

SUFFICIENT CONDITIONS FOR INTERPOLATION AND SAMPLING HYPERSURFACES IN THE BERGMAN BALL

TAMÁS FORGÁCS AND DROR VAROLIN

ABSTRACT. We give sufficient conditions for a closed smooth hypersurface W in the n -dimensional Bergman ball to be interpolating or sampling. As in the recent work [OSV] of Ortega-Cerdà, Schuster and the second author on the Bargmann-Fock space, our sufficient conditions are expressed in terms of a geometric density of the hypersurface that, though less natural, is shown to be equivalent to a Bergman ball analog of the Beurling-type densities used in [OSV]. In the interpolation theorem we interpolate L^2 data from W to the ball using the method of Ohsawa-Takegoshi, extended to the present setting, rather than the Cousin I approach used in [OSV]. In the sampling theorem, our proof is completely different from [OSV]. We adapt the more natural method of Berndtsson and Ortega-Cerdà [] to higher dimensions. This adaptation motivated the notion of density that we introduced. The approaches of [OSV] and of the present paper both work in either the case of the Bergman ball or of the Bargmann-Fock space.

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1. INTRODUCTION

Recall that the Bergman metric on the unit ball $B = \{z \in \mathbb{C}^n ; |z| < 1\}$ is the Kähler metric whose associated $(1, 1)$ -form is $\omega_B = -(n+1)dd^c\lambda$, where

$$\lambda = \log(1 - |z|^2) - \frac{n}{n+1} \log(n+1)$$

and in our convention $d^c = \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)$. The weighted Bergman spaces on the Bergman ball are

$$\mathcal{H}^2(B, \kappa) := \left\{ F \in \mathcal{O}(B) ; \int_B |F|^2 e^{\lambda - \kappa} \omega_B^n < +\infty \right\},$$

where $\mathcal{O}(X)$ denotes the space of holomorphic functions on a complex manifold X . In this paper we assume that κ is \mathcal{C}^2 . The case $\kappa = -(n+1) \log(1 - |z|^2)$ corresponds to the classical Bergman space of holomorphic functions that are square integrable with respect to Lebesgue measure.

Given a nonsingular closed complex hypersurface $W \subset B$, we let

$$\mathfrak{H}^2(W, \kappa) := \left\{ f \in \mathcal{O}(W) ; \int_W |f|^2 e^{\lambda - (\nu + \kappa)} \omega_B^{n-1} < +\infty \right\},$$

where

$$\nu := \begin{cases} \lambda & n \geq 2 \\ 0 & n = 1 \end{cases}.$$

This strange discrepancy between the 1-dimensional and higher dimensional cases will be explained after the proof of Lemma 5.3.

DEFINITION 1.1. (a) We say that W is an interpolation hypersurface if for each $f \in \mathfrak{H}^2(W, \kappa)$ there exists $F \in \mathcal{H}^2(B, \kappa)$ such that $F|_W = f$.

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(b) We say that W is a sampling hypersurface if there is a constant A such that for every $F \in \mathcal{H}^2(B, \kappa)$,

$$(1) \quad \frac{1}{A} \int_B |F|^2 e^{\lambda - \kappa} \omega_B^n \leq \int_W |F|^2 e^{\lambda - (\nu + \kappa)} \omega_B^{n-1} \leq A \int_B |F|^2 e^{\lambda - \kappa} \omega_B^n.$$

Let F_a denote a holomorphic involution of B sending 0 to a (see Section 2).

REMARK. We will often use, without explicit indication, the fact that F_z is an involution. Thus the reader should not be confused if F_z is seen when F_z^{-1} is expected.

We define the *total density tensor* of W in the ball of radius r to be the $(1, 1)$ -form

$$\Upsilon_r^W(z) = \frac{1}{V_n(r)} \left(\int_{B(0,r)} \frac{\partial^2 \log |T(F_z(\zeta))|^2}{\partial z^i \partial \bar{z}^j} \omega_B^n \right) \sqrt{-1} dz^i \wedge d\bar{z}^j.$$

Here T is any holomorphic function such that $W = \{T = 0\}$ with $dT|_W$ nowhere zero, and

$$V_n(r) = \int_{B(0,r)} \omega_B^n$$

is the volume of the Euclidean ball of radius r and center 0, with respect to (our normalization of) the volume induced by the Bergman metric. The total density tensor is a Bergman ball analog of the total density tensor introduced in [OSV] in the case of \mathbb{C}^n . In the case of the Bergman ball, some of the more basic properties of the total density tensor do not follow as readily as their analogs in the \mathbb{C}^n case. For example, at the end of Section 2 we will show that the definition of Υ_r^W is independent of the choice of T .

We define

$$[W]_\varepsilon(z) = dd^c \left(\frac{1}{V_n(\varepsilon)} \int_{|F_z(\zeta)| < \varepsilon} \log |T(\zeta)|^2 \omega_B^n(\zeta) \right).$$

If we denote by $[W]$ the current of integration along W , then $[W]_\varepsilon$ is in some sense the average of $[W]$ over the Bergman-Green ball of radius ε . Note that $[W]_\varepsilon = \Upsilon_\varepsilon^W$ and thus $[W]_\varepsilon$ is independent of the choice of T . Moreover, though not necessarily smooth, the current $[W]_\varepsilon$ is locally bounded, as can be seen by changing variables in the integral and then differentiating under the integral. Finally, it is also clear that, in the sense of currents, $[W]_\varepsilon \rightarrow [W]$ as $\varepsilon \rightarrow 0$.

From here on out we assume that W is uniformly flat (see Section 3 for the definition) and that

$$\frac{1}{C} \omega_B \leq \sqrt{-1} \partial \bar{\partial} \kappa \leq C \omega_B$$

for some constant $C > 1$.

DEFINITION 1.2.

(I) Let $\mathcal{P}_W(B)$ denote the set of $(n-1, n-1)$ -forms θ on B with the following properties.

- (a) $\theta \wedge \omega_B \geq c e^{n\lambda} \omega_B^n$ for some constant $c > 0$.
- (b) For each $\varepsilon > 0$ there exists $C > 0$ such that $[W]_\varepsilon \wedge \theta \leq C [W]_\varepsilon \wedge e^{n\lambda} \omega_B^{n-1}$.
- (c) $dd^c \theta = 0$.

(II) For $\theta \in \mathcal{P}_W(B)$, let

$$\mathcal{D}_B^+(W, \kappa)[\theta] = \limsup_{r \rightarrow 1} \sup_{z \in B} \frac{\left(\Upsilon_r^W + \frac{n}{n+1} \omega_B \right) \wedge \theta}{\sqrt{-1} \partial \bar{\partial} \kappa \wedge \theta}$$

and

$$\mathcal{D}_B^-(W, \kappa)[\theta] = \liminf_{r \rightarrow 1} \inf_{z \in B} \frac{\left(\Upsilon_r^W + \frac{n}{n+1} \omega_B \right) \wedge \theta}{\sqrt{-1} \partial \bar{\partial} \kappa \wedge \theta}$$

(III) The upper and lower densities of W are

$$\mathcal{D}_B^+(W, \kappa) = \sup_{\theta \in \mathcal{P}_W(B)} \mathcal{D}_B^+(W, \kappa)[\theta]$$

and

$$\mathcal{D}_B^-(W, \kappa) = \sup_{\theta \in \mathcal{P}_W(B)} \mathcal{D}_B^-(W, \kappa)[\theta]$$

Our main results are the following two theorems.

THEOREM 1.3. *If $\mathcal{D}_B^+(W, \kappa) < 1$, then W is an interpolation hypersurface.*

THEOREM 1.4. *If $\mathcal{D}_B^-(W, \kappa) > 1$, then W is a sampling hypersurface.*

Theorems 1.3 and 1.4 give generalizations to higher dimensions of results of Seip [Seip-93] and of Berndtsson-Ortega Cerdà [BO-95]. By now Theorems 1.3 and 1.4 carry with them a rich history. Most recently, results analogous to Theorems 1.3 and 1.4 have been established for the case of \mathbb{C}^n in the paper [OSV], which we refer to for further historical remarks regarding interpolation and sampling problems for Bergman spaces.

Though there is a strong similarity between the results of [OSV] and the present paper, the methods of proof are completely different. In fact, the present approach and the approach of [OSV] could be used interchangeably for the case of \mathbb{C}^n and the Bergman ball.

In the case of interpolation, we employ the Ohsawa-Takegoshi technique to extend functions from the submanifold W to the ball in one shot, rather than using the L^2 Cousin I-type approach to extend the function locally and then patch together the resulting local extensions. (We should perhaps remark that if one wants to apply the Cousin I-type method in the case of the ball, then the negativity of the curvature of ω_B requires the use of a sharper version of Hörmander's $\bar{\partial}$ Theorem, due to Ohsawa. The need for Ohsawa's Theorem was already noticed in the 1-dimensional case [BO-95].)

By contrast with [OSV], our approach to sampling is closer in spirit to the technique that has been used in the one-variable case in [BO-95]. Our densities, laid out in Definition 1.2 above, do not directly correspond to those in [OSV] (though we prove in Section 4 that they are actually the same). We feel that the methods of the present paper fit in more naturally with the Hilbert Space approach. The proofs also seem more elementary than the Beurling-inspired approach used in [OSV].

The paper is organized as follows.

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2. RAPID REVIEW OF BERGMAN GEOMETRY

Bergman geometry is one of the oldest and most studied areas of complex geometry. Therefore we content ourselves with stating facts, and provide few proofs.

