Some advanced techniques on PDE

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Abstract

These are notes from (part of) a course taught at *State University of New York at Stony Brook* on Fall 06 by professor Marcus Khuri. Since these are notes from a course, there may be some small gaps or some repetitions. Comments may be sent to disconzi@math.sunysb.edu.

In what follows we use multi-index notation and summation convention. When we do estimates we always absorb all constants into a single one and keep the same letter for such new constant. We do not spend much time introducing standard concepts, although sometimes they are defined for the convenience of the reader (e.g., the definition of Sobolev spaces).

1 Negative Sobolev norms and a necessary and sufficient condition for existence of solutions.

Let $\Omega \subset \mathbb{R}^n$ be open. For any $s = 0, 1, \dots$ define the inner product:

$$(u, v)_s = \sum_{|\alpha| \le s} \int_\Omega \partial^\alpha u \overline{\partial^\alpha y}$$

Notice that $(\cdot, \cdot)_0$ is the standard L^2-inner product, which we write simply as $(\cdot, \cdot)$ when no confusion can arise.

Remark 1. *We introduce the general definition, but in most cases we shall be dealing with real valued functions.*

Definition 1. *The completion of $C^\infty(\Omega)$ w.r.t. the norm $\| \cdot \|_s$ given by the inner product $(\cdot, \cdot)_s$ is the s^{th} Sobolev space $H^s(\Omega)$.*

Even though $H^s(\Omega)$ is naturally a Hilbert space we will be treating it merely as a Banach space, at least for the moment. We first want to characterize the duals of Sobolev spaces; this is the most important dual for applications in PDE.

For any $v \in L^2(\Omega)$ define $F_v : H^s(\Omega) \to \mathbb{C}$ by $F_v(u) = (u, v)$. It follows that $|F_v(u)| \leq \| u \|_0 \| v \|_0 \leq \| u \|_s \| v \|_0$, so F_v is a bounded linear functional.

Definition 2. *Define the negative Sobolev norm $\| \cdot \|_{-s}$ by:*

$$\| v \|_{-s} = \| F_v \| = \sup_{u \in H^s(\Omega)} \frac{|(u,v)|}{\| u \|_s}$$

(clearly $\| \cdot \|_{-s}$ is indeed a norm). Then define $H^{-s}(\Omega)$ as the completion of $L^2(\Omega)$ w.r.t. the norm $\| \cdot \|_{-s}$.

Remark 2. *Since $H^{-s}(\Omega)$ is a completion, it is a Banach space. Later on we shall endow it with an inner product, making it into a Hilbert space.*

Theorem 1. *$H^s(\Omega)^* = H^{-s}(\Omega)$.*

Remark 3. *Of course, since $H^s(\Omega)$ is a Hilbert space, we naturally have $H^s(\Omega)^* = H^s(\Omega)$, so the claim apparently is $H^s(\Omega) = H^{-s}(\Omega)$. However, as it would be clear below, we are not simply interested in the existence of a Banach space isomorphism. Our goal is is actully show that a particular map alpha (to be defined below) happens to be a Banach space isomorphism, so that we can use this map to put a Hilbert space structure on $H^{-s}(\Omega)$.*

Proof. Put $\Lambda = \{F_v \mid v \in L^2(\Omega)\}$, where F_v is as defined above. We claim that Λ is dense in $H^s(\Omega)^*$. Indeed, if this is not the case, take $F \in H^s(\Omega)^* - \overline{\Lambda}$ and then, by (one of the corollaries of) Hahn-Banach there exists a $\ell \in H^s(\Omega)^{**}$ such that $\ell(F) \neq 0$ and $\ell|_{\overline{\Lambda}} \equiv 0$. By reflexivity, there exists $f_\ell \in H^s(\Omega)$ such that $\ell(\tilde{F}) = \tilde{F}(f_\ell)$ for every $\tilde{F} \in H^s(\Omega)^*$. But then $F_v(f_\ell) = \ell(F_v) = 0$ for all $F_v \in \Lambda$. Thus $0 = F_v(f_\ell) = (f_\ell, v)$ for all $v \in L^2(\Omega)$, what implies $f_\ell = 0$, contradiction. So Λ is dense and $\overline{\Lambda} = H^s(\Omega)^*$.

Now define $\alpha : H^{-s}(\Omega) \to H^s(\Omega)^*$ by $\alpha(v) = \lim_n F_{v_n}$, where $v = \lim_n v_n$, $v_n \in L^2(\Omega)$ and $\lim_n v_n$ means limit in $H^{-s}(\Omega)$, and $\lim_n F_{v_n}$ is limit in the operator norm.

First we check that α is well defined. If $v = \lim_n v_n = \lim w_n$ then (since $\| v_n \|_{-s} = \| F_n \|$) we get $\| F_{v_n} - F_{w_n} \| = \| F_{v_n - w_n} \| = \| v_n - w_n \|_{-s} \to 0$ as $n \to \infty$, showing that α is well defined, i.e., $\alpha(v) = \lim_n F_{v_n} = \lim_n F_{w_n}$.

Now we show that α is one-to-one and onto. Assume $\alpha(v) = \alpha(w)$. Then $0 = \lim_n \| F_{v_n} - F_{w_n} \| = \lim_n \| v_n - w_n \|_{-s}$ and so $v = w$, showing injectivity. By the above denseness α is also onto.

Finally, notice that $\| \alpha(v) \| = \| F_v \| = \| v \|_{-s}$, and so α is an isometric isomorphism (i.e., a Banach space isomorphism). $\square$

Using the above theorem we can extend the notation $(u, v)_0$ to pairs where $v \in H^{-s}(\Omega)$ and $u \in H^s(\Omega)$ by letting $(u, v)_0$ denote the action of the functional v on u. More explicitely, following the constrction of theorem 1 we can write $(u, v)_0 = F_v(u)$. Of course, $F_v(u)$ *is* the L^2-inner product if $v \in L^2(\Omega)$, but $F_v(u) = \alpha(v)(u)$ if $v \in H^{-s}(\Omega) - L^2(\Omega)$ (we do have $L^2(\Omega) \subset H^{-s}(\Omega)$, see remark below), where α is the map constructed in theorem 1 given by $\alpha(v) = \lim F_{v_n}$, with $v_n \to v$ in $H^{-s}(\Omega)$. Now if $F_{v_n} \to F_v \equiv \alpha(v)$ in the operator norm topology then $F_{v_n} \equiv (v_n, u)_0 \to F_v(u)$. Indeed

$$F_v(u) = F_v(u) - F_{v_n}(u) + F_{v_n}(u) = F_{v-v_n}(u) + (v_n, u)_0$$

but $|F_{v-v_n}(u)| \leq \| F_{v-v_n} \| \| u \|_s = \| v - v_n \|_{-s} \| u \|_s \to 0$ when $v_n \to v$ in $H^{-s}(\Omega)$. So when we extend the notation $(u, v)_0$ can also write $(u, v)_0 = \lim(v_n, u)_0$.

