ acts semifreely on V .

Suppose first that $(\lambda, \delta) \leq 2$. By claim (a), there exist orthogonal roots $\eta_1, \dots, \eta_k \in \Delta$ so that $\lambda = \sum a_i \eta_i$. Since the roots are orthogonal, for each η_i the associated element of the Weyl group w_{η_i} takes η_i to $-\eta_i$ and leaves η_j fixed for all $j \neq i$. Hence, their product $w = w_{\eta_1} \cdots w_{\eta_n}$ takes λ to $-\lambda$, and so reverses Λ .

So assume instead that $(\lambda, \delta) > 2$. If $-\text{id}$ is in the Weyl group, then statement (i) is trivial. So we assume that it is not. Let T act on V via restriction, and pick any nonzero weight $\alpha \in \tilde{\ell}^*$ in the weight decomposition. By claim (c), we can find some $\sigma \in W$ such that $|(\sigma \cdot \alpha, \lambda)| > 1$. Since $\sigma \cdot \alpha$

also appears in the weight decomposition, this contradicts the assumption that the action of Λ on V is semifree. \square

Proof of Lemma 3.2 (ii). Let $H \subseteq G$ be a proper subgroup which contains Λ . Assume that the adjoint action of Λ on $\mathfrak{g}/\mathfrak{h}$ is semifree. Let L be the set of roots $\eta \in \Delta$ so that the associated weight space $E_\eta \subset \mathfrak{g}_\mathbb{C}$ lies in $\mathfrak{h}_\mathbb{C}$. Clearly, if $|(\eta, \lambda)| > 1$, then $\eta \in L$.

Suppose first that $(\lambda, \delta) \leq 2$. By claim (a), there exist orthogonal roots $\eta_1, \dots, \eta_k \in \Delta$ so that $\lambda = \sum a_i \eta_i$ and so that $(\lambda, \eta_i) = 2$ for every i . Since $(\eta_i, \lambda) = 2$, η_i lies in L for all i . Hence, the associated element of the Weyl group w_{η_i} lies in H for all i . Thus $w = w_{\eta_1} \cdots w_{\eta_k}$ must lie in H .

So assume instead that $(\lambda, \delta) > 2$. We see immediately that δ and $-\delta$ lie in L . If η , η' and $\eta + \eta'$ are all roots, then $[E_\eta, E_{\eta'}] = E_{\eta + \eta'}$. Hence, since $\mathfrak{h}_\mathbb{C}$ is closed under Lie bracket, if η and η' are in L then $\eta + \eta' \in L$ also, that is, L is closed under addition. Additionally, if η and η' are roots such that $\delta = \eta + \eta'$, then either $(\lambda, \eta) > 1$ or $(\lambda, \eta') > 1$. If the former holds, then η and $-\eta$ lie in L . Since L is closed under addition, so do η' and $-\eta'$. The other case is identical. Thus, claim (b) implies that every root lies in L . This contradicts the claim that H is a proper subgroup. \square

Proof of Lemma 3.2 (iii). Let $H \subsetneq G$ be a proper subgroup which contains the maximal torus T , and assume that the natural action of $\Lambda \subset T$ on G/H has at most twofold isotropy.

If $(\lambda, \delta) \leq 2$, then part (iii) follows by the argument used to prove part (i). So assume that $(\lambda, \delta) > 2$. We may also assume that $-\text{id}$ does not lie in the Weyl group, because otherwise the claim is trivial. Since $H \subset G$ is proper, there exists at least one root η so that the associated weight space E_η is not contained in $\mathfrak{h}_\mathbb{C}$. Then there is $\sigma \in W$ so that the root $\sigma \cdot \eta$ lies in the positive Weyl chamber. Hence by (d) $\sigma \cdot \eta = \delta$, and so $|(\sigma \cdot \eta, \lambda)| > 2$. Choose $\tilde{\sigma} \in N(T)$ which descends to σ . Then $\tilde{\sigma}H$ is a fixed point for T , and $\sigma \cdot \eta$ is one of the weights for Λ at this fixed point. This contradicts the fact that the action has at most twofold isotropy. \square

We are now ready to deduce Theorems 1.4 and 1.6. In both cases, we will do this by proving the contrapositive, that is, we will assume that Λ is an inessential circle subgroup and use this to construct a reversor.