Bergman metric. As already mentioned, the Bergman metric is $\omega_B = -(n+1)dd^c\lambda$. It is easy to see that, with $\omega_E = dd^c|z|^2$ denoting the Euclidean metric,

$$\omega_B|_{z=0} = (n+1)\omega_E|_{z=0} \quad \text{and} \quad \omega_B^n = e^{-(n+1)\lambda}\omega_E^n,$$

and in particular,

$$\text{Ricci}(\omega_B) = -\omega_B.$$

Basics of $\text{Aut}(B)$. For the reader's convenience, we recall that $\text{Aut}(B)$ contains the involutions

$$F_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle}, \quad a \in B - \{0\}, \quad F_0(z) = -z,$$

where $P_a = |a|^{-2}aa^\dagger$, $Q_a = I - P_a$ and $s_a = \sqrt{1 - |a|^2}$. Moreover, the Schwarz Lemma shows that any automorphism of B is of the form UF_a for some unitary U . Note that $F_a(0) = a$ and

$$1 - |F_a(z)|^2 = \frac{(1 - |z|^2)(1 - |a|^2)}{|1 - \langle z, a \rangle|^2}.$$

Thus $\text{Aut}(B)$ acts transitively on the ball and ω_B is $\text{Aut}(B)$ -invariant. (For much more detail on this and the next paragraph, the reader is referred to [R-80] or [St-94].)

Basic potential theory of the Bergman metric. Recall that the Bergman Laplacian Δ_B associated to ω_B is the ω_B -trace of dd^c :

$$(\Delta_B g)\omega_B^n = dd^c g \wedge \omega_B^{n-1}.$$

DEFINITION 2.1. The Green's function with pole at $a \in B$ is the function $G_B(z, a)$ satisfying

$$\Delta_B(G_B(\cdot, a))\omega_B^n = \delta_a \quad \text{and} \quad G_B(\cdot, a)|_{\partial B} = 0.$$

Using $\text{Aut}(B)$ -invariance, it is easily seen that $G(z, a) = G(F_a(z), 0)$ and that

$$n(n+1)(\Delta_B g)(a) = \text{trace} \left(DF_a(0)^\dagger D^{1,1}g(a) DF_a(0) \right).$$

Here $D^{1,1}g$ is the matrix of the $(1,1)$ -form $\sqrt{-1}\partial\bar{\partial}g$ in Euclidean coordinates. Setting $\gamma_B = G_B(\cdot, 0)$, we see from unitary invariance that $\gamma_B(z) = f(|z|^2)$ for some function f . Substitution into the Bergman-Laplace equation and solving the resulting ODE shows that

$$f(t) = -C_n \int_t^1 \frac{(1-u)^{n-1}}{u^n} du,$$

where $C_n = (2\pi)^{-n}(n+1)^{-(n-1)}$.

Note that $f'(t) > 0$. It follows that for each $a \in B$ the sublevel sets $G(z, a)$ are also the sublevel sets of $|F_a(z)|$. We use the latter to define distances.

DEFINITION 2.2. (i) The Bergman-Green distance between two points a and b in B is

$$|F_a(b)|.$$

(ii) The Bergman-Green balls with center a and radius r are

$$E(a, r) = F_a(B(0, r)) = \{z \in B ; |F_a(z)| < r\}.$$

By using the Green-Stokes identity

$$(2) \quad \int_{\partial D} (g_1 d^c g_2 - g_2 d^c g_1) \wedge \omega^{n-1} = \int_D (g_1 dd^c g_2 - g_2 dd^c g_1) \wedge \omega^{n-1},$$

where (D, ω) is an n dimensional Kähler manifold with boundary and $g_1, g_2 : D \rightarrow \mathbb{C}$ are functions, we obtain the following Lemma.

LEMMA 2.3. *Let h be a function such that $\Delta_B h \geq 0$. Then*

$$(3) \quad h(0) \leq \frac{1}{(2\pi)^n} \int_{\partial B} h(rz) d^c |z|^2 \wedge \omega_E^{n-1}(z).$$

Moreover, equality holds when $\Delta_B h \equiv 0$.

Proof. Apply (2) with $D = B(0, r)$, $r < 1$, $g_1 = h$ and $g_2 = \gamma_r$, where

$$\gamma_r(z) := \gamma_B(z) + C_n \int_{r^2}^1 \frac{(1-t)^{n-1}}{t^n} dt,$$

observing that $\gamma_r|_{\partial B(0, r)} \equiv 0$ and $dd^c \gamma_r \wedge \omega_B^{n-1} = \delta_0$. The result now follows by direct computation. \square

COROLLARY 2.4. *Let h be a function such that $\Delta_B h \geq 0$. Then for all $r < 1$,*

$$(4) \quad h(0) \leq \frac{1}{V_n(r)} \int_{B(0, r)} h \omega_B^n.$$

Moreover, equality holds in (4) when $\Delta_B h \equiv 0$.

Let us end this section by justifying our claim that Υ_r^W is independent of the choice of holomorphic function T defining W . Suppose \tilde{T} is another function such that $W = \{\tilde{T} = 0\}$ and $d\tilde{T}|_W$ is free of zeros. Then the function \tilde{T}/T is holomorphic and free of zeros in the ball. Since the ball is simply connected, any zero-free holomorphic function is the exponential of some holomorphic function. Thus $\tilde{T} = e^h T$ for some holomorphic function h . It follows that

$$\int_{B(0, r)} \log |\tilde{T}(F_z(\zeta))|^2 \omega_B^n = \int_{B(0, r)} \log |T(F_z(\zeta))|^2 \omega_B^n + 2 \int_{B(0, r)} \operatorname{Re} h(F_z(\zeta)) \omega_B^n.$$

Since $\operatorname{Re} h$ is (pluri)harmonic, its ball average, with respect to a radially symmetric probability measure, is equal to its central value. Since $F_z(0) = z$, we have

$$\int_{B(0, r)} \log |\tilde{T}(F_z(\zeta))|^2 \omega_B^n = \int_{B(0, r)} \log |T(F_z(\zeta))|^2 \omega_B^n + 2V_n(r) \operatorname{Re} h(z).$$

The pluriharmonicity of $\operatorname{Re} h$ thus completes the justification of our claim.

3. UNIFORM FLATNESS

In [OSV] a notion of uniform flatness was developed for closed smooth hypersurfaces in \mathbb{C}^n . Here we define the analogous notion for the ball with its Bergman geometry.

Let

$$N_\varepsilon^B(W) = \left\{ z \in B \mid \inf_{w \in W} |F_z(w)| < \varepsilon \right\}$$

DEFINITION 3.1. We define a smooth divisor W in B to be uniformly flat if there exists an $\varepsilon_0 > 0$ such that $N_{\varepsilon_0}^B(W)$ has the following property: for each $z \in N_{\varepsilon_0}(W)$ there is a unique $w_z \in W$ minimizing the "distance to z " function $w \mapsto |F_z(w)|$ along W .

REMARK. Recall that a pseudohyperbolic disk of radius ε is the image under $F \in \text{Aut}(B)$ of the disk $\{(0, \dots, 0, z) \in B ; |z| < \varepsilon\}$. The uniform flatness hypothesis implies that in fact $N_{\varepsilon_0}^B(W)$ is foliated by pseudohyperbolic disks. Indeed, since our condition is invariant under $\text{Aut}(B)$, it suffices to see this for the case where $W \ni 0$ and $T_{W,0} = \{z_n = 0\}$. In this case, it is clear that the boundary of the disk $\{(0, \dots, 0, z) \in B ; |z| < \varepsilon_0\}$ has distance exactly ε_0 to the origin.

These observations imply the existence of a diffeomorphism

$$\Phi : W \times \mathbb{D}(0, \varepsilon_0) \rightarrow N_{\varepsilon_0}^B(W)$$

that is holomorphic in the disk variable, and sends each disk $\{w\} \times \mathbb{D}(0, \varepsilon_0)$ to the disk with center at w , which minimizes the pseudo-hyperbolic distance and whose tangent vector is orthogonal to $T_{W,w}$ in the Bergman metric.

The following consequence of uniform flatness is useful.

LEMMA 3.2. *If a closed non-singular complex hypersurface $W \subset B$ is uniformly flat, then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for each $z \in W$ the set $F_z(W) \cap B(0, \varepsilon_0)$ is a graph, over the Euclidean ε_0 -neighborhood of the origin in the tangent space $T_{F_z(W),0} = dF_z(T_{W,z})$, of some function f such that*

$$|f(x)| \leq C|x|^2, \quad |x| < \varepsilon_0.$$

Sketch of proof. Since the notion of uniform flatness is invariant with respect to $\text{Aut}(B)$, it suffices to assume that $z = 0 \in W$. Moreover, since we are working in a small neighborhood, we may replace the Bergman metric by the Euclidean metric, and the pseudo-hyperbolic distance $|F_z(w)|$ by Euclidean distance. In this setting, the result follows from Proposition 2.2 in [OSV]. We leave the details to the interested reader. \square

4. THE DENSITY CONDITIONS AGAIN

4.1. **Reformulation of the density conditions.** It will be useful to rewrite the hypotheses on the upper and lower densities in terms of the positivity properties of certain associated differential forms.

LEMMA 4.1. 1. *If $\mathcal{D}_B^+(W, \kappa) < 1$, then there is a positive constant c such that*

$$\sqrt{-1}\partial\bar{\partial}\kappa - \frac{n}{n+1}\omega_B - \Upsilon_r^W \geq c\sqrt{-1}\partial\bar{\partial}\kappa.$$

2. *If $\mathcal{D}_B^-(W, \kappa) > 1$, then there exists $\theta \in \mathcal{P}_W(B)$ and $c > 0$ such that*

$$\left(\Upsilon_r^W + \frac{n}{n+1}\omega_B - \sqrt{-1}\partial\bar{\partial}\kappa \right) \wedge \theta \geq ce^{n\lambda}\omega_B^n.$$

Proof. After using condition (a) in the definition of $\mathcal{P}_W(B)$, assertion 2 is trivially true from the definition of the lower density.

