

Answers for HW 1

9.

$$\lim_{n \rightarrow \infty} \frac{3 + 5n^2}{n + n^2} = \lim_{n \rightarrow \infty} \frac{\frac{3}{n^2} + 5}{\frac{1}{n} + 1} = \frac{0 + 5}{0 + 1} = 5.$$

10.

$$\lim_{n \rightarrow \infty} \frac{n + 1}{3n - 1} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{3 - \frac{1}{n}} = \frac{1 + 0}{3 - 0} = \frac{1}{3}.$$

11.

$$\lim_{n \rightarrow \infty} \frac{2^n}{3^{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = 0.$$

12.

$$\lim_{n \rightarrow \infty} \frac{1}{\frac{1}{\sqrt{n}} + 1} = \frac{1}{0 + 1} = 1.$$

19. Consider  $f(x) = x^2 e^{-x} = \frac{x^2}{e^x}$ . By l'Hospital's Rule twice,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0.$$

So we get  $\lim_{n \rightarrow \infty} a_n = \lim f(n) = 0$ .

20. Since  $\arctan(x)$  is increasing and a one to one continuous map from  $(-\infty, +\infty)$  to  $(-\frac{\pi}{2}, +\frac{\pi}{2})$ ,  $a_n = \arctan(2n)$  has the limit  $\frac{\pi}{2}$ .

21.  $0 \leq a_n = \frac{\cos^2(n)}{2^n} \leq \frac{1}{2^n} \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$ .

22.  $a_{2n} = 2n$  and  $a_{2n-1} = -(2n - 1)$ , so it is divergent.

23. Since  $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e^x$ ,  $\lim a_n = \lim ((1 + \frac{2}{n})^{\frac{n}{2}})^2 = e^2$ .

24.  $0 \leq |a_n| \leq \frac{1}{1 + \sqrt{n}}$ , so  $\lim_{n \rightarrow \infty} a_n = 0$ .

25. If it converges, then it will have a finite limit. Thus  $a_n - a_{n+1}$  should be very small when  $n$  is sufficiently large, i.e., we would have  $(\lim(a_n - a_{n+1}) = 0)$ . However, for any  $n$  in  $N$ , there exists some  $m$  bigger than  $n$  s.t.  $a_m - a_{m+1} = -1$ . So it diverges.

26. Consider  $f(x) = \frac{(\ln x)^2}{x}$ . By l'Hospital's Rule twice,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\frac{2 \ln x}{x}}{1} = \lim_{x \rightarrow \infty} \frac{2}{x} = 0. \text{ So } \lim a_n = 0.$$

27. By the meaning of continuity of  $\ln(x)$ ,  $a_n = \ln(\frac{2n^2+1}{n^2+1})$ , so  $\lim a_n = \ln 2$ .

31. Similar to 27.  $\lim a_n = \arctan 1 = \frac{\pi}{4}$ .

32. Since  $|\sin(n)|$  is bounded by 1 and  $\sqrt{n}$  goes to infinity, the limit is 0.

33. When  $n > 3$ ,

$$0 \leq a_n \leq \frac{n^3}{n(n-1)(n-2)(n-3)} = \frac{1}{n} \left(1 - \frac{1}{n}\right)^{-1} \left(1 - \frac{2}{n}\right)^{-1} \left(1 - \frac{3}{n}\right)^{-1}.$$

Thus the limit is 0 by the squeeze principle.

36. Note that  $a_n$  is always an integer. A famous conjecture, Collatz conjecture (also called Kakutani, Shizuo conjecture), is that no matter which positive integer  $a_1$  is, there will be some finite  $n$  s.t.  $a_n = 1$ . For more information, go to [http://en.wikipedia.org/wiki/Collatz\\_conjecture](http://en.wikipedia.org/wiki/Collatz_conjecture).

42.  $a_n = \frac{2n-3}{3n+4} = \frac{2}{3} + \frac{-3-\frac{2}{3} \times 4}{3n+4}$ , so it is increasing to  $2/3$  and bounded.

46. First to prove  $a_n \leq 2$  by induction, then  $2a_{n+1} \geq a_{n+1}^2 \geq a_n + a_n$ . Hence it is increasing. So the limit  $b$  exists. Take limit to both sides of the equation to get  $b^2 = 2 + b \Rightarrow b = 2$  or  $-1$ . So the limit is 2.