Let \tilde{G} denote the universal cover of G . Then \tilde{G} is the direct product of a compact simply connected semisimple Lie group and a vector space. Since Λ is inessential, it lifts to a circle subgroup of \tilde{G} . Since this lift must lie in the compact part of \tilde{G} , we may assume without loss of generality that \tilde{G} is a compact simply connected semisimple Lie group.

In fact, it is enough to prove these claims for the universal cover of G , as long as we no longer insist on an effective action but instead allow a finite number of elements of the group to act trivially on M . Thus we may assume that G is the product of compact simple and simply connected

groups $G_1 \times \cdots \times G_n$. Let Λ_i be the projection of Λ to G_i . Without loss of generality, we may assume that $\Lambda_i \neq \{\text{id}\}$ for all i .

Proof of Theorem 1.4. Let $G = G_1 \times \cdots \times G_n$ as above. Choose $p \in F$ and let $H \subset G$ be the stabilizer of p . Then $\Lambda \subset H$. There exists a representation V of H , called the **isotropy representation**, so that a neighborhood of the G -orbit through p is equivariantly diffeomorphic to a neighborhood of the zero section of $G \times_H V$. Fix some simple factor G_i , and let $H_i = H \cap G_i$.

Assume first that H_i is a proper subgroup. Note that \mathfrak{g}_i is invariant under the action of Λ . Thus, since Λ acts semifreely on $\mathfrak{g}/\mathfrak{h}$ via the adjoint action, Λ_i acts semifreely on $\mathfrak{g}_i/\mathfrak{h}_i$. Thus, by Lemma 3.2 (ii) there exists an element $h_i \in H_i$ that reverses Λ_i .

So assume on the contrary that $H_i = G_i$. Let Λ' be the projection of Λ onto the product of all the simple factors except G_i . Since $\Lambda_i \subset G_i \subset H$ and $\Lambda \subset H$, we must have $\Lambda' \subset H$. Hence Λ' acts on V . For any integer k , let V_k denote the subspace of V on which Λ' acts with weight k . Since Λ' commutes with G_i , V_k is a representation of G_i . Since only a finite number of elements of G act trivially on M , G_i must act nontrivially on $G \times_H V$, and hence also on V . Therefore, there is some k so that the representation of G_i on V_k is nontrivial. Because G_i is simple, Λ_i must act with both positive and negative weights on V_k . But the weights for the action of Λ on V_k are the weights for the action of Λ_i shifted by k . Hence, because F is a semifree fixed point component, $k = 0$ and the action of Λ_i on V_k is itself semifree. Therefore by Lemma 3.2 (i) there exists $h_i \in G_i = H_i$ that reverses Λ_i .

Since h_i reverses Λ_i for each i , $g = (h_1, \dots, h_n)$ reverses Λ , as required. Moreover, since $H_1 \times \cdots \times H_n \subset H$ (in general they are not equal), g lies in H , and hence fixes p . \square

Proof of Theorem 1.6. Fix some simple factor G_i . Let W be the Weyl group of G_i . Let $T \subset G_i$ be a maximal torus of G_i containing Λ_i . Let $\Phi : M \rightarrow \mathfrak{t}^*$ be the moment map for the T -action. Pick any $\xi \in \mathfrak{t}$ so that the one parameter subgroup generated by ξ is dense in T . Let p be any point which maps to the minimum value of Φ^ξ , the component of Φ in the direction ξ . By construction, p is a fixed point for T . Assume first that $\Phi^\xi(p) = 0$, that is, the function Φ^ξ is nonnegative on M . Since the moment polytope $\Phi(M)$ is invariant under the Weyl group W , this implies that $\Phi^{\sigma \cdot \xi}$ is also nonnegative on M for all $\sigma \in W$. Because G_i is simple and ξ is a generic point of \mathfrak{t} , for any nonzero $x \in \mathfrak{t}^*$ there exists an element $\sigma \in W$ such that $(\sigma \cdot \xi, x) < 0$. Applying this to $x \in \Phi(M) \setminus \{0\}$, we see that $\Phi(M)$ must be the single point $\{0\}$, which is impossible, because the action is effective. Therefore, $\Phi(p) \neq 0$.