To see assertion 1, choose any $v \in T_{B,p}$ having norm 1, say with respect to the Bergman metric. After a unitary change of coordinates in \mathbb{C}^n (where the ball lies) we may assume that that $v = c\frac{\partial}{\partial x^1}$, where x^1, \dots, x^n are coordinates in \mathbb{C}^n . Consider the $(n-1, n-1)$ -form

$$\theta = \theta_v := (\sqrt{-1})^{n-1} dx^2 \wedge d\bar{x}^2 \wedge \dots \wedge dx^n \wedge d\bar{x}^n.$$

We claim that $\theta \in \mathcal{P}_W(B)$. Indeed, $\sqrt{-1}\partial\bar{\partial}\theta = 0$ so condition (c) in the definition of $\mathcal{P}_W(B)$ holds. Condition (b) is clear in view of the local boundedness of $[W]_\varepsilon$. Condition (a) can be seen as follows:

$$\begin{aligned}\theta \wedge \omega_B &= C\theta \wedge \left(\frac{\omega_E}{1-|x|^2} + \frac{\sqrt{-1}\partial|x|^2 \wedge \bar{\partial}|x|^2}{(1-|x|^2)^2} \right) \\ &\geq Ce^{-\lambda}\omega_E^n \\ &= Ce^{n\lambda}\omega_B^n.\end{aligned}$$

By the density condition there exists $\delta > 0$ and $r_o \gg 0$ such that for all $r > r_o$,

$$\begin{aligned}1 - \delta &> \frac{\left(\Upsilon_r^W + \frac{n}{n+1}\omega_B \right) \wedge \theta}{\sqrt{-1}\partial\bar{\partial}\kappa \wedge \theta} \\ &= \frac{\left(\Upsilon_r^W(v, \bar{v}) + \frac{n}{n+1}\omega_B(v, \bar{v}) \right)}{\sqrt{-1}\partial\bar{\partial}\kappa(v, \bar{v})}.\end{aligned}$$

Observe that the density condition says this inequality holds uniformly on B . Clearly if we rotate our original v a little, this bound will still hold. Since the unit sphere is compact, we can choose r_o and δ so that the result holds for all v in the unit sphere in $T_{B,p}$. (Here, for the sake of simplifying the argument, we are exploiting the triviality of the tangent bundle T_B .) This completes the proof. \square

4.2. A seemingly better notion of density. In the paper [OSV], a different notion of density was used. The purpose of this section is to demonstrate the equivalence of the density notions of the present paper and those in [OSV].

Let us define the Bergman ball analogues of the densities used in [OSV]. One first sets

$$D_{z,r}(W, \kappa) := \sup_{v \neq 0} \frac{\Upsilon_r^W(v, \bar{v}) + \frac{n}{n+1}\omega_B(v, \bar{v})}{\sqrt{-1}\partial\bar{\partial}\kappa(v, \bar{v})}.$$

Then one takes

$$D^+(W, \kappa) := \limsup_{r \rightarrow 1} \sup_{z \in B} D_{z,r}(W, \kappa)$$

and

$$D^-(W, \kappa) := \liminf_{r \rightarrow 1} \inf_{z \in B} D_{z,r}(W, \kappa).$$

Note that $D_{z,r}(W, \kappa)$ is the maximum eigenvalue of the $(1, 1)$ -form $\Upsilon_r^W + \frac{n}{n+1}\omega_B$ with respect to the positive $(1, 1)$ -form $\sqrt{-1}\partial\bar{\partial}\kappa$ at the point z .

LEMMA 4.2. *Let (M, ω) be a Hermitian manifold of complex dimension n , and let α be a non-negative $(n-1, n-1)$ form on M . Then for each p there exists a vector $v \in T_{M,p}^{1,0}$ such that for any real $(1, 1)$ -form β , one has*

$$\alpha \wedge \beta_p = \beta_p(v, \bar{v})\omega^n.$$

The Lemma says that the mapping $\sqrt{-1}v \wedge \bar{v} \mapsto \theta_v$, with θ_v as in the proof of Lemma 4.1, is a pointwise isomorphism.

Proof. We shall use multi-linear algebra on $T_{M,p}$. To this end, choose a unitary basis e^1, \dots, e^n for $(T_{M,p}^*)^{1,0}$ and e_1, \dots, e_n its dual basis. Let $\alpha^{i\bar{j}}$ be a basis for $\Lambda^{n-1, n-1}(T_{M,p}^*)$ such that

$$\sqrt{-1}e^k \wedge \bar{e}^\ell \wedge \alpha^{i\bar{j}} = \delta^{ik}\delta^{\bar{j}\bar{\ell}} \frac{\omega^n}{n!}.$$

Let A (resp. B) be the Hermitian matrix with entries $a_{i\bar{j}}$ (resp. $b_{i\bar{j}}$) such that at the point p ,

$$\alpha = a_{i\bar{j}}\alpha^{i\bar{j}} \quad (\text{resp. } \beta = b_{i\bar{j}}\sqrt{-1}e^i \wedge \bar{e}^j).$$

Then

$$\alpha \wedge \beta_p = \text{Trace}(AB^\dagger) \frac{\omega^n}{n!} \quad \text{and} \quad \beta_p(v, \bar{v}) = v^\dagger B v.$$

After a unitary rotation, we may assume that the basis e^1, \dots, e^n diagonalizes A . Thus, since α is positive, there exist non-negative numbers $\lambda_1, \dots, \lambda_n$ such that

$$\text{Trace}(AB^\dagger) = \sum_{k=1}^n \lambda_k b_{kk}.$$

Taking

$$v = \sum_{k=1}^n \sqrt{\lambda_k} e_k$$

completes the proof. □

We can now obtain the following proposition.

PROPOSITION 4.3.

- (a) $\mathcal{D}^+(W, \kappa) = D^+(W, \kappa)$.
- (b) $\mathcal{D}^-(W, \kappa) \leq D^-(W, \kappa)$.

Proof. (a) Fix $z \in B$ and $r \in [0, 1)$. By definition of $D^+(W, \kappa)$, we have that for any $\theta \in \mathcal{P}_W(B)$,

$$\begin{aligned} D^+(W, \kappa) &\geq D_{z,r}(W, \kappa) \\ &\geq \frac{\Upsilon_r^W \wedge \theta(z) + \frac{n}{n+1} \omega_B \wedge \theta(z)}{\sqrt{-1} \partial \bar{\partial} \kappa \wedge \theta(z)}. \end{aligned}$$

(In the second inequality we have used Lemma 4.2.) Taking the supremum over z and then the lim sup as $r \rightarrow 1$, we see that

$$D^+(W, \kappa) \geq \mathcal{D}^+(W, \kappa)[\theta].$$

Finally, taking the supremum of the right hand side over all $\theta \in \mathcal{P}_W(B)$ shows that $D^+(W, \kappa) \geq \mathcal{D}^+(W, \kappa)$.

To obtain the reverse inequality, fix $\varepsilon > 0$. Then for each $r < 1$ sufficiently close to 1 there exist $z \in B$ and $v \in \mathbb{C}^n$ such that

$$\begin{aligned} D^+(W, \kappa) - \varepsilon &\leq \frac{\Upsilon_r^W(v, \bar{v}) + \frac{n}{n+1} \omega_B(v, \bar{v})}{\sqrt{-1} \partial \bar{\partial} \kappa(v, \bar{v})} \\ &= \frac{\Upsilon_r^W \wedge \theta_v(z) + \frac{n}{n+1} \omega_B \wedge \theta_v(z)}{\sqrt{-1} \partial \bar{\partial} \kappa \wedge \theta_v(z)} + \varepsilon \\ &\leq \mathcal{D}^+(W, \kappa)[\theta_v] \leq \mathcal{D}^+(W, \kappa) + \varepsilon, \end{aligned}$$

where θ_v is defined as in the proof of Lemma 4.1. The second-to-last inequality follows since $0 \ll r < 1$. Since ε is arbitrary, 1 is proved.

(b) Fix $\varepsilon > 0$. By definition of $\mathcal{D}^-(W, \kappa)$, there exists $\theta \in \mathcal{P}_W(B)$ such that

$$\mathcal{D}^-(W, \kappa) \leq \mathcal{D}^-(W, \kappa)[\theta] + \frac{\varepsilon}{2}.$$

Moreover, by the definition of $\mathcal{D}^-(W, \kappa)[\theta]$ we have that for all $z \in B$ and all $r \in [0, 1)$ sufficiently large,

$$\mathcal{D}^-(W, \kappa)[\theta] \leq \frac{\Upsilon_r^W \wedge \theta(z) + \frac{n}{n+1} \omega_B \wedge \theta(z)}{\sqrt{-1} \partial \bar{\partial} \kappa \wedge \theta(z)} + \frac{\varepsilon}{2}.$$

But by Lemma 4.2 and the definition of $D_{z,r}(W, \kappa)$,

$$\frac{\Upsilon_r^W \wedge \theta(z) + \frac{n}{n+1} \omega_B \wedge \theta(z)}{\sqrt{-1} \partial \bar{\partial} \kappa \wedge \theta(z)} \leq D_{z,r}(W, \kappa).$$

This proves (b). □

THEOREM 4.4. $\mathcal{D}^-(W, \kappa) \geq D^-(W, \kappa)$.

Proof. We introduce the notation

$$\Omega_\delta := \Upsilon_r^W + \frac{n}{n+1} \omega_B - (D^-(W, \kappa) - \delta) \sqrt{-1} \partial \bar{\partial} \kappa.$$

Let $\delta > 0$ be given. For $r \gg 0$ we are going to construct a form $\theta \in \mathcal{P}_W(B)$ such that

$$\Omega_\delta \wedge \theta \geq 0.$$

If this is done, the proof is complete.

