MAT 545: COMPLEX GEOMETRY (FALL 2017)
PROBLEM SET 1

(1) Prove Corollary 1.5.2.
(2) Prove that if $F \in \mathcal{O}(\mathbb{C}^n)$ and
$$\int_{\mathbb{C}^n} |F|^2 dV < +\infty$$
then $F \equiv 0$.
(3) For each $k \geq 0$, find all holomorphic functions $G \in \mathcal{O}(\mathbb{C}^n)$ such that
$$\int_{\mathbb{C}^n} \frac{|G(z)|^2}{(1+|z|^2)^{k+n+1}} dV(z) < +\infty.$$
(4) For the holomorphic function $\tan(z^3w^2 - w^3)$ defined in a small neighborhood of the origin, find the Weierstrass polynomial in some non-degenerate coordinate system of your choice.
(5) Let $f \in \mathcal{O}(\mathbb{C}^n)$ and let
$$U := \{z \in \mathbb{C}^n \ ; \ |z| < 1 \text{ and } |f(z)| \neq 0\} \quad \text{and} \quad \mathbb{B} := \{z \in \mathbb{C}^n \ ; \ |z| < 1\}.$$
Show that if $g \in \mathcal{O}(U)$ is bounded then there exists $G \in \mathcal{O}(\mathbb{B})$ such that $G|_U = g$. Does the statement remain true if g is unbounded? Justify your answer with proof.
(6) Let $f_1, f_2 \in \mathcal{O}(\mathbb{C}^n)$ be entire holomorphic functions such that $df_1 \wedge df_2 \not\equiv 0$ on the set
$$Z := \{z \in \mathbb{C}^n \ ; \ f_1(z) = f_2(z) = 0\},$$
and let
$$\Omega := \mathbb{C}^n - Z = \{z \in \mathbb{C}^n \ ; \ |f_1(z)|^2 + |f_2(z)|^2 \neq 0\}.$$
Prove that if $g \in \mathcal{O}(\Omega)$ then there is an entire function $G \in \mathcal{O}(\mathbb{C}^n)$ such that $G|_\Omega = g$.
(7) Show that a holomorphic function $f \in \mathcal{O}(\Omega)$, as a map $f : \Omega \to \mathbb{C}$, is an open mapping. On the other hand, show that the map $F : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $F(z, w) = (z, zw)$ is not an open mapping.