

Answers to the MAT127 Homework No.12

Chapter 7 Section 4 Problem 2-4,7,10 & Section 5 Problem 2,5,7,8,11,13,14,15

Section 7.4

2.(a) Let $P(t)$ be the population of bacteria w.r.t. time t in the unit of hour, and let k be the relative growth rate in the unit of hour^{-1} . One bacterium divides to two in 20 minutes, i.e. $1/3$ hour, hence $P(0) = 1$, $P(1/3) = 2$. $P(t)$ satisfies the following differential equation

$$\frac{dP}{dt} = kP.$$

So, $P(t) = Ce^{kt}$, where C is a constant. It follows that

$$1 = P(0) = Ce^{k \cdot 0} = C,$$

hence $C = 1$. And

$$2 = P(1/3) = e^{k \cdot 1/3},$$

so $k = 3 \log 2$.

(b) Here, we change $P(0) = 1$ to $P(0) = 60$. Then

$$60 = P(0) = C e^{k \cdot 0} = C, \quad \text{i.e. } C = 60.$$

So

$$P(t) = 60e^{t \cdot 3 \log 2} = 60 \cdot 2^{3t}.$$

(c) $P(8) = 60 \cdot 2^{3 \cdot 8} = 60 \cdot 2^{24} \approx 10^9$

(d) The growth rate after 8 hour is

$$\frac{dP(8)}{dt} = kP(8) = 3 \log 2 P(8) \approx 2.07 \times 10^9.$$

(e) Suppose at time T , the population reach 20,000. Then we have

$$20000 = P(T) = 60 \cdot 2^{3T}.$$

Take log in both side, we have

$$\log 20000 = \log 60 + 3T \log 2.$$

It follows that

$$T = \frac{\log 20000 - \log 60}{3 \log 2} \approx 2.79 \quad \text{hour}$$

3.(a) Let $P(t)$ be the population of bacteria w.r.t. time t in the unit of hour, and let k be the relative growth rate in the unit of hour^{-1} . Then, $P(0) = 100$, $P(1) = 420$. $P(t)$ satisfies the following differential equation

$$\frac{dP}{dt} = kP.$$

So, $P(t) = Ce^{kt}$, where C is a constant. It follows that

$$100 = P(0) = Ce^{k \cdot 0} = C,$$

hence $C = 100$. And

$$420 = P(1) = 100e^{k \cdot 1},$$

so $k = \log(21/5)$. Therefore, $P(t) = 100e^{t \log(21/5)} = 100 \cdot (21/5)^t = 4.2^t$.

(b)

$$P(3) = 100 \cdot 4.2^3 \approx 7409$$

(c)

$$\frac{dP(3)}{dt} = kP(3) = k = \log(21/5)P(3) = 10632 \quad \text{hour}^{-1}.$$

(d)

$$10000 = P(T) = 100 \cdot 4.2^T.$$

Hence,

$$T = \log 100 / \log 4.2 \approx 3.21 \quad \text{hour}.$$

4.(a) The meaning of P , t , and k is the same with in previous problem. Hence we have

$$P(t) = P(0)e^{kt}.$$

Then

$$\frac{75000}{600} = \frac{P(8)}{P(2)} = \frac{P(0)e^{8k}}{P(0)e^{2k}} = e^{6k}.$$

So, $125 = e^{6k}$, i.e. $k = \log 125/6 = \log 5/2$. Plug the value of k into the following equation

$$600 = P(2) = P(0)e^{2t},$$

we get $P(0) = 600/5 = 120$, which is the initial population.

