Answers for HW 1

9.

$$\lim_{n\to\infty}\frac{3+5n^2}{n+n^2}=\lim_{n\to\infty}\frac{\frac{3}{n^2}+5}{\frac{1}{n}+1}=\frac{0+5}{0+1}=5.$$

10.

$$\lim_{n\to\infty}\frac{n+1}{3n-1}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{3-\frac{1}{n}}=\frac{1+0}{3-0}=\frac{1}{3}.$$

11.

$$\lim_{n\to\infty}\frac{2^n}{3^{n+1}}=\lim_{n\to\infty}\frac{1}{3}(\frac{2}{3})^n=0.$$

12.

$$\lim_{n\to\infty}\frac{1}{\frac{1}{\sqrt{n}}+1}=\frac{1}{0+1}=1.$$

19. Consider $f(x)=x^2e^{-x}=\frac{x^2}{e^x}$. By l'Hospital's Rule twice,

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0.$$

So we get $\lim_{n\to\infty}a_n=\lim f(n)=0$.

20. Since $\arctan(x)$ is increasing and a one to one continuous map from $(-\infty,+\infty)$ to $(-\frac{\pi}{2},+\frac{\pi}{2})$, $a_n=\arctan(2n)$ has the limit $\frac{\pi}{2}$.

21. $0\leqq a_n=\frac{cos^2(n)}{2^n}\leqq\frac{1}{2^n}\Rightarrow\lim_{n\to\infty}a_n=0$.

22. $a_{2n}=2n\ and\ a_{2n-1}=-(2n-1)$, so it is divergent.

23. Since $\lim_{x\to\infty}(1+\frac{1}{x})^x=e^x, \lim a_n=\lim((1+\frac{2}{n})^{\frac{n}{2}})^2=e^2$.

24. $0\leqq|a_n|\leqq\frac{1}{1+\sqrt{n}}, so\ \lim_{n\to\infty}a_n=0$.

25. If it converges, then it will has a finite limit. Thus a_n-a_{n+1} should be very small when n is sufficiently large, i.e., we would have $(\lim(a_n-a_{n+1})=0)$. However, for any n in N, there exists some m bigger than n s.t. $a_m-a_{m+1}=-1$. So it diverges.

26. Consider $f(x)=\frac{(\ln\ x)^2}{x}$. By l' Hospital's Rule twice,
$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\frac{2\ln\ x}{x}}{1}=\lim_{x\to\infty}\frac{2}{x}=0.So\ \lim a_n=0$.

27. By the meaning of continuity of $\ln(x)$, $a_n=\ln(\frac{2n^2+1}{n^2+1})$, so $\lim a_n=\ln\ 2$.

31. Simil to 27. $\lim a_n=\arctan 1=\frac{\pi}{4}$.

32. Since $|\sin(n)|$ is bounded by 1 and $\sqrt{n}$ goes to infinity, the limit is 0.

33. When $n > 3$,

$$0 \leq a_n \leq \frac{n^3}{n(n-1)(n-2)(n-3)} = \frac{1}{n}(1-\frac{1}{n})^{-1}(1-\frac{2}{n})^{-1}(1-\frac{3}{n})^{-1}.$$

Thus the limit is 0 by the squeeze principle.

36. Note that a_n is always an integer. A famous conjecture, Collatz conjecture (also called Kakutani, Shizuo conjecture), is that no matter which positive interger a_1 is, there will be some finite n s.t. $a_n = 1$. For more information, go to `http://en.wikipedia.org/wiki/Collatz_conjecture`.

42. $a_n = \frac{2n-3}{3n+4} = \frac{2}{3} + \frac{-3-\frac{2}{3}\times 4}{3n+4}$, so it is increasing to 2/3 and bounded.

46. First to prove $a_n \leqq 2$ by induction, then $2a_{n+1} \geqq a_{n+1}^2 \geqq a_n + a_n$. Hence it is increasing. So the limit b exists. Take limit to both sides of the equation to get $b^2 = 2 + b \Rightarrow b = 2 \; or \; -1$. So the limit is 2.