Remark 4. *We have the following facts: (i)* $\| v \|_{-s} \leq \| v \|_0$, *so* $H^s(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega)$ *(of course,* $H^s(\Omega) \subset L^2(\Omega)$ *is a standard fact).*
(ii) Generalized Cauchy-Schwartz inquality $|(u, v)_0| \leq \| u \|_s \| v \|_{-s}$ *(recall that we extended the notation* $(u, v)_0$*).*

Putting all of above together we obtain:

Theorem 2. *Every bounded linear funcional on* $H^s(\Omega)$ *can be represented by* F_v, *for some* $v \in H^{-s}(\Omega)$, *i.e., if* $F \in H^s(\Omega)^*$ *then* $F(u) = (u, v)_0$ *for all* $u \in H^s(\Omega)$.

Definition 3. *Let* $\beta : H^s(\Omega)^* \to H^s(\Omega)$ *be given by the Riesz representation theorem. Then it is easly checked that the following defines an inner product on* $H^{-s}(\Omega)$*:*

$$(u, v)_{-s} = (\beta \circ \alpha(u), \beta \circ \alpha(v))_s$$

where α *is as in theorem 1.*

Notice that if $v_n \to v$ in $H^{-s}(\Omega)$ then $F_{v_n} \to F_v$ in the operator norm topology since $\| F_{v_n} - F_v \| = \| v_n - v \|_{-s} \to 0.$, i.e., $F_v(u) = (u,v) = \lim_n (u, v_n)$ since $|(u, v - v_n)| \leq \| u \|_s \| v - v_n \|_{-s} \to 0$. So

$$\sqrt{(v,v)_{-s}} = \sqrt{(\beta \circ \alpha(v), \beta \circ \alpha(v))_s} = \| \beta \circ \alpha(v), \beta \circ \alpha(v) \|_s =$$
$$= \| \alpha(v) \| = \| F_v \| = \sup_{u \in H^s(\Omega)} \frac{|(u,v)|}{\| u \|_s} = \| v \|_{-s}$$

where we used that β is an isometry. Hence $(\cdot,\cdot)_{-s}$ gives (generates) $\| \cdot \|_{-s}$. Because Hilbert spaces are reflexives we could conclude from the above that $H^{-s}(\Omega)^* = H^s(\Omega)^{**} = H^s(\Omega)$, but we want a specific form of this result.

Theorem 3. $H^{-s}(\Omega)^* = H^s(\Omega)$. *Moreover, any $G \in H^{-s}(\Omega)^*$ can be represented by a unique $u \in H^s(\Omega)$ through $G(v) = G_u(v) = (u,v)$ for all $v \in H^{-s}(\Omega)$.*

Proof. For $u \in H^s(\Omega)$ set $G_u(v) = (u,v)$, $v \in H^{-s}(\Omega)$. By generalized Cauchy-Schwartz inequatlity we have that $G_u \in H^{-s}(\Omega)^*$. We claim that $\| G_u \| = \| u \|_s$. First we have $|G_u(v)| \leq \| u \|_s \| v \|_{-s}$ for every $v \in H^{-s}(\Omega)$, so $\| G_u \| \leq \| u \|_s$. For the opposite inequality, consider

$$\| G_u \| = \sup_{v \in H^{-s}(\Omega)} \frac{|G_u(v)|}{\| v \|_{-s}} = \sup_{v \in H^{-s}(\Omega)} \frac{|(u,v)|}{\| v \|_{-s}}$$

For all $v \in H^{-s}(\Omega)$ we have, by theorem 2, $\frac{|(u,v)|}{\|v\|_{-s}} = \frac{|F_v(u)|}{\|F_v\|}$. Choose F such that $F(u) = \| u \|_s$, then $\| F \| = 1$ thus

$$\| G_u \| \geq \sup_{F \in H^s(\Omega)^*} \frac{|F(u)|}{\| F \|} \geq \| u \|_s$$

Let $\Lambda = \{G_u \mid u \in H^s(\Omega) \subset H^{-s}(\Omega)^*\}$. An argument similiar to that of theorem 1 shows that Λ is dense in $H^{-s}(\Omega)^*$. If a sequence $\{u_n\}$ converges to u in $H^s(\Omega)$ then, for any $v \in H^{-s}(\Omega)$, we have $G_{u_n}(v) = (u_n, v) \to (u,v) = G_u(v)$. Define $\gamma : H^s(\Omega) \to H^{-s}(\Omega)$ by $\gamma(u) = G_u$. Clearly γ is one-to-one and onto, and $\| \gamma(u) \| = \| G_u \| = \| u \|_s$, so γ is an isometric isomorphism. □

Definition 4. *Consider a linear PDE given by $Lu = f$ on Ω, $f \in H^t(\Omega)$, $t \in \mathbb{Z}$. We say that it has a weak solution $u \in H^s(\Omega)$, $s \in \mathbb{Z}$, if $(u, L^* v) = (f, v)$ for all $v \in C_c^\infty(\Omega)$, where L^* is the formal adjoint of L.*

Remark 5. *Notice that no boundary conditions are imposed on u since $v \in C_c^\infty(\Omega)$.*

Recall that if $Lu = a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu$ (which is the case for most applications) then $L^*u = a^{ij}u_{x^ix^j} + (a^{ij}_{x^j} - b^i)u_{x^i} + (c - b^i_{x^i} + a^{ij}_{x^ix^j})u$. The idea is that we can integrate by parts to get L^*.