(b)

$$P(t) = P(0)e^{kt} = 120e^{t \log 5/2} = 120 \cdot 5^{\frac{t}{2}}.$$

(c)

$$P(5) = 120 \cdot 5^{\frac{5}{2}} \approx 6708$$

(d)

$$\frac{dP(5)}{dt} = kP(5) \approx 5398 \quad \text{hour}^{-1}$$

(e)

$$200,000 = P(T) = 120 \cdot 5^{\frac{T}{2}}.$$

Hence

$$T = \frac{2 \log(200000/120)}{\log 5} \approx 9.21 \text{ hour.}$$

7.(a) For convenience, let us denote $[N_2O_5]$ by R . Then the differential equation becomes

$$-\frac{dR}{dt} = 0.0005R.$$

So, $R(t) = Ce^{-0.0005t}$.(b) Suppose at time T , we get 90% of the original value. Then

$$0.9C = P(T) = Ce^{-0.0005T}$$

Hence

$$T = \frac{\log 0.9}{-0.0005} \approx 211s.$$

10.(a) Assume at time t , the percentage of remaining sample to the original one is $m(t)$. Obviously, $m(0) = 1$ The relative rate of decaying is k . Then the differential equation satisfied by $m(t)$ is

$$\frac{dm}{dt} = km.$$

Hence,

$$m(t) = m(0)e^{kt} = e^{kt}.$$

And after 1 year, the remaining is 94.5%, i.e. $m(1) = 94.5\%$.

$$0.945 = m(1) = e^k,$$

So, $k = \log 0.945$. Suppose half-life of tritium-3 is T . In other words, at time T , the remaining is 50%. Therefore,

$$0.5 = m(T) = e^{T \log 0.945}$$

Then

$$T = \frac{\log 0.5}{\log 0.945} \approx 14.7 \text{ year.}$$

(b) Suppose at time T' , the sample decays to 20%. Then

$$0.2 = m(T') = e^{T' \log 0.945}$$

Therefore

$$T' = \frac{\log 0.2}{\log 0.945} \approx 28.4 \text{ year.}$$

Section 7.5

Note: we omit all problems need to draw pictures.

2.(a) Let P , k , K and t have the same meaning as in your textbook. Then $K = 6000$ and $k = 0.0015$, hence the differential equation is

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K} \right) = 0.0015P \left(1 - \frac{P}{6000} \right).$$

(e) The solution to the differential equation is

$$P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{6000}{1 + Ae^{-0.0015t}}.$$

Let $t = 0$, then

$$1000 = P(0) = \frac{6000}{1 + A}.$$

So, $A = 5$. Therefore

$$P(t) = \frac{6000}{1 + 5e^{-0.0015t}}.$$

After 50 years, the population is

$$P(50) = \frac{6000}{1 + 5e^{-0.0015 \cdot 50}} \approx 1064.$$

5.(a) The increasing rate in 1990s, which is the difference of the birth rate and the death rate, is approximately 20 million per year. By the hint in the problem, we can estimate k by

$$k = \frac{1}{P} \frac{dP}{dt} \approx \frac{1}{5.3} \cdot 20 \cdot 10^{-3} = \frac{1}{265} \approx 0.0038$$

And $K = 100$ billion, the differential equation is

$$\frac{dP}{dt} = \frac{1}{265} P \left(1 - \frac{P}{100} \right).$$

(b) The solution to the differential equation is

$$P(t) = \frac{K}{1 + Ae^{-kt}}.$$

Let $P(0)$ be the population in 1990. Then

$$5.3 = P(0) = \frac{100}{1 + A},$$

hence, $A = 100/5.3 - 1 = 17.9$. The population in year 2000 is

$$P(10) = \frac{100}{1 + 17.9e^{-0.0038 \cdot 10}} \approx 5.48 \quad \text{billion}$$

(c) The Population in year 2100 is $P(110) \approx 7.82$ billion, in 2500 is $P(510) \approx 27.95$ billion. The calculation is similar to the last step in (b).

(d) The method is the same as in (a) and (b). The answers are 5.48, 7.61, 22.41, units are billion.

7.(a) In one hand, the rate of spread is $\frac{dy}{dt}$, in the other hand, the rate is proportional to the product of y and $1 - y$. Let k be the proportional constant, then we have

$$\frac{dy}{dt} = ky(1 - y).$$

(b) The equation is a Logistic equation with $K = 1$. Hence the solution is

$$y(t) = \frac{1}{1 + Ae^{-kt}}.$$

(c) $y(0) = 80/1000 = 0.08$, so

$$0.08 = \frac{1}{1 + A}.$$

Hence $A = 11.5$. And

$$0.5 = y(4) = \frac{1}{1 + 11.5e^{-k \cdot 4}},$$

so, $k = 0.437$. Suppose at time T , 90% of the residences have heard the rumor. Then we have

$$0.9 = y(T) = \frac{1}{1 + 11.5e^{-kT}}.$$

Hence $T = 10.6$, i.e. at about 6 : 36pm, 90% of the population have heard the rumor.