Now let us reconsider the action of G on M . Let H be the stabilizer of p in G , and let $H_i = H \cap G_i$. Since Λ acts with at most twofold isotropy on $G/H \subset M$, Λ_i acts with at most twofold isotropy on G_i/H_i . Since $\Phi(p)$ is not zero, the stabilizer of $\Phi(p)$ in G_i is a proper subgroup of G_i . Since Φ is equivariant, this implies that H_i is a proper subgroup of G_i . By Lemma

3.2 (iii), this implies that there exists $g_i \in G_i$ which reverses Λ_i . Then (g_1, \dots, g_n) reverses Λ . \square

Proof of Lemma 3.1. We now prove claims (a)-(d) on a case by case basis, using the classification of compact simple Lie groups. We will use the notation of §2. Note, however, that here $G = \tilde{G}$ since G is simply connected.

(I) Recall that for the group A_n , where $n \geq 1$, $\mathfrak{t} = \mathfrak{t}^* = \{\xi \in \mathbb{R}^n \mid \sum \xi_i = 0\}$, the roots are $\epsilon_i - \epsilon_j = e_i - e_j$ for $i \neq j$, and the integral lattice is $\tilde{\ell} = \mathbb{Z}^{n+1} \cap \mathfrak{t}$. The positive Weyl chamber is¹

$$\{\xi \in \mathfrak{t} \mid \xi_1 \geq \dots \geq \xi_{n+1}\}.$$

The highest root is $\delta = e_1 - e_{n+1}$.

If $(\lambda, \delta) = \lambda_1 - \lambda_{n+1} \leq 2$, then $|\lambda_i| \leq 1$ for all i . Since $\sum_i \lambda_i = 0$ and $\lambda_i \in \mathbb{Z}$ for all i , there are an equal number of +1's and -1's, and the rest are 0's. Hence, λ is the sum of orthogonal roots of the form $\eta = e_i - e_j$. Since $(\eta, \eta) = 2$, this proves claim (a).

Since $\delta = (e_1 - e_k) + (e_k - e_{n+1})$, the roots $\pm(e_1 - e_k)$ and $\pm(e_k - e_{n+1})$ lie in L for all $1 < k < n + 1$. If neither i nor j is equal to 1, then $e_i - e_j = -(e_1 - e_i) + (e_1 - e_j)$ is also in L . This proves claim (b).

We now prove (c). The weight lattice is $\tilde{\ell}^* = \{\alpha \in \mathfrak{t} \mid \alpha_i - \alpha_j \in \mathbb{Z} \ \forall i, j\}$. By permuting the coordinates of α , we may assume $\alpha_1 \geq \dots \geq \alpha_n$. Since $\alpha \neq 0$, there exists $k \in (1, \dots, n)$ such that $\alpha_k - \alpha_{k+1} > 0$; since this difference lies in \mathbb{Z} , it must be at least 1. Since $\lambda_1 - \lambda_{n+1} = \lambda_1 + \sum_{i=1}^n \lambda_i > 2$ and $\lambda_i \geq \lambda_{i+n-k}$, $\sum_{i=1}^k \lambda_i + \sum_{i=1}^{n+1-k} \lambda_i > 2$. Therefore, either $\sum_{i=1}^k \lambda_i > 1$ or $\sum_{i=1}^{n+1-k} \lambda_i > 1$. In the former case,

$$(\alpha, \lambda) = \sum_{j=1}^n \left((\alpha_j - \alpha_{j+1}) \sum_{i=1}^j \lambda_i \right) \geq (\alpha_k - \alpha_{k+1}) \sum_{i=1}^k \lambda_i > 1.$$