Theorem 4. *A necessary and sufficient condition for $Lu = f$ to have a weak solution $u \in H^s(\Omega)$ for each $f \in H^t(\Omega)$ is that there exists a constant C such that*

$$\| v \|_{-t} \leq C \| L^*v \|_{-s}$$

for all $v \in C_c^\infty(\Omega)$.

Proof. Assume the estimate. Put $X = L^*C_c^\infty(\Omega)$ and consider $F : X \to \mathbb{R}$ given by $F(L^*v) = (f, v)$ (notice that this is well defined because the estimate says that if $\| L^*v \|_{-s} = 0$ then $\| v \|_{-t} = 0$). We claim that F is bounded on the subspace $X \subset H^{-s}(\Omega)$. Estimate

$$|F(L^*v)| \leq |(f, v)| \leq \| f \|_t \| v \|_{-t} \leq C \| f \|_t \| L^*v \|_{-s}$$

which shows boundness. By Hanh-Banach F extends to $\tilde{F}$ on all $H^{-s}(\Omega)$. Because $H^{-s}(\Omega)^* = H^s(\Omega)$, there exists a $u \in H^s(\Omega)$ such that $\tilde{F}(w) = (u, w)$ for all $w \in H^{-s}(\Omega)$. Now restrict w back to X to get

$$(u, L^*v) = \tilde{F}(L^*v) = F(L^*v) = (f, v)$$

What shows the existence.

Assume now that we can solve $Lu = f$ for all $f \in H^t(\Omega)$. Let $u \in H^s(\Omega)$ be a solution for a given $f \in H^t(\Omega)$. We have

$$|(f, v)| = |(u, L^*v)| \leq \| u \|_s \| L^*v \|_{-s}$$

The above inequality means that there exists a constant C_f (depending on f) such that

$$|(f, \frac{v}{\| L^*v \|_{-s}})| \leq C_f$$

By the identification $H^t(\Omega) = H^{-t}(\Omega)^*$, we have that $f \in H^t(\Omega)$ defines an element of $H^{-t}(\Omega)^*$, i.e., $(f, v) = G_f(v)$. On the other hand, by Riesz we have $G_f(v) = (\alpha(f), v)_{-t}$ for some $\alpha(f) \in H^{-t}(\Omega)$. Since both these identifications ($f \in H^t(\Omega) \mapsto G_f \in H^{-t}(\Omega)^*$ given by theorem 3 and $G_f \in H^{-t}(\Omega)^* \mapsto \alpha(f) \in H^{-t}(\Omega)$ given by Riesz) are isometries, we have

$$|(f, \frac{v}{\| L^* v \|_{-s}})| = |(\alpha(f), \frac{v}{\| L^* v \|_{-s}})_{-t}| \leq C_f$$

Therefore we have a familiy of bounded linear functionals $\beta_v \in H^{-t}(\Omega)^*$ given by $\beta_v(u) = (\alpha(f), \frac{v}{\|L^* v\|_{-s}})_{-t}$, where $u = \alpha(f)$ for some $f \in H^t(\Omega)$. This famility is pointwise bounded for all $v \in C_c^\infty(\Omega)$ and therefore by Banach-Steinhaus the family is uniformly bounded, i.e., $\| \beta_v \| \leq C$ for all $v \in C_c^\infty(\Omega)$. But

$$|\beta_v(u)| \leq \| \alpha(f) \|_{-t} \| \frac{v}{\| L^* v \|_{-s}} \|_{-t} \leq C_f$$

so $\| \beta \| \leq \| \frac{v}{\|L^* v\|_{-s}} \|_{-t}$ as $\| u \|_{-t} = \| \alpha(f) \|_{-t}$. Also, choosing $\alpha(f) = \frac{v}{\|L^* v\|_{-s}}$ gives

$$|\beta_v(\frac{v}{\| L^* v \|_{-s}})| = \| \frac{v}{\| L^* v \|_{-s}} \|_{-t} \quad \text{and then} \quad \| \beta_v \| \geq \| \frac{v}{\| L^* v \|_{-s}} \|_{-t}$$

Hence $\| \beta_v \| = \| \frac{v}{\|L^* v\|_{-s}} \|_{-t}$. Therefore the uniform bound gives

$$\| v \|_{-t} \leq C \| L^* v \|_{-s} \tag{1}$$

□

2 A PDE which is not locally solvable at the origin.

As an immediate application of the above theorem we shall give an example of a PDE which has no solution in any neighborhood of the origin. The result is remarkable once we take into account that such PDE is linear and has smooth coefficients. This example is due to Egorov (1993).

Definition 5. *Consider a linear differential operator L. We say that L is locally solvable at the origin if given $f \in C_c^\infty(\Omega_1)$, $\Omega_1 \subset \mathbb{R}^n$ containing the origin, there exists $u \in H^{-s}(\Omega_2)$ solving $Lu = f$ in the weak sense, for some $s \in \mathbb{N}$ and some $\Omega_2 \subset \Omega_1$.*

Consider the operator $Lu = u_{tt} - a^2(t)u_{xx} + b(t)u_x$, where $a, b \in C^\infty(\mathbb{R})$ (notice that this is a degenerate hyperbolic equation if a vanishes at some point). The goal is to construct a, b such that the necessary and sufficient condition of theorem 4 is violated. We need he following preliminary lemma which refines the necessary condition for this particular L.

Lemma 1. *Consider the above operator L. If $Lu = f$ always has a solution in fixed $\Omega \subset \mathbb{R}^2$, $0 \in \Omega$, then there exist a constant C and a $N \in \mathbb{N}$ such that*

$$\| v \|_0 \leq \| L^* v \|_N$$

for all $v \in C_c^\infty(\Omega)$.

