Quiz 4 - Solutions

Question 1. The integral $\int_0^1 \frac{dx}{2x-1}$ is improper because $\frac{1}{2x-1}$ is not defined at $\frac{1}{2}$.

Question 2. Notice that $e^{2x}+3 > e^{2x}$, so $\frac{1}{e^{2x}+3} < \frac{1}{e^{2x}}$. We then obtain:

$$\int_0^\infty \frac{e^x}{e^{2x}+3}dx < \int_0^\infty \frac{e^x}{e^{2x}}dx = \int_0^\infty e^{-x}dx$$

This last integral is easily computed:

$$\int_0^\infty e^{-x}dx = \lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty} -e^{-x}\Big|_0^t = -\lim_{t\to\infty}(e^{-t}-1) = 1$$

So the original integral is convergent by the comparison test. Now, for computing it, do the substitution $u = e^x$, so $du = e^x dx$. Then:

$$\int_0^t \frac{e^x}{e^{2x}+3}dx = \int_0^t \frac{e^x}{(e^x)^2+3}dx = \int_1^{e^t} \frac{du}{u^2+3} = \frac{1}{3}\int_1^{e^t} \frac{du}{\frac{u^2}{3}+1}$$

Now write $\frac{u^2}{3}$ as $\left(\frac{u}{\sqrt{3}}\right)^2$, make the substitution $\frac{u}{\sqrt{3}} = v$, so that $du = \sqrt{3}dv$ and obtain:

$$\frac{1}{3}\int_1^{e^t} \frac{du}{\frac{u^2}{3}+1} = \frac{\sqrt{3}}{3}\int_{\frac{1}{\sqrt{3}}}^{\frac{e^t}{\sqrt{3}}} \frac{dv}{v^2+1} = \frac{\sqrt{3}}{3}\arctan v\Big|_{\frac{1}{\sqrt{3}}}^{\frac{e^t}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}}{3}(\arctan\frac{e^t}{\sqrt{3}} - \arctan\frac{1}{\sqrt{3}})$$

Recall that $\arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. Now we need to take the limit $t \to \infty$:

$$\int_0^\infty \frac{e^x}{e^{2x}+3}dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{e^{2x}+3}dx = \lim_{t\to\infty}\frac{\sqrt{3}}{3}(\arctan\frac{e^t}{\sqrt{3}} - \frac{\pi}{6})$$

$$= \frac{\sqrt{3}}{3}(\lim_{t\to\infty}\arctan\frac{e^t}{\sqrt{3}} - \frac{\pi}{6}) = \frac{\sqrt{3}}{3}(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{\sqrt{3}\pi}{9}$$

Here we used the following: $\lim_{t\to\infty}\frac{e^t}{\sqrt{3}} = \infty$. Now, recall that *the tangent of some angle θ approaches infinity when the angle approaches* $\frac{\pi}{2}$, therefore $\lim_{t\to\infty}\arctan\frac{e^t}{\sqrt{3}} = \frac{\pi}{2}$.