By definition of $D^-(W, \kappa)$, there exists a locally finite open cover U_j of B and constant $(n-1, n-1)$ -forms (i.e., forms of the type θ_v defined in the proof of Lemma 4.1) θ_j on U_j such that

$$\Omega_{\delta/2} \wedge \theta_j \geq 0 \quad \text{on } U_j.$$

By the uniform flatness of W we may choose the cover $\{U_j\}$ such that any point of B is contained in some finite number of neighborhoods, this number depending only on the dimension. Moreover, by the continuity of the forms Ω_δ we may choose the forms θ_j so that if $U_j \cap U_k \neq \emptyset$ then $\theta_j - \theta_k$ is as small as we like. In fact, by elementary anti-differentiation we may take forms μ_j depending quadratically on the (global) coordinates in B such that $\theta_j = \sqrt{-1} \partial \bar{\partial} \mu_j$ and if $U_j \cap U_k \neq \emptyset$ then $\|\mu_j - \mu_k\|_{\mathcal{C}^2(U_j \cap U_k)}$ is as small as we like, where $\|\cdot\|_{\mathcal{C}^2}$ denotes \mathcal{C}^2 -norm.

The argument we present here requires a little more precision. Later we will have to control the size of the neighborhoods U_j in order to make the $\theta_j - \theta_k$ small enough. To this end, we choose the U_j to be balls (or polydisks) of diameter ε , measured with respect to the Bergman-Green distance. We momentarily indicate this dependence on ε by writing $U_{j,\varepsilon}$, $\mu_{j,\varepsilon}$ and $\theta_{j,\varepsilon}$. Observe that if we take $\mu_{j,\varepsilon}$ to be bihomogeneous quadratic in the Euclidean coordinates with origin that of $U_{j,\varepsilon}$, then the uniform estimates for $\mu_{j,\varepsilon}$ scale by ε^2 , those for $D\mu_{j,\varepsilon}$ by ε , and those from $\theta_{j,\varepsilon}$ are invariant with respect to ε .

Let $\{\psi_{j,\varepsilon}\}$ be a partition of unity subordinate to the cover $\{U_{j,\varepsilon}\}$. We may choose this partition so that

$$\sum_j \|\psi_{j,\varepsilon} \mu_{j,\varepsilon}\|_{\mathcal{C}^2} \leq C$$

for some constant C independent of ε . Indeed, as the neighborhoods $U_{j,\varepsilon}$ scale by ε , the estimates for $D\psi_{j,\varepsilon}$ scale by ε^{-1} while those for $D^2\psi_{j,\varepsilon}$ scale by ε^{-2} . Thus the desired estimate follows from the product rule

$$D^2(\psi_{j,\varepsilon} \mu_{j,\varepsilon}) = \mu_{j,\varepsilon} D^2 \psi_{j,\varepsilon} + (D\mu_{j,\varepsilon})(D\psi_{j,\varepsilon}) + \psi_{j,\varepsilon} D^2 \mu_{j,\varepsilon}.$$

Thus it is clear that we have scale invariant estimates. To simplify the notation, we can now drop the notational dependence on ε .

We would like to correct the local forms θ_j so that they can be pieced together to give us an element of $\mathcal{P}_W(B)$ with the desired density. We shall use cocycles to do this. To this end, the obstruction to the θ_j piecing together to give a global form is carried by the 1-cocycle

$$\alpha_{jk} = \theta_j - \theta_k = \sqrt{-1} \partial \bar{\partial} (\mu_j - \mu_k)$$

supported on $U_j \cap U_k$. By our choice of the θ_j , the α_{jk} are small in \mathcal{C}^0 -norm. We now define

$$\eta_j = \sqrt{-1} \partial \bar{\partial} \left(\sum_k \psi_k (\mu_j - \mu_k) |_{U_j \cap U_k} \right).$$

By modifying our choices of the μ_j we may make the η_j as small as we like. Moreover, $\sqrt{-1}\partial\bar{\partial}\eta_j = 0$ and

$$\begin{aligned}\eta_j - \eta_\ell &= \sqrt{-1}\partial\bar{\partial} \sum_k \psi_k(\mu_j - \mu_k + \mu_k - \mu_\ell) \\ &= \sqrt{-1}\partial\bar{\partial} \sum_k \psi_k(\mu_j - \mu_\ell) \\ &= \alpha_{j\ell}.\end{aligned}$$

It follows that

$$\theta = \theta_j - \eta_j \quad \text{on } U_j$$

is well defined and belongs to $\mathcal{P}_W(B)$. Moreover, by choosing the $\mu_j - \mu_k$ even smaller if necessary, we see that

$$\Omega_\delta \wedge \theta \geq 0,$$

as desired. □

5. INTERPOLATION

5.1. A negative function singular along a hypersurface. Recall that

$$V_n(r) := \int_{B(0,r)} \omega_B^n.$$

As the Bergman metric is invariant under automorphisms, one sees that for each $a \in B$, $V_n(r)$ is also the Bergman volume of Bergman-Green balls $E(a, r) = F_a(B(0, r))$.

Let

$$\Gamma_r(z, \zeta) = G_B(z, \zeta) - \frac{1}{V_n(r)} \int_{E(z,r)} G_B(x, \zeta) \omega_B^n(x).$$

Since $\Delta_B G(\cdot, \zeta) \equiv 0$ on $B - \{\zeta\}$, we see from Corollary 2.4 that Γ_r is non-negative and is supported on the set

$$\{(z, \zeta) \in B \times B \mid |F_z(\zeta)| \leq r\},$$

which contains a neighborhood of the diagonal in $B \times B$.

We define the function

$$\begin{aligned}s_r(z) &:= \int_B \Gamma_r(z, \zeta) \omega_B^{n-1}(\zeta) \wedge dd^c \log |T|^2(\zeta) \\ &= \int_{\{\zeta \mid |F_z(\zeta)| < r\}} \Gamma_r(z, \zeta) \omega_B^{n-1}(\zeta) \wedge dd^c \log |T|^2(\zeta).\end{aligned}$$

By the Lelong-Poincaré identity,

$$(5) \quad s_r(z) = 2\pi \int_{W_{z,r}} \left(G_B(z, \zeta) - \frac{1}{V(r)} \int_{E(z,r)} G_B(x, \zeta) \omega_B^n(x) \right) \omega_B^{n-1}(\zeta),$$

where

$$W_{z,r} = W \cap \{\zeta \mid |F_z(\zeta)| < r\}$$

PROPOSITION 5.1. *Let $T \in \mathcal{O}(B)$ be a holomorphic function so that $W = T^{-1}(0)$ and dT is nowhere zero on W . Then*

$$s_r(z) = \log |T(z)|^2 - \frac{1}{V_n(r)} \int_{E(z,r)} \log |T(\zeta)|^2 \omega_B^n(\zeta).$$

In particular,

$$\frac{1}{2\pi} dd^c s_r(z) = [W] - \Upsilon_r^W(z).$$

Proof. Let $\varphi \in \mathcal{C}_0^\infty(B)$ be a function whose total integral with respect to Euclidean volume is 1, and let χ_ε be the characteristic function of the set $\{z \in B; |z| < 1 - 2\varepsilon\}$. Let $\varphi_\varepsilon(x) = \varepsilon^{-2n}\varphi(\varepsilon^{-1}x)$, and set

$$f_\varepsilon = (\chi_\varepsilon \log |T|^2) * \varphi_\varepsilon.$$

Then f_ε is smooth with compact support in B , and

$$s_r(z) = \lim_{\varepsilon \rightarrow 0} \int_B \Gamma_r(z, \zeta) \omega_B^{n-1}(\zeta) \wedge dd^c f_\varepsilon(\zeta).$$

But by definition of Green's function,

$$\int_B G(z, \zeta) \omega_B^{n-1}(\zeta) \wedge dd^c f_\varepsilon(\zeta) = \int_B G(z, \zeta) dd^c (f_\varepsilon \omega_B^{n-1})(\zeta) = f_\varepsilon(z).$$

The proof is completed by letting $\varepsilon \rightarrow 0$. □

LEMMA 5.2. *The function $s_r(z)$ has the following properties:*

1. *It is non-positive.*
2. *For each $r, \epsilon > 0$ there exist a constant $C_{r,\epsilon}$ such that if $\delta_B(z, W) > \epsilon$ then $s_r(z) > -C_{r,\epsilon}$.*
3. *The function e^{-sr} is not locally integrable at any point of W .*

Here $\delta_B(z, W) = \inf\{|F_z(w)|; w \in W\}$.

Proof. By Corollary 2.4 and the fact that $\Delta_B G(\cdot, \zeta) \equiv 0$ on $B - \{\zeta\}$, $\Gamma_r \leq 0$ and 1 follows. Moreover, 3 is an immediate consequence of Proposition 5.1.

To see 2, we first note that since $\delta_B(z, W) > \varepsilon$, $G_B(z, \zeta) > A_\varepsilon$. Thus it suffices to obtain an estimate

$$-\int_{E(z,r)} G_B(x, \zeta) \omega_B^n(x) = -\int_{B(0,r)} G_B(x, y) \omega_B^n(x) \leq D_r$$

for some $D_r > 0$ and all $y = F_z(\zeta) \in B(0, r)$. To do this, it is enough to estimate the integral

$$I(r) := -\int_{B(0,(r+1)/2)} G_B(x, y) \omega_B^n(x).$$

Fix $y \in B(0, r)$. Let $\rho > 0$ be the largest number such that

$$B(y, \rho) \subset B(0, (r+1)/2).$$

One has

$$n_r \leq \rho \leq \frac{r+1}{2}$$

for some $n_r > 0$ depending on r but not on y .