Proof. By the theorem 4 there exist $s, t \in \mathbb{Z}$ and a constant C such that

$$\| v \|_s \leq C \| L^* v \|_t$$

for all $v \in C_c^\infty(\Omega)$. If $s \geq 0$ then choose $N > t$ and we are done because in this case $\| v \|_0 \leq \| v \|_s$. Otherwise, notice that $\partial_x^\alpha v \in C_c^\infty(\Omega)$ if $v \in C_c^\infty(\Omega)$, so $\| \partial_x^\alpha v \|_s \leq C \| \partial_x^\alpha L^* v \|_t$. Since $L^* v = v_{tt} - a^2(t)v_{xx} + b(t)v_x$ we have

$$\begin{gathered} \| v_{tt} \|_{s-1} \leq C(\| L^* v \|_{s-1} + \| v_{xx} \|_{s-1} + \| v_x \|_{s-1}) \leq \\ C(\| L^* v \|_{s-1} + \| v_x \|_s + \| v \|_s) \leq C \| L^* v \|_{t+1} \end{gathered}$$

where we used that we may assume $t \geq s$. Hence

$$\begin{gathered} \| v \|_{s+1} \leq C(\| v \|_s + \| v_{xx} \|_{s-1} + \| v_{tt} \|_{s-1}) \leq \\ C(\| L^* v \|_t + \| L^* v \|_{t+1} \leq C \| L^* v \|_{t+1} \end{gathered}$$

We may repeat these arguments to obtain

$$\| v \|_{s+2} \leq C \| L^* v \|_{t+2}, \ldots, \| v \|_{s+n} \leq C \| L^* v \|_{t+n}$$

Then chose n such that $s + n = 0$ and set $N = t + n$ □

Theorem 5. *Consider again the operaor L defined above. There exist $a, b \in C^\infty(\mathbb{R})$ such that given $f \in C^\infty(\Omega)$ with $\Omega \subset \mathbb{R}^2$ some neighborhood of the origin, $Lu = f$ has no solution $u \in H^{-s}(\Omega)$ for any $s \in \mathbb{N}$.*

Remark 6. *Notice that this does not quite say yet that L is not locally solvable.*

Proof. Define

$$a(t) = \begin{cases} e^{-t^2 - \sin^{-2}(\frac{1}{t})} & t > 0 \\ 0 & t \le 0 \end{cases}$$
$$b(t) = \begin{cases} 2\xi'(t) - a'(t) & t > 0 \\ 0 & t \le 0 \end{cases}$$

where $\xi = \sin^{-4}(\frac{1}{t})$. We left as an exercices to check that these functions are smooth. Notice that a oscillates very fast with the intervals $I_\mu = (\frac{1}{\pi(\mu+1)}, \frac{1}{\pi\mu})$. The idea is to violate the inequality of lemma 1 by constructing a sequence of functions $v_{\mu\lambda} \in C_c^\infty(I_\mu \times (-\frac{2}{\lambda}, \frac{2}{\lambda}))$ which makes he right-hand side of the necessary condition on the above lemma smaller than the left-hand side for large μ, λ.

Recall that $L^*v = v_{tt} - a^2(t)v_{xx} + b(t)v_x$. We look for approximate solutions to $L^*v = 0$ with $v \in C_c^\infty(I_\mu \times (-\frac{2}{\lambda}, \frac{2}{\lambda}))$. It turns out that this can be easily done with separation of variables if we make a simplifying coordinate change to get rid of the v_{xx} term. Namely, set $\bar{t} = t$ and $\bar{x} = x - \int_0^t a(t')dt'$. Clearly this is a smooth change of coordinates in a neighborhood of the origin. Then

$$\begin{aligned} v_t &= \frac{\partial v}{\partial \bar{t}}\frac{d\bar{t}}{dt} + \frac{\partial v}{\partial \bar{x}}\frac{d\bar{x}}{dx} = \frac{\partial v}{\partial \bar{t}} - a\frac{\partial v}{\partial \bar{x}} \\ v_{tt} &= \frac{\partial^2 v}{\partial \bar{t}^2} - 2a\frac{\partial^2 v}{\partial \bar{x}\partial \bar{t}} + a^2\frac{\partial^2 v}{\partial \bar{x}^2} - a'\frac{\partial v}{\partial \bar{x}} \\ v_x &= \frac{\partial v}{\partial \bar{x}}\frac{d\bar{x}}{dx} + \frac{\partial v}{\partial \bar{t}}\frac{d\bar{t}}{dx} = \frac{\partial v}{\partial \bar{x}} \\ v_{xx} &= \frac{\partial^2 v}{\partial \bar{x}^2} \end{aligned}$$

So in the new coordinates

$$\begin{aligned} L^*v &= \frac{\partial^2 v}{\partial \bar{t}^2} - 2a\frac{\partial^2 v}{\partial \bar{x}\partial \bar{t}} - (b + a')\frac{\partial v}{\partial \bar{x}} \\ &= \frac{\partial^2 v}{\partial \bar{t}^2} - 2a\frac{\partial^2 v}{\partial \bar{x}\partial \bar{t}} - 2\xi' a\frac{\partial v}{\partial \bar{x}} \end{aligned}$$

From now on we drop the bars on the new coordinates in order to simplify notation. We look for an approximiate solution of the form

$$v_{\mu\lambda}(t, x) = \sum_{i=1}^{n} \frac{1}{\lambda^i} z_i(t) w_i(\lambda x)$$

in $C_c^\infty(J_{\mu\lambda})$, where $J_{\mu\lambda} = I_\mu \times (-\frac{1}{\lambda}, \frac{1}{\lambda})$; that is $z_i \in C_c^\infty(I_\mu)$ and $w_i \in C_c^\infty(-1,1)$. To see how to choose z_i and w_i we calculate

$$L^* v_{\mu\lambda} = -\lambda 2aw_0'(z_0' + \xi' z_0) + [z_0'' w_0 - 2aw_1'(z_1' + \xi' z_1)] + \frac{1}{\lambda}[z_1'' w_1 - 2aw_2'(z_2' + \xi' z_2)] + \cdots + \frac{1}{\lambda^n} z_n'' w_n$$

It is clear that we should choose w_i so that $w_{i+1}' = w_i$. This can be accomplished by taking $w_n \in C_c^\infty(-1,1)$ and setting

$$w_i = \left(\frac{d}{d\tilde{x}}\right)^{n-i} w_n(\tilde{x}), \text{ where } \tilde{x} = \lambda x$$

Then inductively set

$$z_0(t) = e^{-\xi(t)}$$
$$z_{i+1}(t) = e^{-\xi(t)} \int_0^t \frac{z_i''(s)}{2a(s)} e^{\xi(s)} ds$$

Observe that each $z_i \in C_c^\infty(I_\mu)$ since $\exp(-\sin^{-4}(\frac{1}{t}))$ dominates $\frac{z_i''}{2a} e^\xi$ because this last expression is of order $o(t^{-\alpha} \exp(\beta \sin^{-2}(\frac{1}{t}))$ for every $\alpha, \beta > 0$.