In the latter case, let α' be obtained from α by the permutation which reverses the coordinates, so that $\alpha'_i = \alpha_{n+2-i}$. Then

$$(\alpha', \lambda) = \sum_{j=1}^n \left((\alpha_{n+2-j} - \alpha_{n+1-j}) \sum_{i=1}^j \lambda_i \right) \leq (\alpha_{k+1} - \alpha_k) \sum_{i=1}^{n+1-k} \lambda_i < -1.$$

The only facts we have used are that $\mathfrak{t} = \{\xi \in \mathbb{R}^n \mid \sum_i \xi_i = 0\}$, that the Weyl group contains the permutation group S_n , and that $\alpha_i - \alpha_j \in \mathbb{Z}$ for any $\alpha \in \tilde{\ell}^*$.

Finally, δ is the only root in the positive Weyl chamber.

(II) Recall that for the group B_n , where $n \geq 2$, $\mathfrak{t} = \mathfrak{t}^* = \mathbb{R}^n$, the roots are $\pm e_i$ and $\pm e_i \pm e_j$ for $i \neq j$, and the integral lattice is $\tilde{\ell} = \{\xi \in \mathbb{Z}^n \mid \sum_i \xi_i \in \mathbb{Z}\}$.

¹For uniformity, we shall always use the lexicographical order to choose the positive Weyl chamber.

$2\mathbb{Z}$ }. The positive Weyl chamber is $\{\xi \in \mathfrak{t} \mid \xi_1 \geq \cdots \geq \xi_n \geq 0\}$. The highest root is $\delta = e_1 + e_2$.

If $(\lambda, \delta) = \lambda_1 + \lambda_2 \leq 2$, then either $\lambda_1 = 2$ and $\lambda_i = 0$ for all $i \neq 1$, or $\lambda_i \leq 1$ for all i . Either way, since $\sum_i \lambda_i \in 2\mathbb{Z}$, we can write λ as the sum of orthogonal roots η_i such that $(\eta_i, \eta_i) = 2$.

Since $\delta = (e_1 - e_k) + (e_2 + e_k) = (e_1 + e_k) + (e_2 - e_k)$, the roots $\pm e_1 \pm e_k$ and $\pm e_2 \pm e_k$ lie in L for $k \neq 1$ or 2 . Since $\delta = (e_1) + (e_2)$, the roots $\pm e_1$ and $\pm e_2$ lie in L . Every root can be written as a sum of these roots.

Since $-\text{id}$ lies in the Weyl group, we are done.

(III) Recall that for the group C_n , where $n \geq 3$, $\mathfrak{t} = \mathfrak{t}^* = \mathbb{R}^n$, the roots are $\pm 2e_i$ and $\pm e_i \pm e_j$ for $i \neq j$, and the integral lattice is $\tilde{\ell} = \mathbb{Z}^n$. The positive Weyl chamber is $\{\xi \in \mathfrak{t} \mid \xi_1 \geq \cdots \geq \xi_n \geq 0\}$. The highest root is $\delta = 2e_1$.

If $(\lambda, \delta) = 2\lambda_1 \leq 2$, then $\lambda_i \leq 1$ for all i . Since $\lambda \in \mathbb{Z}^n$, we can write λ as half the sum of orthogonal roots of the form $2e_i$. Note that $(\lambda, 2e_i) = 2$.

Since $\delta = (e_1 - e_k) + (e_1 + e_k)$, the roots $\pm e_1 \pm e_k$ lie in L for $k \neq 1$. Every root can be written as a sum of these roots.

Since $-\text{id}$ lies in the Weyl group, we are done.

(IV) Recall that for the group D_n , where $n \geq 4$, $\mathfrak{t} = \mathfrak{t}^* = \mathbb{R}^n$, the roots are $\pm e_i \pm e_j$ for $i \neq j$, and the integral lattice is $\tilde{\ell} = \{\xi \in \mathbb{Z}^n \mid \sum \xi_i \in 2\mathbb{Z}\}$. The positive Weyl chamber is $\{\xi \in \mathfrak{t} \mid \xi_1 \geq \cdots \geq \xi_{n-1} \geq |\xi_n|\}$. The highest root is $\delta = e_1 + e_2$.