8.(a) $P(0) = 400$, $P(1) = 3 \cdot 400 = 1200$, and $K = 10000$. Then by standard procedure, we get $A = 24$ and $k = 1.18$. Hence the solution is

$$P(t) = \frac{10000}{1 + 24e^{-1.18t}}.$$

(b) Suppose that at time T the population increase to 5000. Then

$$5000 = P(T) = \frac{10000}{1 + 24e^{-1.18T}}.$$

Hence $T \approx 2.70$ year.

11.(a) The term -15 means the constant rate of death of fish for some reason.

(c) The equilibrium solutions satisfies

$$0 = \frac{dP}{dt} = 0.08P \left(1 - \frac{P}{1000} \right) - 15$$

Solve this equation, we get $P \equiv 750$ or $P \equiv 250$.

(d) See the answers in your textbook.

(e) By factoring the right hand side of the differential equation, we have (note that we already know two roots of this quadratic polynomial)

$$k(P - 250)(P - 750), \text{ where } k = 0.08/1000.$$

Imitating the method in your textbook, we integrate the differential equation

$$\int \frac{dP}{(P - 250)(P - 750)} = \int k dt$$

Note that

$$\frac{1}{(P - 250)(P - 750)} = \frac{1}{500} \left(\frac{1}{P - 750} - \frac{1}{250} \right).$$

Then, the left hand side is

$$\begin{aligned} \int \frac{1}{500} \left(\frac{1}{P - 750} - \frac{1}{250} \right) &= \int k dt \\ \frac{1}{500} (\ln |P - 250| - \ln |P - 750|) &= kt + C \\ \ln \left| \frac{P - 250}{P - 750} \right| &= k't + C' \quad \text{where } k' = 500k = 0.04 \\ \frac{P - 250}{P - 750} &= Ae^{k't} \end{aligned}$$

Solving above equation, we get

$$1 + \frac{500}{P - 750} = Ae^{k't} \quad \Rightarrow \quad P = 750 + \frac{500}{Ae^{k't} - 1}$$

so

$$P = \frac{250 - 750e^{0.04t}}{1 - Ae^{0.04t}}.$$

Plug $P(0) = 200$ or $P(0) = 300$ in, we get $A = 1/11$ or $A = -1/9$.

13.(a) Observe the right hand side of the equation, we find that when $m < P < K$, $\frac{dP}{dt} > 0$, and when $0 < P < m$, $\frac{dP}{dt} < 0$

(c) Note that $P(1 - m/P) = (P - m)$. We get a similar equation to the one in Problem 11. C.f. the answers in your textbook.

(d) Note the last term in the numerator goes to $-\infty$ when $t \rightarrow \infty$ provided $P_0 < m$.

14.(a) Integrating the equation, we get

$$\begin{aligned} \int \frac{dP}{P \ln \frac{K}{P}} &= \int c dt \\ \int \frac{d \ln P}{\ln K - \ln P} &= ct + C \\ -\ln |\ln K - \ln P| &= ct + C \\ \ln K - \ln P &= Ae^{-ct} \\ P &= Ke^{-A \exp(-ct)} \end{aligned}$$

(b) Suppose $c > 0$,

$$\lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} Ke^{-A \exp(-ct)} = Ke^{-A \lim_{t \rightarrow \infty} \exp(-ct)} = Ke^0 = K.$$

(d) It suffices to show that $f(P) = \ln(K/P)P$ achieves maximal at $P = K/e$. In fact,

$$f'(P) = \ln K - \ln P - 1 \quad f''(P) = -\frac{1}{P}.$$

So, $f'(K/e) = 0$ and $f''(K/e) = -\frac{1}{K/e} < 0$. It follows that this point is maximal.