With these choices of z_i, w_i we have that $v_{\mu\lambda} \in C_c^\infty(J_{\mu\lambda})$. Moreover

$$\| L^* v_{\mu\lambda} \|_{H^N(J_{\mu\lambda})} \leq C_{\mu n N} \lambda^{-N}$$

where $C_{\mu n N}$ is independent of λ. By choosing λ sufficiently large we also have

$$\| v_{\mu\lambda} \|^2_{L^2(J_{\mu\lambda})} \geq \int_{J_{\mu\lambda}} e^{-\xi(t)} w_0(\lambda x) dt dx - D_{\mu n} \lambda^{-1}$$

where $D_{\mu n}$ does not depend on λ nor N. But

$$\int_{\frac{1}{\pi(\mu+1)}}^{\frac{1}{\pi\mu}} e^{-2\sin^{-4}(\frac{1}{t})} dt = \int_{\pi(\mu+1)}^{\pi\mu} e^{-2\sin^{-4}(\tilde{t})} d\tilde{t}$$
$$\geq k^{-4} \int_0^\pi e^{-2\sin^{-4}(\tilde{t})} d\tilde{t} \quad \text{where } \tilde{t} = \frac{1}{t}$$

We also havve

$$\int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} w_0^2(\lambda x) dx = \int_{-1}^{1} w_0^2(\tilde{x}) \lambda^{-1} d\tilde{x} \geq \lambda^{-1} C_n \quad \text{where } \tilde{x} = \lambda x$$
$$\text{so that } \| v_{\mu\lambda} \|^2_{L^2(J_{\mu\lambda})} \geq E_{\mu n} \lambda^{-1}$$

It follows that for large λ and μ we can violate the inequality on lemma 1 for any given C and N. $\square$

Now we can refine this result in order to obtain that L is not locally solvable at the origin. This is an application of Baire's Category Theorem.

Corollary 1. *There exists $f \in C_c(\mathbb{R}^2)$ such that $Lu = f$ has no solution $u \in H^{-s}(\Omega)$ for any $s \in \mathbb{N}$ and any $\Omega \subset \mathbb{R}^2$ containing the origin.*

Proof. For a fixed Ω which contains the origin define

$$X(\Omega) = \{f \in C_c^\infty(\mathbb{R}^2) \mid Lu = f \text{ has a solution } u \in H^{-s}(\Omega) \text{ for some } s \in \mathbb{N}\}$$
$$X_s(\Omega) = \{f \in C_c^\infty(\mathbb{R}^2) \mid Lu = f \text{ has a solution } u \in H^{-s}(\Omega) \text{ such that } \parallel u \parallel_{-s} \leq |s| + 1\}$$

(throughout the proof the equality $Lu = f$ is to be understood in the week sense). Notice that $X(\Omega) = \cup_{s=-\infty}^{\infty} X_s(\Omega)$. We are going to show that for each s the space $X_s(\Omega)$ is nowhere dense. For this we first recall the topological properties of $X_s(\Omega)$. The topology on $X_s(\Omega)$ is that given due to the fact that $X_s(\Omega)$ is a Fréchet space and it is generated by open sets

$$\mathcal{U}_{\alpha_1,\ldots,\alpha_k;\epsilon} = \{x \in X_s(\Omega) \mid \parallel x \parallel_{\alpha_1} < \epsilon, \ldots, \parallel x \parallel_{\alpha_k} < \epsilon\}$$

where $\parallel \cdot \parallel_{\alpha_i}$ is the collection of semi-norms. This means that in this topology a sequence $\{x_n\}$ converges to an element x if and only if $\lim_n \parallel x_n - x \parallel_\ell = 0$ for every semi-norm $\parallel \cdot \parallel_\ell$. The collection of semi-norms on $X_s(\Omega)$ is given by the Sobolev norms $\parallel \cdot \parallel_r$, $r \in \mathbb{N}$.

First we claim that $X_s(\Omega)$ is closed. To see this, suppose that $\{f_j\} \subset X_s(\Omega)$ is such that $f_j \to f$. For each j there exists a u_j such that $Lu_j = f_j$ and $\parallel u \parallel_{-s} \leq |s| + 1$. Since bounded sequences on Hilbert spaces have weakly convergent subsequences, we have that u_{j_k} converges weakly in $H^{-s}(\Omega)$ to some u, so $(u_{j_k}, L^* v) \to (u, L^* v)$ for all $v \in C_c^\infty(\Omega)$; we also have $(f_{j_k}, v) \to (f, v)$ for all $v \in C_c^\infty(\Omega)$. Then $(u, L^* v) = (f, v)$ and so $Lu = f$ weakly. Hence$f \in X_s(\Omega)$ because $\parallel u \parallel_{-s} \leq |s| + 1$.

We know show that $X_s(\Omega)$ has empty interior. Let $f \in X_s(\Omega)$ be arbitrary. By the theorem, there exists $\hat{f} \in C_c^\infty(\Omega)$ such that $Lu = \hat{f}$ has no solution for any $s \in \mathbb{N}$. Now, if $f + t\hat{f} \in X_s(\Omega)$ then there exists $w \in H^{-s}(\Omega)$ with $Lw = f + t\hat{f}$. Also, since $f \in X_s(\Omega)$, there exists $z \in H^{-s}(\Omega)$ such that $Lz = f$. But then $tL(w - z) = t\hat{f}$, contradiction. Thus, since $f + t\hat{f} \to 0$ as $t \to 0$ and $f + t\hat{f} \notin X_s(\Omega)$ for any $t \neq 0$ we see that f can not be an interior point.