If $(\lambda, \delta) = \lambda_1 + \lambda_2 \leq 2$, then either $\lambda_1 = 2$ and $\lambda_i = 0$ for all $i \neq 1$, or $|\lambda_i| \leq 1$ for all i . Either way, since $\sum_i \lambda_i \in 2\mathbb{Z}$, we can write λ as the sum of orthogonal roots η_i such that $(\eta_i, \eta_i) = 2$.

Since $\delta = (e_1 - e_k) + (e_2 + e_k) = (e_1 + e_k) + (e_2 - e_k)$, the roots $\pm e_1 \pm e_k$ and $\pm e_2 \pm e_k$ lie in L for $k \neq 1$ or 2 . Every root can be written as a sum of these roots.

Now assume that $(\delta, \lambda) = \lambda_1 + \lambda_2 > 2$. Consider a nonzero weight $\alpha \in \tilde{\ell}^* = \{\alpha \in \mathbb{R}^n \mid \alpha_i \pm \alpha_j \in \mathbb{Z} \forall i, j\}$. By applying the Weyl group, we may assume α lies in the positive Weyl chamber. Since λ also lies in the positive Weyl chamber, $\alpha_i \lambda_i \geq 0$ for all $i \neq n$. Moreover, since $\alpha_{n-1} \geq |\alpha_n|$, and $\lambda_{n-1} \geq |\lambda_n|$, $\alpha_{n-1} \lambda_{n-1} + \alpha_n \lambda_n \geq 0$. Therefore, $\alpha_3 \lambda_3 + \cdots + \alpha_n \lambda_n \geq 0$. (Here, we have used that $n \geq 4$.) Since α is nonzero, either $\alpha_1 \geq 1$, or $\alpha_1 = \alpha_2 = \frac{1}{2}$. In either case, $\alpha_1 \lambda_1 + \alpha_2 \lambda_2 > 1$. (In the first case, we use the fact that $\lambda_1 + \lambda_2 > 2$ and $\lambda_1 \geq \lambda_2$ implies that $\lambda_1 > 1$.) Therefore, $(\alpha, \lambda) \geq \alpha_1 \lambda_1 + \alpha_2 \lambda_2 > 1$. This proves claim (c).

Finally, δ is the only root in the positive Weyl chamber.

(V, a) Recall that for the group E_6 , $\mathfrak{t} = \mathfrak{t}^* = \mathbb{R}^6$ and the roots are 2ϵ , $\epsilon_i - \epsilon_j$, and $\epsilon_i + \epsilon_j + \epsilon_k \pm \epsilon$ for i, j , and k distinct, where $\epsilon = \frac{1}{2\sqrt{3}}(1, 1, 1, 1, 1, 1)$. Therefore

$$\tilde{\ell} = \left\{ n\epsilon + (\xi_1, \dots, \xi_6) \mid \sum_{i=1}^6 \xi_i = 0, n \in \mathbb{Z}, \text{ and } \frac{n}{2} + \xi_i \in \mathbb{Z} \forall i \right\}.$$

The positive Weyl chamber is

$$\left\{ n\epsilon + (\xi_1, \dots, \xi_6) \in \mathfrak{t} \mid \sum_{i=1}^6 \xi_i = 0, \xi_2 \geq \dots \geq \xi_6, \xi_1 + \xi_5 + \xi_6 \geq n/2 \geq 0 \right\}.$$

(Note that these conditions imply $\xi_1 \geq \xi_2$.) The highest root is $\delta = \epsilon_1 - \epsilon_6$.

