Answers to the MAT127 Homework No.6

Chapter 8 Section 6 Problem 3-6, 9, 10, 17, 18-20, 30

3.

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

The series converges when $|x| < 1$, so the the interval of convergence is $(-1, 1)$.

4.

$$f(x) = 3 \cdot \frac{1}{1-x^4} = \sum_{n=0}^{\infty} 3 \cdot (x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}$$

The series converges when $x^4 < 1$, so the interval of convergence is $(-1, 1)$.

5.

$$f(x) = \sum_{n=0}^{\infty}(x^3)^n = \sum_{n=0}^{\infty} x^{3n}$$

The series converges when $|x^3| < 1$, so the interval of convergence is $(-1, 1)$.

6.

$$f(x) = \frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-1)^n 9^n x^{2n}$$

The series converges when $|9x^2| < 1$, i.e. $|x| < 1/3$, so the interval of convergence is $(-1/3, 1/3)$.

9.

$$f(x) = \frac{x}{9} \cdot \frac{1}{1-(-(x/3)^2)} = \frac{x}{9}\sum_{n=0}^{\infty}(-1)^n \left(\frac{x}{3}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3}\left(\frac{x}{3}\right)^{2n+1}$$

The series converges when $(x/3)^2 < 1$, i.e. $|x| < 3$, so the interval of convergence is $(-3, 3)$.

10.

$$f(x) = \frac{x^2}{a^3} \cdot \frac{1}{1-(x^3/a^3)} = \frac{x^2}{a^3}\sum_{n=0}^{\infty} \frac{x^{3n}}{a^{3n}} = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{a^{3n+3}}$$

The series converges when $|x^3/a^3| < 1$, i.e $|x| < |a|$, so the interval of convergence is $(-|a|, |a|)$.

17.

$$\begin{aligned}
\ln(3+x) &= \int \frac{1}{3+x}\mathrm{d}x + C \\
&= \int \left(\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}x^n\right)\mathrm{d}x + C \\
&= \frac{1}{3}\sum_{n=0}^{\infty}\int \frac{(-1)^n}{3^n}x^n\mathrm{d}x + C \\
&= \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}\frac{x^{n+1}}{n+1} + C
\end{aligned}$$

Let $x = 0$, then $\ln 3 = C$. So

$$\ln(3+x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}\frac{x^{n+1}}{n+1} + \ln 3$$

18.

$$f(x) = \frac{1}{25}\cdot\frac{1}{1-(-x^2/5^2)} = \frac{1}{25}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{5^{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^2n}{5^{2n+2}}x^2n$$

19.

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\
&= \int \frac{1}{1+x}\mathrm{d}x + \int \frac{1}{1-x}\mathrm{d}x + C \\
&= 2\int \frac{1}{1-x^2}\mathrm{d}x + C \\
&= 2\int \sum_{n=0}^{\infty} x^{2n}\mathrm{d}x + C \\
&= 2\sum_{n=0}^{\infty}\int x^{2n}\mathrm{d}x + C \\
&= \sum_{n=0}^{\infty}\frac{2x^{2n+1}}{2n+1} + C
\end{aligned}$$

Let $x = 0$, then $C = 0$. So

$$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty}\frac{2x^{2n+1}}{2n+1}.$$

20. Since $(\tan^{-1}(2x))' = \frac{2}{1+4x^2}$,

$$\begin{aligned}\tan^{-1}(2x) &= \int \frac{2}{1+4x^2}\mathrm{d}x + C \\ &= 2\sum_{n=0}^{\infty}\int (-1)^n 4^n x^{2n}\mathrm{d}x + C \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{2n+1}x^{2n+1} + C\end{aligned}$$

Let $x = 0$, then $C = 0$. So

$$\tan^{-1}(2x) = \sum_{n=0}^{\infty}(-1)^n \frac{4^n}{2n+1}x^{2n+1}$$

30.

$$f''(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(x^{2n})''}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n)!}$$

So$f''(x) + f(x) = 0$.