We conclude from the above that $X(\Omega) = \cup_{s=-\infty}^{\infty} X_s(\Omega)$ is of first category. Set

$$Y_i = \{f \in C_c^\infty(\mathbb{R}^2) \mid Lu = f \text{ has a solution } u \in H^{-s}(\Omega_i) \text{ for some } s \in \mathbb{N}\}$$

here, $\{\Omega_i\}_{i=1}^\infty$ is a countable basis of open sets containing the origing (for example, take all open disks containing the origin with rational radius and rational center). By above each Y_i is of first category. By Baire category theorem $C_c^\infty(\mathbb{R}^2) \neq \cup_{i=1}^\infty Y_i$. Therefore there exists a nonnull $f \in C_c^\infty(\mathbb{R}^2) - \cup_{i=1}^\infty Y_i$ and such a f has the desired property. $\square$

3 Techniques for solving boundary value problems.

We now give another application of theorem 4. Consider the BVP

$$\begin{cases} Lu = f \\ Bu|_{\partial\Omega} = 0 \end{cases}$$

where L is linear and Bu could be Dirichlet, Neumann, Robin or something different for a particular kind of equation

Definition 6. *Consider the boundary value problems:*

$$\begin{cases} Lu = f \\ Bu|_{\partial\Omega} = 0 \end{cases} (*) \qquad\qquad \begin{cases} L^*v = g \\ B^*v|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

and let

$$C_B^\infty(\Omega) = \{u \in C^\infty(\Omega) \mid Bu|_{\partial\Omega} = 0\}$$
$$C_{B^*}^\infty(\Omega) = \{v \in C^\infty(\Omega) \mid B^*v|_{\partial\Omega} = 0\}$$

We say that $B^*v|_{\partial\Omega} = 0$ is the adjoint boundary condition for $Bu|_{\partial\Omega} = 0$ if the two following conditions hold: (i) $(u, L^*v) = (f, v)$ for all $v \in C_{B^*}^\infty(\Omega)$ implies $u \in C_B^\infty(\Omega)$; (ii) $(v, L^*u) = (g, u)$ for all $u \in C_B^\infty(\Omega)$ imples $v \in C_{B^*}^\infty(\Omega)$

As an example, consider $Lu = \Delta u = f$ with boundary condition $Bu|_{\partial\Omega} = u|_{\partial\Omega} = 0$. We claim that in this case $B^*v|_{\partial\Omega} = v|_{\partial\Omega} = 0$. Indeed, suppose that condition (i) of the above definition holds. Then

$$(f, v) = (u, L^*v) = (Lu, v) + \int_{\partial\Omega} (u\partial_\nu v - v\partial_\nu u) = (f, v) + \int_{\partial\Omega} (u\partial_\nu v - v\partial_\nu u)$$

where ∂_ν is the normal derivative. Since u vanishes on the boundary we get $\int_{\partial\Omega} v\partial_\nu u$ for all $v \in C^\infty_{B^*}(\Omega)$, then we must have $B^*v|_{\partial\Omega} = 0$. Analogously we verify condition (ii).

The general ideia for solving $(*)$ is the following. Look for an operator M such that we can solve

$$Mu = v,\ v \in C^\infty_{B^*}(\Omega)$$

in order to get a $u \in C^\infty_B(\Omega)$ (in general, we can have more restrictions on u, such as $Bu|_{\partial\Omega} = 0$ + extra higher order conditions) such that $(Mu, Lu) \geq C \parallel u \parallel_s^2$. Then

$$\parallel y \parallel_s \parallel L^*v \parallel_{-s} \geq (u, L^*v) = (Lu, v) = (Mu, Lv) \geq \parallel u \parallel_s^2$$

Usually $\parallel v \parallel_{-s} \leq C \parallel u \parallel_s$ as M is of order $\geq s$. We obtain then $\parallel v \parallel_{-s} \leq C \parallel L^*v \parallel_{-s}$.

Now consider the linear functinal $F : L^*C^\infty_{B^*}(\Omega) \to \mathbb{R}$ given by $F(L^*v) = (f, v)$, f is in $H^s(\Omega)$ or maybe even smooth, depending on the particular problem. Then $F(L^*v) \leq \parallel f \parallel_s \parallel v \parallel_{-s} \leq C \parallel f \parallel_s \parallel L^*v \parallel_{-s}$, so F is bounded on the subspace $L^*C^\infty_{B^*}(\Omega) \subset H^{-s}(\Omega)$. By Hahn-Banach it can be extended to $\tilde{F}$ on all $H^{-s}(\Omega)$. By $H^s(\Omega) = H^{-s}(\Omega)^*$ we get that there exists a $u \in H^s(\Omega)$ such that $\tilde{F}(w) = (u, w)$ for all $w \in H^{-s}(\Omega)$. Restricting w bach to $L^*C^\infty_{B^*}(\Omega)$ we have

$$(u, L^*v) = \tilde{F}(L^*v) = F(L^*v) = (f, v)$$

for all $C^\infty_{B^*}(\Omega)$. By definition of $C^\infty_B(\Omega)$, assuming s large enough we can integrate by parts to force u to solve $Lu = f$ and also to satisfy $Bu|_{\partial\Omega} = 0$.

We want to do some examples now. But before that we need one more definition.

Definition 7. *Define the doulbe (s, r)-Sobolev norm as*

$$\parallel u \parallel_{(s,r)} \quad = \sum_{\substack{0\leq\alpha\leq s,\ 0\leq\beta\leq r \\ \alpha+\beta\leq\max(s,r)}} \int_\Omega (\partial_x^\alpha \partial_y^\beta u)^2$$

Then the (s, r)-Sobolev space$H^{(s,r)}(\Omega)$ is the completion of $C^\infty(\Omega)$ w.r.t. the norm $\parallel \cdot \parallel_{(s,r)}$.