Write $\lambda = n\epsilon + (\xi_1, \dots, \xi_6)$, where $\sum_i \xi_i = 0$. Assume that $(\lambda, \delta) = \xi_1 - \xi_6 \leq 2$. Combining the inequalities $\xi_1 - \xi_6 \leq 2$, $\xi_4 \geq \xi_5$, $\xi_4 \geq \xi_6$, and $\xi_1 + \xi_5 + \xi_6 \geq \frac{n}{2}$, we see that $\xi_4 \geq \frac{n-4}{6}$. Since also $\xi_2 + \xi_3 + \xi_4 \leq 0$, $\xi_2 \geq \xi_4$, and $\xi_3 \geq \xi_4$, we have $\xi_4 \leq 0$. Moreover, in both cases, if the final inequality in the sentence is an equality, so are all the preceding ones. Since $n \geq 0$, $0 \geq \xi_4 \geq -\frac{4}{6}$. Since $\lambda \in \tilde{\ell}$, $\xi_4 = 0$ or $\xi_4 = -\frac{1}{2}$. In the former case, $\xi_2 = \xi_3 = \xi_4 = 0$, so $\xi_1 + \xi_5 + \xi_6 = 0$, so $n = 0$. Hence, $\lambda = (\epsilon_1 - \epsilon_6)$. In the latter case, n is odd, so $\xi_4 = -\frac{1}{2} \geq \frac{n-4}{6}$ implies that $n = 1$. In this case, $\lambda = (\epsilon_1 - \epsilon_6) + (\epsilon + \epsilon_1 + \epsilon_2 + \epsilon_6)$. This proves claim (a).

Since $\delta = (\epsilon_1 - \epsilon_i) + (\epsilon_i - \epsilon_6)$, the roots $\pm(\epsilon_1 - \epsilon_i)$ and $\pm(\epsilon_i - \epsilon_6)$ lie in L for all $1 < i < 6$. Moreover, $\delta = (\epsilon + \epsilon_1 + \epsilon_i + \epsilon_j) - (\epsilon + \epsilon_i + \epsilon_j + \epsilon_6)$, so the roots $\pm(\epsilon + \epsilon_1 + \epsilon_i + \epsilon_j)$ and $\pm(\epsilon + \epsilon_i + \epsilon_j + \epsilon_6)$ lie in L for all $1 < i < j < 6$. Since, for example, $\epsilon + \epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon - \epsilon_4 - \epsilon_5 - \epsilon_6$, it follows easily that L contains all roots.

Let $\alpha \in \tilde{\ell}^*$ be a nonzero weight. Write $\lambda = n\epsilon + \xi$ as before. By applying the Weyl group, we may assume that $\alpha = m\epsilon + (\zeta_1, \dots, \zeta_6)$ is in the positive Weyl chamber. Since $(\alpha, \lambda) \geq (\zeta, \xi)$, it is enough to show that $(\zeta, \xi) > 1$. This fact now follows from the argument from A_5 , since $(\delta, \lambda) = (\delta, \xi) > 2$, since the Weyl group contains the permutation group S_5 , and since ζ must satisfy $\zeta_i - \zeta_j \in \mathbb{Z}$.

Finally, δ is the only root in the positive Weyl chamber.

(V, b) Recall that for the group E_7 , $\mathfrak{t} = \mathfrak{t}^* = \{\xi \in \mathbb{R}^8 \mid \sum \xi_i = 0\}$, the roots are $\epsilon_i - \epsilon_j$ and $\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l$ for i, j, k , and l distinct, and the integral lattice is $\tilde{\ell} = \{\xi \in \mathfrak{t} \mid \xi_i \pm \xi_j \in \mathbb{Z} \forall i, j\}$. The positive Weyl chamber is

$$\{\xi \in \mathfrak{t} \mid \xi_2 \geq \dots \geq \xi_8 \text{ and } \xi_1 + \xi_6 + \xi_7 + \xi_8 \geq 0\}.$$

(Note that this automatically implies that $\xi_1 \geq \xi_2$.) The highest root is $\delta = \epsilon_1 - \epsilon_8$.