Write

$$I(r) = I_1(r) + I_2(r),$$

where

$$I_1(r) := -\int_{E(y,\rho)} G_B(x, y) \omega_B^n(x)$$

and

$$I_2(r) := -\int_{B(0,(r+1)/2) - E(y,\rho)} G_B(x, y) \omega_B^n(x).$$

Now

$$I_1(r) = -\int_{B(0,\rho)} \gamma_B(x) \omega_B^n(x)$$

is clearly bounded by a constant independent of y .

Next, note that for

$$x \in B(0, (r+1)/2) - E(y, \rho)$$

one has the estimate

$$|F_y(x)| \geq \rho \geq n_r.$$

It follows that for such x ,

$$G(x, y) \geq -N_r$$

for some $N_r \in \mathbb{R}$ independent of y . Thus

$$\begin{aligned} I_2(r) &\leq N_r \int_{B(0, (r+1)/2) - E(y, \rho)} \omega_B^n(x) \\ &\leq N_r \int_{B(0, (r+1)/2)} \omega_B^n(x), \end{aligned}$$

and the latter is independent of y . Thus 2 follows. \square

REMARK. There is a direct proof of Lemma 5.2.3 that does not use the formula of Proposition 5.1. Since we will make use of the calculation needed, we present this proof now.

We may assume that W is the coordinate hyperplane $z_n = 0$ and $z = z^n e_n$ for $|z^n| \leq \varepsilon$ with ε sufficiently small. (Though we do not use it here, later we will exploit the fact that, by the uniform flatness of W , $\varepsilon > 0$ may be taken independent of the point on W which has been translated to the origin.) Let U be a sufficiently small neighborhood of the origin. Using the formula (5), we estimate that

$$s_r(z) = 2\pi \int_{U \cap W} \Gamma_r(z, \zeta) \omega_B^{n-1}(\zeta) + O(1).$$

The same method used in the proof of Lemma 5.2.2 allows us to estimate the part of Γ_r involving the integral, so we may replace Γ_r by the Green's function. Letting $\omega_B^{n-1} = 2r^{2n-3} dr ds_{2n-3}$ be the decomposition into polar coordinates and setting

$$A_{n-1} = \int_{S^{2n-3}} ds_{2n-3} = (2\pi)^{n-1},$$

we obtain from the form of the Green's function that

$$s_r(z) = 2\pi C_n (n+1)^{n-1} A_{n-1} \int_0^\alpha -\frac{2r^{2n-3} dr}{(r^2 + |z^n|^2)^{n-1}} + O(1) = \log |z^n|^2 + O(1)$$

where $\alpha > 0$ is a sufficiently small number depending on ε . \square

5.2. The proof of Theorem 1.3. We fix a compact subset $\Omega \subset\subset B$. This set will be fixed until the last part of the argument, when we let $\Omega \rightarrow B$.

Tubular limits. For each $\Omega \subset\subset B$, let

$$\Omega_\varepsilon := \Omega \cap \{s_r < \log \varepsilon^2\}.$$

LEMMA 5.3. *Let $W \subset B$ be uniformly flat. Then there exists a positive constant $C > 0$ such that for all $\Omega \subset\subset B$ and all f holomorphic in a neighborhood of $\bar{\Omega}$,*

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} |f|^2 e^{\lambda - \kappa} \omega_B^n \leq C \int_{\Omega \cap W} |f|^2 e^{\lambda - (\nu + \kappa)} \omega_B^{n-1}.$$

Sketch of proof. We may assume the right hand side is finite. Moreover, we can take $\Omega = E(a, \delta)$ for some $a \in W$, with δ so small that $W \cap \Omega$ is the graph of a quadratic hypersurface. By uniform flatness, δ can be taken independent of a .

Consider first the case $a = 0$. Then $\Omega = B(0, \delta)$, and the result follows after an elementary analysis of the properties of s_r as in the proof of Lemma 5.2, and the remark following that proof.

If we now apply the automorphism F_a to $B(0, \delta)$, then Lemma 3.2 and the $\text{Aut}(B)$ -invariance of ω_B show that the same estimates hold on $E(a, \delta)$. \square

REMARK. We have introduced the weight $e^{-\nu}$ because in a sense we want to compare the integral of a function on Ω_ε with the integral on that function on W , multiplied by the area of the disks transverse to W in the tubular neighborhood. In the 1-dimensional case where W is a sequence of uniformly separated points, we find that

$$\int_{\Omega_\varepsilon} |f|^2 e^{\lambda-\kappa} \omega_B \approx \pi \varepsilon^2 \sum_{x \in W \cap \Omega} |f(x)|^2 e^{\kappa(x)-\lambda(x)},$$

while in the higher dimensional case one has

$$\int_{\Omega_\varepsilon} |f|^2 e^{\lambda-\kappa} \omega_B^n \approx \pi \varepsilon^2 \int_{W \cap \Omega} |f|^2 e^{-\kappa} \omega_B^{n-1}.$$

The latter phenomenon happens essentially because unlike Euclidean Lebesgue measure, the Bergman volume element does not respect the local product structure of Ω_ε .

The twisted Bochner-Kodaira Technique. We fix a smoothly bounded pseudoconvex domain $\Omega \subset\subset B$. Let us denote by $\bar{\partial}_\nu^*$ the formal adjoint of $\bar{\partial}$ in the Hilbert space of $(0, 1)$ -forms on Ω , square integrable with respect to a weight $e^{-\nu} \omega_B^n$. For a $(0, 1)$ -form $u = u_{\bar{\alpha}} d\bar{z}^\alpha$, one has

$$\bar{\partial}_\nu^* u = -e^{\nu+(n+1)\lambda} \partial_\alpha \left(e^{-(\nu+(n+1)\lambda)} u^\alpha \right).$$

Recall that for $(0, 1)$ -forms u in the domains of $\bar{\partial}$ and $\bar{\partial}_\nu^*$, Bochner-Kodaira Identity is

$$\begin{aligned} & \int_{\Omega} |\bar{\partial}_\nu^* u|^2 e^{-\nu} \omega_B^n + \int_{\Omega} |\bar{\partial} u|^2 e^{-\nu} \omega_B^n \\ (6) \quad & = \int_{\Omega} \left((\partial_\alpha \partial_{\bar{\beta}} (\nu + (n+1)\lambda)) u^\alpha \bar{u}^\beta \right) e^{-\nu} \omega_B^n + \int_{\Omega} |\bar{\nabla} u|^2 e^{-\nu} \omega_B^n \\ & + \int_{\partial\Omega} (\partial_\alpha \partial_{\bar{\beta}} \rho) u^\alpha \bar{u}^\beta e^{-\nu} d^c(-(n+1)\lambda) \wedge \omega_B^{n-1}, \end{aligned}$$

where ρ is a defining function for Ω such that $|d\rho| \equiv 1$ on $\partial\Omega$. (See, for example, [Siu-82].) The term $(n+1)\lambda$ in the first integral on the right hand side of (6) comes from the Ricci curvature of ω_B . Writing

$$e^{-\psi} = \frac{e^{-\nu}}{\tau}$$

we obtain

$$\bar{\partial}_\nu^* u = \bar{\partial}_\psi^* u - \frac{(\partial_\alpha \tau) u^\alpha}{\tau} \quad \text{and} \quad \partial_\alpha \partial_{\bar{\beta}} \psi = \partial_\alpha \partial_{\bar{\beta}} \nu + \frac{\partial_\alpha \partial_{\bar{\beta}} \tau}{\tau} - \frac{(\partial_\alpha \tau) (\partial_{\bar{\beta}} \tau)}{\tau^2}.$$

Substitution into (6), followed by some simple manipulation, gives the

TWISTED BOCHNER-KODAIRA IDENTITY for $(0, 1)$ -forms: If u is a $(0, 1)$ -form in the domain of $\bar{\partial}^*$, then

$$\begin{aligned} (7) \quad & \int_{\Omega} \tau |\bar{\partial}_\psi^* u|^2 e^{-\psi} \omega_B^n + \int_{\Omega} \tau |\bar{\partial} u|^2 e^{-\psi} \omega_B^n \\ & = \int_{\Omega} \left(\tau (\partial_\alpha \partial_{\bar{\beta}} (\psi + (n+1)\lambda)) u^\alpha \bar{u}^\beta - \left(u^\alpha \bar{u}^\beta \partial_\alpha \partial_{\bar{\beta}} \tau \right) \right. \\ & \quad \left. + 2\text{Re} \left((\partial_\alpha \tau) u^\alpha \bar{\partial}_\psi^* u \right) \right) e^{-\psi} \omega_B^n + \int_{\Omega} \tau |\bar{\nabla} u|^2 e^{-\psi} \omega_B^n \\ & \quad + \int_{\partial\Omega} \tau (\partial_\alpha \partial_{\bar{\beta}} \rho) u^\alpha \bar{u}^\beta e^{-\psi} d^c(-(n+1)\lambda) \wedge \omega_B^{n-1}. \end{aligned}$$

We now use positivity of the last two integrals on the right hand side, together with the Cauchy-Schwarz inequality applied to the first term in the third line, to obtain the so-called

TWISTED BASIC ESTIMATE: If u is a $(0, 1)$ -form in the domain of $\bar{\partial}^*$, then

$$(8) \quad \begin{aligned} & \int_{\Omega} (\tau + A) |\bar{\partial}_{\psi}^* u|^2 e^{-\psi} \omega_B^n + \int_{\Omega} \tau |\bar{\partial} u|^2 e^{-\psi} \omega_B^n \\ & \geq \int_{\Omega} \left(\tau (\partial_{\alpha} \partial_{\bar{\beta}} (\psi + (n+1)\lambda)) u^{\alpha} \bar{u}^{\bar{\beta}} \right. \\ & \quad \left. - \partial_{\alpha} \partial_{\bar{\beta}} \tau u^{\alpha} \bar{u}^{\bar{\beta}} - \frac{1}{A} |(\partial_{\alpha} \tau) u^{\alpha}|^2 \right) e^{-\psi} \omega_B^n. \end{aligned}$$

