Answers to the MAT127 Homework No.8

Chapter 8 Section 8 Problem 1-3, 9, 10, 13, 14
1.

$$\begin{aligned}\sqrt{1+x} &= \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n \\ &= 1+\sum_{n=1}^{\infty}\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-n)\cdots(\frac{1}{2}-n+1)}{n!}x^n \\ &= 1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\dots\cdot(2n-3)}{2^n n!}x^n\end{aligned}$$

So $a_n = \frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\dots\cdot(2n-3)}{2^n n!}$, then the radius of convergence is

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{2^{n+1}(n+1)!}{2^n n!(2n-1)} = \lim_{n\to\infty}\frac{2n}{2n-1} = 1.$$

2.

$$\begin{aligned}\frac{1}{(1+x)^4} &= \sum_{n=0}^{\infty}\binom{-4}{n}x^n \\ &= 1+\sum_{n=1}^{\infty}\frac{(-4)(-4-1)(-4-2)\cdots(-4-n+1)}{n!}x^n \\ &= 1+\sum_{n=1}^{\infty}\frac{(-1)^n 4\cdot 5\cdot 6\cdot\dots\cdot(n+3)}{n!}x^n \\ &= 1+\sum_{n=1}^{\infty}\frac{(-1)^n(n+3)(n+2)(n+1)}{2\cdot 3}x^n\end{aligned}$$

So $a_n = \frac{(-1)^n(n+3)(n+2)(n+1)}{2\cdot 3}$. Then the radius of convergence is

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)(n+2)(n+3)}{(n+2)(n+3)(n+4)} = 1$$

3.

$$\begin{aligned}\frac{1}{(2+x)^3} &= \frac{1}{2^3}\frac{1}{(1+\frac{x}{2})^3} = \sum_{n=0}^{\infty}\frac{1}{2^3}\binom{-3}{n}\left(\frac{x}{2}\right)^n \\ &= \frac{1}{2^3} + \sum_{n=1}^{\infty}\frac{1}{2^3}\frac{(-3)(-4)\cdots(-3-n+1)}{2^n n!}x^n \\ &= \frac{1}{2^3} + \sum_{n=1}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2^{n+4}}x^n\end{aligned}$$

So $a_n = \frac{(-1)^n(n+1)(n+2)}{2^{n+4}}$. Then the radius of convergence is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)(n+2)2^{n+5}}{2^{n+4}(n+2)(n+3)} = 2$$

9.(a)

$$\begin{aligned}\frac{1}{\sqrt{1-x^2}} &= (1+(-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-x^2)^n \\ &= 1 + \sum_{n=1}^{\infty}\frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1}{n!}(-x^2)^n \\ &= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!}(-1)^n x^{2n} \\ &= 1 + \sum_{n=1}\infty\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!}x^{2n}\end{aligned}$$

(b) Since$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \sin^{-1}x + C$,

$$\begin{aligned}\sin^{-1}x &= \int\left(1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!}x^{2n}\right)\mathrm{d}x + C \\ &= x + \sum_{n=1}^{\infty}\int\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!}x^{2n}\mathrm{d}x + C \\ &= x + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{(2n+1)2^n n!}x^{2n+1} + C\end{aligned}$$

Let $x=0$, then $C=0$. So

$$\sin^{-1}x = x + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{(2n+1)2^n n!}x^{2n+1}$$

10.(a)

$$\begin{aligned}\frac{1}{\sqrt[4]{1+x}} &= \sum_{n=0}^{\infty}\binom{-\frac{1}{4}}{n}x^n \\ &= 1+\sum_{n=1}^{\infty}\frac{(-\frac{1}{4})(-\frac{1}{4}-1)\cdots(-\frac{1}{4}-n+1)}{n!}x^n \\ &= 1+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 5\cdot 9\cdot\cdots\cdot(4n-3)}{4^n n!}x^n\end{aligned}$$

(b) Let $f^{(k)}(x) = \left(\frac{1}{\sqrt[4]{1+x}}\right)^{(k)}$. Then

$$|f^{(k)}| = \frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4k-3)}{4^k}\frac{1}{(1+x)^{1/4+n}} \leq \frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4k-3)}{4^k}\frac{1}{0.9^{1/4+k}},$$

if $|x| \leq 0.1$. So, when $k = 3$, $|f^{(3)}(x)| < 0.991$. Hence the remainder $|R_2(0.1)| < \frac{0.991}{3!}0.1^3 < 0.001$. Therefore we need to calculate the first three terms so as to estimate $1/\sqrt[4]{1.1}$, i.e.,

$$\frac{1}{\sqrt[4]{1.1}} = 1 - \frac{1}{4}0.1 + \frac{1\cdot 5}{4^2\cdot 2}0.1^2 = 0.977.$$

13.(a)

$$\begin{aligned}f(x) &= \sqrt{1+x^2} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^{2n} \\ &= 1+\sum_{n=1}^{\infty}\frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}x^{2n} \\ &= 1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot 1\cdot 3\cdot\cdots\cdot(2n-3)}{2^n n!}x^{2n}\end{aligned}$$

(b) The coefficient of Maclaurin series of $f(x)$ is $a_{2n} = \frac{(-1)^{n-1}\cdot 1\cdot 3\cdot\cdots\cdot(2n-3)}{2^n n!}$, $f^{(10)}(0) = a_{10}\cdot 10! = \frac{(-1)^{5-1}\cdot 1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}\cdot 10! = 99225$

14.(a)

$$\begin{aligned} f(x) &= \sqrt{1+x^2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{2n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} x^{2n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot 1\cdot 3\cdot\cdots\cdot(2n-3)}{2^n n!} x^{2n} \end{aligned}$$

(b) The coefficient of Maclaurin series of $f(x)$ is $a_{3n} = \frac{(-1)^{n-1}\cdot 1\cdot 3\cdot\cdots\cdot(2n-3)}{2^n n!}$, $f^{(9)}(0) = a_9 \cdot 9! = \frac{(-1)^{3-1}\cdot 1\cdot 3}{2^3\cdot 3!} \cdot 9! = 22680$