Assume that $(\delta, \lambda) = \lambda_1 - \lambda_8 \leq 2$. Combining the inequalities $\lambda_1 - \lambda_8 \leq 2$, $\lambda_1 + \lambda_6 + \lambda_7 + \lambda_8 \geq 0$, and $\lambda_5 \geq \lambda_i$ for $i = 6, 7$ and 8 , we see that $\lambda_5 \geq -\frac{1}{2}$. Since also $\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \geq 0$ and $\lambda_i \geq \lambda_5$ for $i = 2, 3$ and 4 , $\lambda_5 \leq 0$. Moreover, in both cases, if the last inequality in the sentence is an equality, all the inequalities are equalities. Since $\lambda \in \ell$, the only possibilities are $\lambda_5 = 0$ or $\lambda_5 = -\frac{1}{2}$. In the former case, we must have $\lambda = \epsilon_1 - \epsilon_8$. In the latter case, the only possibilities are $\lambda = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) + (\epsilon_1 - \epsilon_4)$, or $\lambda = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) + (\epsilon_1 - \epsilon_4) + (\epsilon_2 - \epsilon_3)$. This proves claim (a).

Since $\delta = (\epsilon_1 - \epsilon_i) + (\epsilon_i - \epsilon_8)$, the roots $\pm(\epsilon_1 - \epsilon_i)$ and $\pm(\epsilon_i - \epsilon_8)$ lie in L for all $1 < i < 8$. Since $\delta = (\epsilon_1 + \epsilon_i + \epsilon_j + \epsilon_k) - (\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_8)$, the roots $\pm(\epsilon_1 + \epsilon_i + \epsilon_j + \epsilon_k)$ and $\pm(\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_8)$ also lie in L for all

$1 < i < j < k < 8$. All roots can be written as a sum of these roots. This proves claim (b).

Since $\tilde{\ell} \subset \mathbb{Z}^8 \cap \mathfrak{t}$, $\alpha_i - \alpha_j \in \mathbb{Z}$ for every $\alpha \in \tilde{\ell}^* \subset \mathfrak{t}^*$. Hence, the argument for claim (c) follows from the argument for A_7 .

Finally, δ is the only root in the positive Weyl chamber.

(V, c) For the group E_8 , $\mathfrak{t} = \mathfrak{t}^* = \{\xi \in \mathbb{R}^9 \mid \sum \xi_i = 0\}$ and the roots are $\epsilon_i - \epsilon_j$, and $\pm(\epsilon_i + \epsilon_j + \epsilon_k)$ for i, j and k distinct. Hence the integral lattice is

$$\tilde{\ell} = \{\xi \in \mathfrak{t} \mid 3\xi_i \in \mathbb{Z} \text{ and } \xi_i - \xi_j \in \mathbb{Z} \forall i, j\}.$$

The positive Weyl chamber is

$$\{\xi \in \mathfrak{t} \mid \xi_2 \geq \dots \geq \xi_9 \text{ and } \xi_2 + \xi_3 + \xi_4 \leq 0\}.$$

(Note that these conditions imply that $\xi_1 \geq \xi_2$.) The highest root is $\delta = \epsilon_1 - \epsilon_9$.

Assume that $(\delta, \lambda) = \lambda_1 - \lambda_9 \leq 2$. Combining the inequalities

$$\lambda_1 - \lambda_9 \leq 2, \quad \lambda_1 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9 \geq 0, \quad \lambda_i \leq \lambda_4, i > 4,$$

we see that $\lambda_4 \geq -\frac{1}{3}$. Since $\lambda_2 + \lambda_3 + \lambda_4 \leq 0$ and $\lambda_2 \geq \lambda_3 \geq \lambda_4$, $\lambda_4 \leq 0$. Moreover, in both cases, if the last inequality in the sentence is an equality, all the inequalities are equalities. Since $\lambda \in \ell$, the only possibilities are $\lambda_4 = 0$ or $\lambda_4 = -\frac{1}{3}$. In the former case, $\lambda = \epsilon_1 - \epsilon_9$. In the latter case, $\lambda = (\epsilon_1 - \epsilon_9) + (\epsilon_1 + \epsilon_2 + \epsilon_9)$. Claim (a) follows.