3.1 Elliptic equations.

Consider $Lu = \Delta u - u = f$ in $\Omega\{(x,y) \in \mathbb{R}^2 \mid |x| < 1, |y| < 1\}$. We shall prove existence of regular solutions for the Dirichlet and Neumann problems. To determine the correct BVP notice that

$$(u, L^* v) - (Lu, v) = \int_{\partial\Omega} (u\partial_\Omega v - v\partial_\Omega u)$$

so that we have $Bu = u \Leftrightarrow B^* v = v$ for Dirichlet BC and $Bu = \partial_\nu u \Leftrightarrow B^* v = \partial_\nu v$ for Neumann BC. Let

$$Mu = \sum_{k=0}^{s} (-1)^k \partial_x^{2k} u$$

and solve $Mu = v$, $v \in C^\infty_{B^*}(\Omega)$, with

$$\begin{aligned} \partial_x^{2k} u|_{\partial\Omega_v^\pm} &= 0,\ k = 0, 1, \ldots, s-1 \text{ for Dirichlet} \\ \partial_x^{2k+1} u|_{\partial\Omega_v^\pm} &= 0,\ k = 0, 1, \ldots, s-1 \text{ for Neumann} \end{aligned}$$

and

$$\begin{aligned} u|_{\partial\Omega_h^\pm} &= 0, \text{ for Dirichlet} \\ u_y|_{\partial\Omega_h^\pm} &= 0, \text{ for Neumann} \end{aligned}$$

where $\partial\Omega_h^-$ ($\partial\Omega_h^-$) denotes the bottom (top) horizontal boundary and where $\partial\Omega_v^-$ ($\partial\Omega_v^-$) denotes the left (right) vertical boundary. Notice that this is possible since for an ODE of order $2s$ we can prescribe $2s$ boundary conditions. Also on $\partial\Omega_h^\pm$ $v = 0$ or $v_y = 0$, so we get $u = 0$ or $u_y = 0$, respectively. Furthermore, notice that $(Mu, Lu) \geq C \parallel u \parallel^2_{(s+1,1)}$ for all u having the above boundary contitions. To see that this is indeed the cae, it is illustrative to look at the following example

$$(-u_{xxxx} + u_{xx} - u, u_{xx} + u_{yy} - u) = \int_\Omega (u_{xxx}^2 + u_{xx}^2 + u_x^2 + u^2 + u_{xxy}^2 + u_{xy}^2 + u_y^2)$$

$$+ \int_{\partial\Omega} (uu_{xxx}\nu_1 - u_x u_{xx}\nu_1 - uu_x\nu_1 - u_{xx}u_{xxx} - uu_x\nu_1 - u_{yy}u_{xxx}\nu_1 + u_{xyy}u_{xx}\nu_1)$$

$$+ \int_{\partial\Omega} (-u_{xxy}u_{xx}\nu_2 + u_{yy}u_x\nu_1 - u_{xy}\nu_2 - u_y u\nu_2) \geq \parallel u \parallel^2_{(3,1)}$$

since all boundary terms vanish (ν_j is the normal in the j-direction).

Now observe that if we set

$$\| v \|_{(-s,-r)} = \sup_{w \in H^{(s,r)}(\Omega)} \frac{|(w,v)|}{\| w \|_{(s,r)}}$$

and $H^{(-s,-r)}(\Omega)$ =completion of $L^2(\Omega)$ w.r.t. the norm $\| \cdot \|_{(-s,-r)}$ in the usual way, then

$$\| v \|_{(-s,0)} \leq \sup_{w \in H^{(s,0)}(\Omega)} \frac{|(w,u)_{(s,0)}|}{\| w \|_{(s,0)}} \leq \| u \|_{(s,0)}$$

As an illustration of how to get this inequality, consider

$$\begin{aligned} -(w,v) = (w, -u_{xxxx} + u_{xx} - u) = (w,u)_{(2,0)} + \int_{\partial\Omega} (w u_{xxx} \nu_1 - w_x u_{xx} \nu_1 - w u_x \nu_1) \\ = (w,u)_{(2,0)} \end{aligned}$$

since we can assume that $w|_{\partial\Omega^{\pm}} = 0$, i.e.,

$$\begin{aligned} H^{(3,1)}(\Omega) = & \text{ completion of } C^\infty(\Omega) \text{ in the norm } \| \cdot \|_{(3,1)} = \\ = & \text{ completion of } C^\infty(\Omega) \cap \{ w \mid w|_{\partial\Omega_v^{\pm}} = 0 \} \text{ in the norm } \| \cdot \|_{(3,1)} \end{aligned}$$

Therefore $|(w,v)| \leq \| w \|_{(2,0)} \| u \|_{(2,0)}$, which is the desired inequality in this particular case.

Continuing the computations, now we have

$$\| u \|_{(s+1,1)} \| L^* v \|_{(-s-1,-1)} \geq (u, L^* v) = (Lu, v) = (Mu, Lu) \geq \| u \|^2_{(s+1,1)}$$

for all $v \in C^\infty_{B^*}(\Omega)$. Hence

$$\| v \|_{(-s,0)} \leq \| u \|_{(s+1,1)} \leq \| L^* v \|_{(-s-1,-1)}$$

for all $v \in C^\infty_{B^*}(\Omega)$. Summarizing the above, we have the following result

Theorem 6. *For each $f \in H^{(s,0)}(\Omega)$ there exists a weak solution $u \in H^{(s+1,1)}(\Omega)$ of $Lu = f$, i.e, $(u, L^* v) = (f, v)$ for all $v \in C^\infty_{B^*}(\Omega)$.*

In order to obtain higher regularity in the y-direction we need the following lemma concerning difference quotients, which are defined as

$$u^h(x,y) = \frac{u(x, y+h) - u(x,y)}{h}$$

Lemma 2. *(i) If $\Omega' \subset\subset \Omega$ and $u \in H^{(0,1)}(\Omega)$ then*

$$\| u^h \|_{L^2(\Omega')} \leq \| u_y \|_{L^2(\Omega)}$$

(ii) If $u \in L^2(\Omega')$ and $\| u^h \|_{L^2(\Omega')} \leq C$ for all $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$ then $u \in H^{(0,1)}(\Omega')$ with $\| u_y \|_{L^2(\Omega')} \leq C$.

Proof. Assume first that u is smooth. Then for $0 < |h| < \frac{1}{2}\operatorname{dist}(\Omega', \partial\Omega)$ we have

$$u(x, y+h) - u(x,y) = h \int_0^1 u_y(x, y+th)dt$$

So

$$|u(x, y+h) - u(x,y)| \leq h \int_0^1 |u_y(x, y+th)|dt$$

Hence, by Jensen's inequality

$$\begin{aligned} \int_{\Omega'} |u^h|^2 &\leq \int_{\Omega'} (\int_0^1 |u_y(x, y+th)|dt)^2 dxdy \\ &\leq \int_0^1 (\int_{\Omega'} |u_y(x, y+th)|^2 dxdy)dt \end{aligned}$$

$\leq \int_{\Omega'} |u_y(x, y+t_0h)|^2 dxdy$ for t_0 where the integrand achieves its maximum

$$\leq u_y \|_{L^2(\Omega)}$$

Now, by approximation this holds for $u \in H^{(0,1)}(\Omega)$.

