MAT211 - Introduction to Linear Algebra
SUMMER 07

Practice Final

Question 1. Find the rank of the matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Question 2. Is the matrix below invertible? In case yes, find its inverse:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Question 3. Define a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T(\vec{x}) = A\vec{x}$, where A is the matrix of exercise 2. Notice that $\mathrm{Im}(T)$ and $\ker(T)$ are both subspaces of $\mathbb{R}^4$. What can you say about $\ker(T) \cap \mathrm{Im}(T)$?

Question 4. Let V be a finite dimensional vector space and $T : V \to V$ a linear transformation. Suppose that $\mathcal{B} = \{f_1, f_2, f_3, f_4\}$ is a basis for V and that $T(f_1) = 3f_1 - f_3$, $T(f_2) = f_4$, $T(f_3) = 6f_1 - 2f_3 + f_4$, $T(f_4) = 3f_1 - 3f_2$. Find bases for $\mathrm{Im}(T)$ and $\ker(T)$.

Question 5. Consider again the linear transformation of question 4. Suppose that the space V is $\mathbb{R}^4$ and that the basis $\mathcal{B}$ is the canonical basis. Find a basis for the orthogonal complement of $\mathrm{Im}(T)$.

Question 6. Find the matrix of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect the the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, where:

$$A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \text{ and } \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

What is the matrix S which changes from the basis $\mathcal{B}$ to the canonical basis?

Question 7. Consider the linear transformation $\frac{d}{dt} : P_3 \to P_3$.
(a) Find the matrix of this linear transformation with respect to basis $\mathcal{B} = \{t^3, t^2, t, 1\}$.
(b) Consider now the basis $\mathcal{C} = \{1, t, t^2, t^3\}$. Find the matrix $S_{\mathcal{B}\to\mathcal{C}}$.
(c) Find the matrix of $\frac{d}{dt}$ with respect to the basis $\mathcal{C}$.
(d) Is $\frac{d}{dt}$ an isomorphism?

Question 8. Consider the linear transformation $\frac{d}{dt} : P_3 \to P_3$ of the previous exercise. What is the meaning of the expression $\det(\frac{d}{dt})$? Generelly, if V is a vector space and we have a linear transformation $T : V \to V$, is there any way you can make sense of the expression $\det(T)$?

Question 9. Consider the bases $\mathcal{B} = \{1, t-1, t^2-2t+1\}$ and $\mathcal{C} = \{1+t, -t, t^2\}$ for P_2 and the map $T : P_2 \to P_2$ give by $T(f(t)) = \frac{df(t)}{dt} + 2f(t)$.
(a) Check that T is linear.
(b) Find the matrix of T with respect to the basis $\mathcal{B}$.
(c) Find the matrix $S_{\mathcal{B}\to\mathcal{C}}$ of change of basis from $\mathcal{B}$ to $\mathcal{C}$.
(d) Using (b) and (c), find the matrix of T with respect to the basis $\mathcal{C}$.

Question 10. Consider the linear transformations $T : P_2 \to M_{2\times 2}$, $p : M_{2\times 2} \to \mathbb{R}^3$, $L : \mathbb{R}^3 \to P_2$ give by:

$$T(a_2t^2+a_1t+a_0) = \begin{bmatrix} a_2 & a_1 \\ 0 & -a_0 \end{bmatrix}, \; T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ c \\ d \end{bmatrix}, \; L\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = v_1t^2+v_2t+v_3$$

(a) Which of these transformations are isomorphisms?
(b) Find the matrix of the composition $L \circ p \circ T$ with respect to the basis $\mathcal{B} = \{1, t-1, t^2-2t+1\}$.
(c) Is $L \circ p \circ T$ an isomorphism?

Question 11. Let $T : M_{2\times 2} \to M_{2\times 2}$ be given by:

$$T(M) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M - M \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Find the matrix of this linear transformation with respect to the basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

Question 12. Below bases for some subspaces of $\mathbb{R}^n$ are given. Perform the Gram-Schmidt process and find orthonormal bases for these subspaces.

$$\text{(a)} \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 7 \end{bmatrix} \right\} \qquad \text{(b)} \left\{ \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} \right\}$$

Question 13. Find the least-square solutions of $A\vec{x} = \vec{b}$ in the following cases:

$$\text{(a) } A = \begin{bmatrix} 6 & 9 \\ 3 & 8 \\ 2 & 10 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 0 \\ 49 \\ 0 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$$

$$\text{(c) } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Compute the error on the above least-square solutions.

Question 14. Fit a linear function of the form $f(t) = a_0 + a_1 t$ to the data points $(0,0), (0,1), (1,1)$. Sketch your solution.

Question 15. Fit a quadratic polynomial to the data points $(0,27), (1,0), (2,0), (3,0)$. Sketch your solution.

Question 16. Find the function of the form:

$$f(t) = a_0 + a_1 \sin(t) + a_2 \cos(t) + a_3 \sin(2t) + a_4 \cos(2t)$$

that best fits the data points:

$$\{(0,0), (0.5,0.5), (1,1), (1.5,1.5), (2,2), (2.5,2.5), (3,3)\}$$

(you may use a calculator to copute the values of sin and cos). Sketch your solution.

Question 17. Compute the determinant of:

$$A = \begin{bmatrix} -1 & 0 & 2 & -2 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & -4 & -3 \\ 1 & 6 & 10 & 1 \end{bmatrix}$$

Question 18. Find the eigenvalues and eigenvectors of the matrices below. Find bases for the eigenspaces. If possible, find a basis of eigenvectors, i.e., an eigenbasis.

(a) $A = \begin{bmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix}$ (b) $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}$ (c) $A = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}$ (e) $A = \begin{bmatrix} 2 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{bmatrix}$ (f) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Question 19. Decide which of the matrices in the previous exercise are diagonalizable. For those which are, diagolize them, i.e., find the matrices S and D.

Question 20. Is every invertible matrix diagonalizable?

Question 21. Now that the course is finishing, you should be able to have a more or less global picture of Linear Algebra. Write a few paragraphs about the most important things you learned in the course and how they can be useful in different branches of science. Do not forget to include some main points such as: the concept of *linearity* (after all, why is linearity so important and how does it make things easier?); what the main tools or Linear Algebra are; how abstract concepts such as vector spaces come into play and turn out to be useful; how different parts of Linear Algebra are connected among themselves and which other parts of Mathematics and sicence in general. Please, try to write objectively and concisely. Notice that your answer is not better just because it is longer.