We now notice that $\delta = \epsilon_1 - \epsilon_9 = (\epsilon_1 - \epsilon_k) + (\epsilon_k - \epsilon_9) = (\epsilon_1 + \epsilon_i + \epsilon_j) - (\epsilon_i + \epsilon_j + \epsilon_9)$ for all $1 < k < 9$ and $1 < i < j < 9$. Therefore, the corresponding roots $\pm(\epsilon_1 - \epsilon_k)$, $\pm(\epsilon_k - \epsilon_9)$, $\pm(\epsilon_1 + \epsilon_i + \epsilon_j)$, and $\pm(\epsilon_i + \epsilon_j + \epsilon_9)$ all lie in L . Since every root can be written as a sum of these roots, claim (b) follows.

Since $\tilde{\ell} \subset \mathbb{Z}^9 \cap \mathfrak{t}$, $\alpha_i - \alpha_j \in \mathbb{Z}$ for every $\alpha \in \tilde{\ell}^* \subset \mathfrak{t}^*$. Hence, the argument for claim (c) carries over from the argument for the group A_8 .

Finally, δ is the only root in the positive Weyl chamber.

(VI) For the group F_4 , $\mathfrak{t} = \mathfrak{t}^* = \mathbb{R}^4$. The roots are $\pm e_i$, $e_i \pm e_j$ for $i \neq j$, and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. Hence the integral lattice is $\tilde{\ell} = \{\xi \in \mathbb{Z}^4 \mid \sum \xi_i \in 2\mathbb{Z}\}$. The positive Weyl chamber is

$$\{\xi \in \mathfrak{t} \mid \xi_2 \geq \xi_3 \geq \xi_4 \geq 0 \text{ and } \xi_1 \geq \xi_2 + \xi_3 + \xi_4\}.$$

(Note that automatically $\xi_1 \geq \xi_2$.) The highest root is $\delta = e_1 + e_2$.

The argument for claim (a) carries over word for word from the argument for B_4 .

Notice that if $k = 3$ or 4

$$\begin{aligned} \delta &= e_1 + e_2 = (e_1) + (e_2) = (e_1 - e_k) + (e_2 + e_k) = (e_1 + e_k) + (e_2 - e_k) \\ &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4) + \frac{1}{2}(e_1 + e_2 - e_3 - e_4) \\ &= \frac{1}{2}(e_1 + e_2 - e_3 + e_4) + \frac{1}{2}(e_1 + e_2 + e_3 - e_4). \end{aligned}$$

Hence, the corresponding roots all lie in L . Since every root can be written as the sum of these roots, this proves claim (b).

Since $-\text{id}$ lies in the Weyl group, we are done.

(VII) For the group G_2 , $\mathfrak{t} = \mathfrak{t}^* = \{\xi \in \mathbb{R}^3 \mid \sum \xi_i = 0\}$. The roots are $\pm\epsilon_i$ and $\epsilon_i - \epsilon_j$ for i and j distinct. The positive Weyl chamber is $\{\xi \in \mathfrak{t}^* \mid 0 \geq \xi_2 \geq \xi_3\}$. (Note that automatically $\xi_1 \geq \xi_2$.) The integral lattice is $\tilde{\ell} = \mathbb{Z}^3 \cap \mathfrak{t}$. The highest root is $\delta = \epsilon_1 - \epsilon_3$.

The argument for claim (a) follows the argument for A_3 word for word.

Since $\delta = (\epsilon_1 - \epsilon_2) + (\epsilon_2 - \epsilon_3) = (\epsilon_1) + (-\epsilon_3)$, the roots $\pm(\epsilon_1 - \epsilon_2)$, $\pm(\epsilon_2 - \epsilon_3)$, $\pm\epsilon_1$ and $\pm\epsilon_3$ all lie in L . Since every root can be written as a sum of these roots, claim (b) follows.

Since $-\text{id}$ lies in the Weyl group, we are done.

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