Choice of ψ , τ and A . From the very beginning, we choose

$$\psi = \kappa - \lambda + s_r.$$

By the density hypothesis (via Lemma 4.1.1) and the fact that $\sqrt{-1} \partial \bar{\partial} s_r = [W] - \Upsilon_r^W$, one has

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} (\psi + (n+1)\lambda) &= \sqrt{-1} \partial \bar{\partial} (\kappa + n\lambda + s_r) \\ &\geq c \sqrt{-1} \partial \bar{\partial} \kappa. \end{aligned}$$

Next, fix $\gamma > 1$. We define

$$\xi = \log(e^{s_r} + \varepsilon^2),$$

with $\varepsilon > 0$ so small that $\gamma - \xi \geq 1$. One has

$$\begin{aligned} & \sqrt{-1} \partial \bar{\partial} \xi \\ &= \sqrt{-1} \partial \left(\frac{e^{s_r}}{e^{s_r} + \varepsilon^2} \bar{\partial} s_r \right) \\ &= \frac{e^{s_r}}{e^{s_r} + \varepsilon^2} \sqrt{-1} \partial \bar{\partial} s_r + \frac{\varepsilon^2}{(e^{s_r} + \varepsilon^2)^2} e^{s_r} \sqrt{-1} \partial s_r \wedge \bar{\partial} s_r \\ &= -\frac{e^{s_r}}{e^{s_r} + \varepsilon^2} \Upsilon_r^W + \frac{4\varepsilon^2}{(e^{s_r} + \varepsilon^2)^2} \sqrt{-1} \partial \left(e^{\frac{1}{2}s_r} \right) \wedge \bar{\partial} \left(e^{\frac{1}{2}s_r} \right), \end{aligned}$$

where the last equality follows since $\sqrt{-1} \partial \bar{\partial} s_r = [W] - \Upsilon_r^W$ and $e^{s_r} |W \equiv 0$.

Let $0 < \alpha < 1$ and set

$$a = \gamma - \alpha \xi.$$

Observe that $a \geq 1$. Moreover, we have

$$\begin{aligned} & -\sqrt{-1} \partial \bar{\partial} a \\ &= \alpha \sqrt{-1} \partial \bar{\partial} \xi \\ &= -\frac{\alpha e^{s_r}}{e^{s_r} + \varepsilon^2} \Upsilon_r^W + \frac{4\alpha \varepsilon^2}{(e^{s_r} + \varepsilon^2)^2} \sqrt{-1} \partial \left(e^{\frac{1}{2}s_r} \right) \wedge \bar{\partial} \left(e^{\frac{1}{2}s_r} \right). \end{aligned}$$

Now let

$$\tau = a + \log a \quad \text{and} \quad A = (1 + a)^2.$$

Then $\tau \geq 1$ and we have

$$\partial \tau = \left(1 + \frac{1}{a} \right) \partial a \quad \text{and} \quad \sqrt{-1} \partial \bar{\partial} \tau = \left(1 + \frac{1}{a} \right) \sqrt{-1} \partial \bar{\partial} a - \frac{1}{a^2} \sqrt{-1} \partial a \wedge \bar{\partial} a,$$

and thus

$$-\sqrt{-1} \partial \bar{\partial} \tau - \frac{\sqrt{-1} \partial \tau \wedge \bar{\partial} \tau}{A} = \left(1 + \frac{1}{a} \right) (-\sqrt{-1} \partial \bar{\partial} a) \geq -\sqrt{-1} \partial \bar{\partial} a.$$

It follows that

$$\begin{aligned}
& \tau\sqrt{-1}\partial\bar{\partial}(\psi + (n+1)\lambda) - \sqrt{-1}\partial\bar{\partial}\tau - \frac{|\partial\tau|^2}{A} \\
& \geq c\sqrt{-1}\partial\bar{\partial}\kappa - \frac{\alpha e^{s_r}}{e^{s_r} + \varepsilon^2} \Upsilon_r^W + \frac{4\alpha\varepsilon^2}{(e^{s_r} + \varepsilon^2)^2} \sqrt{-1}\partial\left(e^{\frac{1}{2}s_r}\right) \wedge \bar{\partial}\left(e^{\frac{1}{2}s_r}\right) \\
& \geq \frac{4\alpha\varepsilon^2}{(e^{s_r} + \varepsilon^2)^2} \sqrt{-1}\partial\left(e^{\frac{1}{2}s_r}\right) \wedge \bar{\partial}\left(e^{\frac{1}{2}s_r}\right),
\end{aligned}$$

provided we take α sufficiently small. Substituting into the twisted basic estimate (8), we obtain the following lemma.

LEMMA 5.4. *If u is a $(0, 1)$ -form in the domain of $\bar{\partial}^*$, then*

$$\begin{aligned}
& \int_{\Omega} (\tau + A) |\bar{\partial}_{\psi}^* u|^2 e^{-\psi} \omega_B^n + \int_{\Omega} \tau |\bar{\partial}u|^2 e^{-\psi} \omega_B^n \\
& \geq c \int_{\Omega} \frac{4\varepsilon^2}{(e^{s_r} + \varepsilon^2)^2} \left| \partial\left(e^{\frac{1}{2}s_r}\right)(u) \right|^2 e^{-\psi} \omega_B^n.
\end{aligned}$$

An a priori estimate. We write $\Omega_j = B\left(0, \frac{j}{1+j}\right)$. Suppose we are given $f \in \mathfrak{H}^2(W, \kappa)$. Since W is a closed submanifold of B , there exists a holomorphic extension \tilde{f} of f to B . We write

$$W_j = W \cap \Omega_j, \quad f_j = f|_{W_j} \quad \text{and} \quad \tilde{f}_j = \tilde{f}|_{\Omega_j}.$$

Observe that

$$\int_{W_j} |f_j|^2 e^{\lambda - \kappa} \omega_B^{n-1} \leq \int_W |f|^2 e^{\lambda - \kappa} \omega_B^{n-1} < +\infty.$$

Let $\chi \in \mathcal{C}_0^\infty([0, 1])$ be such that

$$0 \leq \chi \leq 1, \quad \chi|_{[0, 1/3]} \equiv 1 \quad \text{and} \quad \sup_{[0, 1]} |\chi'| \leq 2.$$

We set

$$\chi_\varepsilon = \chi\left(\frac{e^{s_r}}{\varepsilon^2}\right)$$

and define the 1-forms $\alpha_{\varepsilon, j}$ on Ω_j by

$$\alpha_{\varepsilon, j} = \bar{\partial}\chi_\varepsilon \tilde{f}_j.$$

We note that for ε sufficiently small, $\alpha_{\varepsilon, j}$ is supported on the tubular neighborhood

$$\Omega_{\varepsilon, j} := \Omega_j \cap \left\{ e^{\frac{1}{2}s_r} \leq \varepsilon \right\}$$

of W_j in Ω_j . Thus, for a $(0, 1)$ -form u with compact support on Ω_j , we have

$$\begin{aligned}
|(\alpha_{\varepsilon,j}, u)|^2 &= \left| \int_{\Omega_j} \left\langle \chi' \left(\frac{e^{s_r}}{\varepsilon^2} \right) \frac{\bar{\partial}(e^{s_r})}{\varepsilon^2} \tilde{f}_j, u \right\rangle e^{-\psi} \omega_B^n \right|^2 \\
&\leq \left(\frac{2}{\varepsilon^2} \int_{\Omega_j} \left| \chi' \left(\frac{e^{s_r}}{\varepsilon^2} \right) \right| \left| \partial \left(e^{\frac{1}{2}s_r} \right) (u) \right| |\tilde{f}_j| e^{-\frac{1}{2}s_r} e^{\lambda-\kappa} \omega_B^n \right)^2 \\
&\leq \frac{16}{\varepsilon^4} \left(\int_{\Omega_{\varepsilon,j}} |\tilde{f}_j|^2 \frac{(e^{s_r} + \varepsilon^2)^2}{4\varepsilon^2} e^{\lambda-\kappa} \omega_B^n \right) \\
&\quad \times \int_{\Omega_j} \frac{4\varepsilon^2}{(e^{s_r} + \varepsilon^2)^2} \left| \partial \left(e^{\frac{1}{2}s_r} \right) (u) \right|^2 e^{-\psi} \omega_B^n \\
&\leq \frac{16}{c} C_{\varepsilon,j} (\|T^*u\|^2 + \|Su\|^2),
\end{aligned}$$

where

$$Tu = \bar{\partial} \left(\sqrt{\tau + Au} \right) \quad \text{and} \quad Su = \sqrt{\tau} (\bar{\partial}u),$$

and

$$C_{\varepsilon,j} = \frac{1}{\varepsilon^2} \int_{\Omega_{\varepsilon,j}} |\tilde{f}_j|^2 e^{\lambda-\kappa} \omega_B^n.$$

Thus the last inequality follows from Lemma 5.4 and the fact that $e^{s_r} < \varepsilon^2$ on $\Omega_{\varepsilon,j}$.

By standard Hilbert space methods, we have the following L^2 twisted- $\bar{\partial}$ theorem.

THEOREM 5.5. *There exists a function $h_{\varepsilon,j}$ on Ω_j such that*

$$Th_{\varepsilon,j} = \alpha_{\varepsilon,j} \quad \text{and} \quad \int_{\Omega_j} |h_{\varepsilon,j}|^2 e^{-\psi} \omega_B^n \leq \frac{16}{c} C_{\varepsilon,j}.$$

In particular, $h_{\varepsilon,j}|_W \equiv 0$.

