Answers for HW 5

3. $a_n = \frac{x^n}{\sqrt{n}}$. When $x \neq 0$, $lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = lim_{n\to\infty}|x|\frac{\sqrt{n}}{\sqrt{n+1}} = |x|$.So $R = 1$ When $x = 1$, $p = \frac{1}{2} < 1$ means it diverges. When $x = -1$, by the Alternating Series Test (AST), it is convergent. Finally, the convergent interval is $[-1, 1)$.

7.$lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = 0$. So $R = \infty$.

8.$lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \frac{|x|}{3}$. When $x = 3$, it is divergent and when $x = -3$, by AST, it is convergent. So $[-3, 3)$.

9.$lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = 2|x|$.When $x = -\frac{1}{2}$, it is divergent and when $x = \frac{1}{2}$, by AST, it is convergent. So $(-\frac{1}{2}, \frac{1}{2}]$.

10.$lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \frac{|x|}{5}$.When $x = \pm 5$, it is convergent. But when $|x| > 5$, by l'Hospital's Rule, we have:(fixed x)

$$lim_{n\to\infty}\frac{e^{n\ln\frac{|x|}{5}}}{n^5} = lim_{n\to\infty}\frac{\ln\frac{|x|}{5}e^{n\ln\frac{|x|}{5}}}{5n^4} = \cdots = lim_{n\to\infty}\frac{(\ln\frac{|x|}{5})^5 e^{n\ln\frac{|x|}{5}}}{5!} = \infty$$

i.e. $|a_n| \nrightarrow 0$. So $[-5, 5]$.

11.$lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \frac{|x|}{4}$. (One may use l'Hospital Rule here). When $x = -4$, it is divergent (integral test) and when $x = 4$, by AST, it is convergent. So $(-4, 4]$.

12.$lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = 0$. So $R = \infty$.

18.$lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = 0$.. So $R = \infty$.

19.(a). yes. Since $\sum_{n=0}^{\infty} c_n 4^n$ is convergent. We have $lim_{n\to\infty}c_n 4^n = 0$. So after some large N, $|c_n 4^n| < 1$. Hence $|c_n(-2)^n| < 2^{-n}$. Now we get:

$$\sum_{n=0}^{\infty}|c_n(-2)^n| < \sum_{n=0}^{N}|c_n 2^n| + 2^{-N},$$

which means it is absolutely convergent.

Remark: This means if a series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is divergent at some point, it won't converger any more for further points from the center c.

(b) Let c_n be $\frac{1}{(-4)^n \cdot n}$. Then the answer is no generally.

25. $f(x) = (1+2x)\sum_{n=0}^{\infty} x^{2n}$. Easy to get $R = 1$. Since $(\pm 1)^{2n} \nrightarrow 0$, the convergent interval is $(-1, 1)$ and $f(x) = \frac{1+2x}{1-x^2}$.