Proof. Consider the linear functional

$$\mathcal{L} : T^*u \mapsto (u, \alpha_{\varepsilon,j}),$$

where $u \in \text{Kernel}S \cap \text{Domain}(T^*)$. The estimate

$$|(u, \alpha_{\varepsilon,j})|^2 \leq \frac{16C_{\varepsilon,j}}{c} \|T^*u\|^2$$

($Su = 0$) means \mathcal{L} is continuous on the image of T^* , hence on the closure of that image. Extend \mathcal{L} to be 0 in $\text{Image}(T^*)^\perp$. Then \mathcal{L} is a continuous linear functional in our Hilbert space, and thus, by the Riesz Representation Theorem, is represented by some element $h_{\varepsilon,j}$ having the same norm as \mathcal{L} in the orthogonal direction. Elliptic regularity implies that $h_{\varepsilon,j}$ is smooth.

It remains only to prove the assertion about the vanishing of $h_{\varepsilon,j}$. But by Lemma 5.2.3, $e^{-\psi}$ is not locally integrable at any point of W , and thus the vanishing of $h_{\varepsilon,j}|_W$ follows. \square

Conclusion of the proof of Theorem 1.3. Observe first that by Lemma 5.3 there exists a constant $C > 0$ such that, for all j ,

$$\limsup_{\varepsilon \rightarrow 0} C_{\varepsilon,j} \leq C \int_W |f|^2 e^{\lambda-(\nu+\kappa)} \omega_B^{n-1}.$$

We set

$$F_{\varepsilon,j} = \chi_\varepsilon \tilde{f}_j - \sqrt{(\tau + A)} h_{\varepsilon,j} \quad \text{on } \Omega_j.$$

By Theorem 5.5, $F_{\varepsilon,j}$ is holomorphic on Ω_j and $F_{\varepsilon,j}|_{W_j} - f_j \equiv 0$. Moreover there exists a constant M such that

$$\int_{\Omega_j} |F_{\varepsilon,j}|^2 e^{\lambda-\kappa} \omega_B^n \leq M \left(o(1) + \int_W |f|^2 e^{\lambda-(\nu+\kappa)} \omega_B^{n-1} \right), \quad \varepsilon \sim 0.$$

Indeed, the integral

$$\int_{\Omega_j} |\chi_\varepsilon \tilde{f}_j|^2 e^{\lambda-\kappa} \omega_B^n$$

is negligible for small ε , since the integrand is locally integrable and supported on a set of arbitrarily small measure. On the other hand,

$$\begin{aligned} \int_{\Omega_j} (\tau + A) |h_{\varepsilon,j}|^2 e^{\lambda-\kappa} \omega_B^n &= \int_{\Omega_j} e^{s_r} (\tau + A) |h_{\varepsilon,j}|^2 e^{-\psi} \omega_B^n \\ &\leq \left(\sup_{\Omega_j} e^{s_r} (\tau + A) \right) \int_{\Omega_j} |h_{\varepsilon,j}|^2 e^{-\psi} \omega_B^n \\ &\leq C_{\varepsilon,j} e^{\frac{\gamma}{\alpha}} \left(\sup_{\Omega_j} e^{-\frac{1}{\alpha} a} (a + \log(a) + (1+a)^2) \right) \\ &\leq KC_{\varepsilon,j} \end{aligned}$$

for some universal constant K depending only on the density of W . The last estimate holds since $a \geq 1$.

By Corollary 2.4 and the Lebesgue Dominated Convergence Theorem,

$$F_j = \lim_{\varepsilon \rightarrow 0} F_{\varepsilon,j}$$

exists, uniformly for each fixed j . Moreover, since $F_{\varepsilon,j} = f$ on W_j and $F_{\varepsilon,j} \rightarrow F_j$ pointwise, we have $F_j = f$ on W_j for all j . We thus have a sequence of functions F_j , holomorphic by Montel's Theorem, such that $F_j|_W = f$ and

$$\int_{\Omega_j} |F_j|^2 e^{\lambda-\kappa} \omega^n \leq C \int_W |f|^2 e^{\lambda-(\nu+\kappa)} \omega^{n-1}.$$

Moreover, the constant C does not depend on j . Letting $j \rightarrow \infty$, we obtain (again by corollary 2.4, the Dominated Convergence Theorem and Montel's Theorem) a holomorphic function F that also agrees with f on W , and furthermore satisfies

$$\int_B |F|^2 e^{\lambda-\kappa} \omega_B^n \leq C \int_W |f|^2 e^{\lambda-(\nu+\kappa)}.$$

This completes the proof of Theorem 1.3

6. SAMPLING

6.1. A construction of Berndtsson-Ortega Cerdà.

LEMMA 6.1. *Let φ be a subharmonic function on the unit disk \mathbb{D} . Then there exist a positive constant K and a holomorphic function on $G \in \mathcal{O}(\mathbb{D}(0, 1/2))$ such that $G(0) = 0$ and*

$$\sup_{\mathbb{D}(0,1/2)} |\varphi - \varphi(0) - 2\operatorname{Re} G| \leq K.$$

Moreover, if φ depends smoothly on a parameter, then so does G .

The proof of this lemma, which uses Riesz Potentials, can be found in [BO-95].

6.2. Restriction from tubes and the upper inequality.

PROPOSITION 6.2. *Let W be a uniformly flat smooth hypersurface. Then there exists a constant $C > 0$ such that for all $\varepsilon > 0$ sufficiently small and all $F \in \mathcal{H}^2(N_\varepsilon^B(W), \kappa)$ one has*

$$C\varepsilon^2 \int_W |F|^2 e^{\lambda - \kappa} \omega_B^{n-1} \leq \int_{N_\varepsilon^B(W)} |F|^2 e^{\lambda - \kappa} \omega_B^n.$$

Proof. Let

$$D(0, \varepsilon) = \{(z', z^n) \in \mathbb{C}^{n-1} \times \mathbb{C}; |z'| \leq \varepsilon, |z^n| < \varepsilon\}.$$

Via Lemma 3.2, the uniform flatness of W implies that $N_\varepsilon^W(W)$ is a union of open sets U_j such that for each j there is some $F_{z_j} \in \text{Aut}(B)$ for which

$$F_{z_j}(U_j) \sim D(0, \varepsilon).$$

Moreover, this approximation may be taken uniform in j . Thus it suffices to prove that for some $\varepsilon > 0$ and all $a \in W$,

$$C\varepsilon^2 \int_{D(0, \varepsilon) \cap F_a(W)} |F|^2 e^{\lambda - \kappa} \omega_B^{n-1} \leq \int_{D(0, \varepsilon)} |F|^2 e^{\lambda - \kappa} \omega_B^n.$$

After a change of variables provided by Lemma 3.2, we may assume that $F_a(W) \subset \mathbb{C}^{n-1} \times \{0\}$.

Now, by Lemma 6.1 there exists a function G , holomorphic in z^n , such that

$$G(z', 0) \equiv 0 \quad \text{and} \quad e^{-\kappa(z', 0) + 2\text{Re } G(z', z^n)} \leq ce^{-\kappa(z', z^n)}$$

for some $c > 0$. We then have

$$\begin{aligned} \int_{B(0, \varepsilon) \cap \mathbb{C}^{n-1} \times \{0\}} |F|^2 e^{-(\nu + \kappa)} e^{\lambda} \omega_B^{n-1} &= \int_{B(0, \varepsilon) \cap \mathbb{C}^{n-1} \times \{0\}} |F e^G|^2 e^{-(\nu + \kappa)} e^{\lambda} \omega_B^{n-1} \\ &\leq \frac{C_o}{\varepsilon^2} \int_{D(0, \varepsilon)} |F e^G|^2 e^{-\kappa(z', 0)} e^{\lambda} \omega_B^n \\ &= \frac{C_o}{\varepsilon^2} \int_{D(0, \varepsilon)} |F|^2 e^{-\kappa(z', 0) + 2\text{Re } G} e^{\lambda} \omega_B^n \\ &\leq \frac{1}{C\varepsilon^2} \int_{D(0, \varepsilon)} |F|^2 e^{-\kappa(z', z^n)} e^{\lambda} \omega_B^n. \end{aligned}$$

The first inequality follows from the sub-mean value property for radial measures in the disk (see also Corollary 2.4). This completes the proof. \square

COROLLARY 6.3. *If W is a uniformly flat hypersurface then there exists a constant M such that for all $F \in H^2(B, \kappa)$,*

$$\int_W |F|^2 e^{\lambda - (\nu + \kappa)} \omega_B^{n-1} \leq M \int_B |F|^2 e^{\lambda - \kappa} \omega_B^n.$$

6.3. **Regularization of the singular function s_r .** Consider the function

$$s_{r, \varepsilon}(z) := \frac{1}{V_n(\varepsilon)} \int_{E(z, \varepsilon)} s_r \omega_B^n.$$

In this section we prove the following result.

LEMMA 6.4. *The function $s_{r, \varepsilon}$ enjoys the following properties.*

(1)

$$\lim_{\varepsilon \rightarrow 0} dd^c s_{r, \varepsilon} = [W] - \Upsilon_r^W.$$

(2) For each r there exists a constant C_r such that if $0 < \varepsilon \leq \varepsilon_1 \ll 1$ and $\text{dist}(z, W) < \varepsilon$, then

$$\log \varepsilon^2 - C_r \leq s_{r,\varepsilon} \leq 0.$$

Proof. Property 1 is seen as follows: let f be a test $(n-1, n-1)$ -form. Then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_B (dd^c s_{r,\varepsilon}) \wedge f &= \lim_{\varepsilon \rightarrow 0} \int_B s_{r,\varepsilon} dd^c f \\ &= \int_B s_r dd^c f \\ &= \int_B ([W] - \Upsilon_r^W) \wedge f. \end{aligned}$$

Property 2 may be established locally, and using group invariance and uniform flatness, we need only consider the case $z = 0$. But then by the calculation in the remark following the proof of Lemma 5.2 we may assume that $s_r = \log |\zeta^n|^2$, and thus 2 follows by integration. \square

6.4. The proof of Theorem 1.4.

A positivity lemma. A key idea behind the proof of the lower sampling inequality is a certain positivity lemma, which we now state and prove.

LEMMA 6.5. *Let θ be a positive $(n-1, n-1)$ -form in B such that for some weight ψ and each $h \in \mathcal{H}^2(B, \psi)$,*

$$\int_B |h|^2 e^{-\psi} \sqrt{-1} \partial \bar{\partial} \theta < +\infty.$$

Then

$$\int_B |h|^2 e^{-\psi} \sqrt{-1} \partial \bar{\partial} \psi \wedge \theta \geq - \int_B |h|^2 e^{-\psi} \sqrt{-1} \partial \bar{\partial} \theta.$$

Proof. Letting $S = |h|^2 e^{-\psi}$, one calculates that

$$\frac{\sqrt{-1} \partial \bar{\partial} S}{S} = \frac{\sqrt{-1} \partial S \wedge \bar{\partial} S}{S^2} + \sqrt{-1} \partial \bar{\partial} \log |h|^2 - \sqrt{-1} \partial \bar{\partial} \psi,$$

and thus

$$\sqrt{-1} \partial \bar{\partial} S \wedge \theta \geq -S \sqrt{-1} \partial \bar{\partial} \psi \wedge \theta.$$

Let $f : \mathbb{R} \rightarrow [0, 1]$ be a smooth function supported on $(-\infty, 3/4]$ such that $f|_{(-\infty, 0]} \equiv 1$. Consider the function

$$\chi_a(z) = f\left(\frac{1 - |z|^2}{-a}\right), \quad a > 0.$$

Then

$$\begin{aligned} \int_B \sqrt{-1} \partial \bar{\partial} S \wedge \theta &= \lim_{a \rightarrow 0^+} \int_B \chi_a(z) \sqrt{-1} \partial \bar{\partial} S \wedge \theta \\ &= \lim_{a \rightarrow 0^+} \int_B S \sqrt{-1} \partial \bar{\partial} (\chi_a(z) \wedge \theta) \\ &= \lim_{a \rightarrow 0^+} \left(\int_B \chi_a(z) S \sqrt{-1} \partial \bar{\partial} \theta + O(a) \right) \\ &= \int_B S \sqrt{-1} \partial \bar{\partial} \theta, \end{aligned}$$

where the second equality follows from the Green-Stokes identity (2). \square

Conclusion of the proof of Theorem 1.4. Let

$$\psi = \kappa + n\lambda + \alpha s_{r,\varepsilon}.$$

In view of Lemmas 6.4 and 4.1.2, for some $0 << \alpha < 1$, $c > 0$ and $\theta \in \mathcal{P}_W(B)$ we have

$$\sqrt{-1}\partial\bar{\partial}\psi \wedge \theta \leq -ce^{n\lambda}\omega_B^n + C[W]_\varepsilon \wedge e^{n\lambda}\omega_B^{n-1},$$

where $[W]_\varepsilon$ denotes the regularization of the current $[W]$ in the manner of Lemma 6.4. Let $F \in \mathcal{H}^2(B, \kappa)$. Then

$$\begin{aligned} \int_B |F|^2 e^{\lambda-\kappa}\omega_B^n &\leq \int_B |F|^2 e^{\lambda-\psi+n\lambda}\omega_B^n \\ &\leq C \int_B |F|^2 e^{\lambda-\psi}[W]_\varepsilon \wedge e^{n\lambda}\omega_B^{n-1} - C \int_B |F|^2 e^{\lambda-\psi}\sqrt{-1}\partial\bar{\partial}\psi \wedge \theta \\ &\leq C \int_B |F|^2 e^{\lambda-\psi}[W]_\varepsilon \wedge e^{n\lambda}\omega_B^{n-1}, \end{aligned}$$

where the last inequality follows from Lemma 6.5 and the definition of $\mathcal{P}_W(B)$. Thus we have

$$\begin{aligned} \int_B |F|^2 e^{\lambda-\kappa}\omega_B^n &\leq C \int_B |F|^2 e^{\lambda-\psi}[W]_\varepsilon \wedge e^{n\lambda}\omega_B^{n-1} \\ &\leq \frac{C}{\varepsilon^2} \int_{N_\varepsilon(W)} |F|^2 e^{\lambda-\psi} e^{n\lambda}\omega_B^n \\ &\leq \frac{C}{\varepsilon^{2+2\alpha}} \int_{N_\varepsilon(W)} |F|^2 e^{\lambda-\kappa}\omega_B^n. \end{aligned}$$

Our next task is to compare

$$\int_{N_\varepsilon(W)} |F|^2 e^{\lambda-\kappa}\omega_B^n \quad \text{with} \quad \int_W |F|^2 e^{\lambda-(\nu+\kappa)}\omega_B^{n-1}.$$

To do this, we cover $N_\varepsilon^B(W)$ by domains

$$\Delta_p(\varepsilon) = \Phi((E(p, \varepsilon_p) \cap W) \times \mathbb{D}(0, \varepsilon)), \quad p \in \mathcal{W}.$$

Here Φ is the diffeomorphism defined in the remark in Section 3 following Definition 3.1, $E(p, \varepsilon_p)$ is the Bergman-Green ball of center p and radius ε_p , and $\mathcal{W} \subset W$ is a discrete set that is uniformly separated with respect to the Bergman-Green distance. We now employ Lemma 6.1 once more to obtain a function

$$\Delta_p(\varepsilon) \stackrel{\Phi}{\cong} (W \cap E(p, \varepsilon_p)) \times \mathbb{D}(0, \varepsilon) \ni (x, t) \mapsto H_p(x, t) \in \mathbb{C}$$

that is holomorphic in t and satisfies

$$H_p(x, 0) = 0 \quad \text{and} \quad |2\text{Re}(H_p(x, t)) + \kappa(x, 0) - \kappa(x, t)| \leq C$$

where C is an absolute constant depending only on $\sqrt{-1}\partial\bar{\partial}\kappa$.

Let $F_p = Fe^{-H_p}$. By Taylor's Theorem, for each x we have

$$|F_p(x, t)|^2 \leq C|F(x, 0)|^2 + \varepsilon^2 \sup_{|t| \leq \varepsilon} \left| \frac{\partial F_p}{\partial t} \right|^2.$$

We then obtain

$$\begin{aligned}
& \int_{\Delta_p(\varepsilon)} |F|^2 e^{\lambda-\kappa} \omega_B^n \\
& \leq \int_{\Delta_p(\varepsilon)} |F_p|^2 e^{-\kappa(x,0)} e^\lambda \omega_B^n \\
& \leq C_1 \varepsilon^2 \int_{W \cap E(p, \varepsilon_p)} |F|^2 e^{\lambda-(\nu+\kappa)} \omega_B^{n-1} + \varepsilon^2 \int_{\Delta_p(\varepsilon)} \sup_{|t| \leq \varepsilon} \left| \frac{\partial F_p}{\partial t} \right|^2 e^{-\kappa(x,0)} e^\lambda \omega_B^n \\
& \leq C_1 \varepsilon^2 \int_{W \cap E(p, \varepsilon_p)} |F|^2 e^{\lambda-(\nu+\kappa)} \omega_B^{n-1} + \varepsilon^4 \int_{W \cap E(p, \varepsilon_p)} \sup_{|t| \leq \varepsilon} \left| \frac{\partial F_p}{\partial t} \right|^2 e^{-\kappa(x,0)} e^{\lambda-\nu} \omega_B^{n-1} \\
& \leq C_1 \varepsilon^2 \int_{W \cap E(p, \varepsilon_p)} |F|^2 e^{\lambda-(\nu+\kappa)} \omega_B^{n-1} + C \varepsilon^4 \int_{\Delta_p(\varepsilon_o)} |F_p|^2 e^{-\kappa(x,0)} e^\lambda \omega_B^n \\
& \leq \varepsilon^2 \int_{W \cap E(p, \varepsilon_p)} |F|^2 e^{\lambda-(\nu+\kappa)} \omega_B^{n-1} + C \varepsilon^4 \int_{\Delta_p(\varepsilon_o)} |F|^2 e^{\lambda-\kappa} \omega_B^n,
\end{aligned}$$

where $\varepsilon < \varepsilon_o/2$ and ε_o is as in Definition 3.1. We have used the Cauchy estimates in the penultimate inequality. Combining all of this, and summing over $p \in \mathscr{W}$, we obtain

$$\int_B |F|^2 e^{\lambda-\kappa} \omega_B^n \leq \frac{C}{\varepsilon^{2\alpha}} \int_W |F|^2 e^{\lambda-(\nu+\kappa)} \omega_B^{n-1} + C' \varepsilon^{2-2\alpha} \int_B |F|^2 e^{\lambda-\kappa} \omega_B^n,$$

which establishes the left inequality in (1) as soon as we take ε small enough. Here we are using the fact that, since \mathscr{W} is uniformly separated,

$$\sum_{p \in \mathscr{W}} \int_{\Delta_p(\varepsilon_o)} |F|^2 e^{\lambda-\kappa} \omega_B^n \leq C \int_B |F|^2 e^{\lambda-\kappa} \omega_B^n.$$

The right inequality was already established in Corollary 6.3. The proof of Theorem 1.4 is complete. \square

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS, URBANA, IL 61801
E-mail address: forgacs@uiuc.edu

DEPARTMENT OF MATHEMATICS
STONY BROOK UNIVERSITY, STONY BROOK, NY 11794
E-mail address: dror@math.sunysb.edu