Assume now the hypothesis of (ii). For all $\phi \in C_c^\infty(\Omega')$ we have

$$\int_{\Omega'} u(x,y)\frac{\phi(x,y+h) - \phi(x,y)}{h} dxdy = -\int_{\Omega'} \phi(x,y)\frac{u(x,y) - u(x,y-h)}{h} dxdy$$

where we used integration by parts for difference quotients and a change of variables $y' = y + h$. In other words, we have

$$\int_{\Omega'} u\phi^h = -\int_{\Omega'} u^{-h}\phi$$

By assumption $\sup_h \parallel u^{-h} \parallel_{L^2(\Omega')} \leq C$. Using the weakly compactness of bounded sets in Hilbert spaces we get a h_k converging to zero and a $w \in L^2(\Omega')$ such that u^{-h_k} converges in $L^2(\Omega')$ to w. But then

$$\int_{\Omega'} u\phi_y = \int_{\Omega} u\phi_y = \lim_{h_k \to 0} \int_{\Omega} u\phi^{h_k}$$
$$= -\lim_{h_k \to 0} \int_{\Omega'} u^{-h_k}\phi = -\int_{\Omega'} \phi = -\int_{\Omega} w\phi$$

□

We can now obtain full regularity for any solution in the previous theorem.

Corollary 2. *For each $f \in H^s(\Omega)$ there exists a unique solution $u \in H^{s+1}(\Omega)$ of $Lu = f$.*

Proof. We know that $u \in H^{(s+1,1)}(\Omega)$ and for all $v \in C^\infty_{B^*}(\Omega)$ we have

$$-(u_y, v_y) = (f - u_{xx} + u, v) = (\bar{f}, v)$$

where $\bar{f} = f - u_{xx} + u \in H^(0,1)(\Omega)$. Then $-(u_y^h, v_y) = (\bar{f}^h, v)$, and if we choose smooth functions $\{v_i\} \subset C^\infty_{B^*}$ such that $v_i \to u^h$ in $H^{(0,1)}(\Omega)$ we get

$$\parallel u_y^h \parallel^2 \leq |(\bar{f}^h, u^h)| \leq \parallel \bar{f}^h \parallel \parallel u^h \parallel \leq \parallel \bar{f_y}^h \parallel \parallel u^h \parallel \leq C$$

by (i) of the lemma; and by (ii) we have $u_y \in H^{(0,1)}(\Omega)$, i.e., $u_{yy} \in L^2(\Omega)$. The following then holds in $L^2(\Omega)$

$$u_{yy} = f - u_{xx} - u \in H^{(s-2,1)}(\Omega)$$

We can then differentiate w.r.t. y to get

$$u_{yyy} = (f - u_{xx} - u)_y \in H^{(s-3,1)}(\Omega)$$

and continue to boot strap to get $u \in H^{s+1}(\Omega)$.

Uniqueness follows since now $u \in C^{s-1}(\Omega)$ and $(-u, Lu) \geq \parallel u \parallel_1^2$ holds as $v \in C^\infty_{B^*}(\Omega)$ imples that $u|_{\partial\Omega} = 0$ or $\partial_\nu u|_{\partial\Omega} = 0$. □

Remark 7. *One draw back of this method is that it does not give the best result, i.e., the best is $f \in C^{k+2,\alpha}(\Omega)$ implying $u \in C^{k,\alpha}(\Omega)$, or $f \in H^{s+2}(\Omega)$ implying $u \in H^s(\Omega)$.*

3.2 Hyperbolic equations

Here we consider the Cauchy problem for the wave equation

$$
\begin{gathered}
Lu = u_{tt} - u_{xx} + u = f \\
u(x,0) = u_t(x,0) = 0,\ u(\pm 1, t) = 0 \text{ or } u_x(\pm 1, t) = 0
\end{gathered}
$$

and its adjoint boundary value problem

$$
\begin{gathered}
L^* v = v_{tt} - v_{xx} + v = g \\
v(x,1) = v_t(x,1) = 0,\ v(\pm 1, t) = 0 \text{ or } v_x(\pm 1, t) = 0
\end{gathered}
$$

The wave equation represents the motion of a string (or waves) with fixed ends (for the Dirichlet boundary conditions) under the influence of an external force f. If we tried the full Dirichlet or Neumann problem as for the Laplace equation we would not be able to prove existence using this method.

Let

$$
Mu = a(t) \sum_{k=0}^{s} (-1)^k \partial_x^{2k} u_t
$$

where $a(t) = -e{-}t$ (we really only need $a'(t) > 0$, $a(t) < 0$) and solve

$$
\begin{gathered}
Mu = v,\ v \in C^\infty_{B^*}(\Omega) \text{ with } u(x,0) = u_t(x,0) = 0 \\
\partial_x^{2k} u(\pm 1, t) = 0,\ k = 0, 1, \ldots, s-1 \text{ fot Dirichlet} \\
\partial_x^{2k+1} u(\pm 1, t) = 0,\ k = 0, 1, \ldots, s-1 \text{ fot Neumann}
\end{gathered}
$$

Notice that this can be done using standard theory for ODEs. Then observe that

$$
(Mu, Lu) \geq \| u \|^2_{(s+1,1)}
$$

for all $u \in C^\infty_B(\Omega) = \{u$ having the above boundary conditions $\}$. For example:

$$\begin{gathered}
(\ u_{tt}-u_{xx}+u\ ,\ a(t)(-u_{txxxx}+u_{txx}-u_t)\ )= \\
\int_\Omega \frac{a'(t)}{2}\Big(u_{xxx}^2+u_{xxt}^2+u_{xt}^2+2u_{xx}^2+u_t^2+2u_x^2+u^2\Big)+ \\
\int_{\partial\Omega}\Big(au_xu_t\nu_1-\frac{a}{2}u_t^2\nu_2-\frac{a}{2}u_x^2\nu_2-\frac{a}{2}u^2\nu_2-\frac{a}{2}u_{xx}^2\nu_2+au_{tt}u_{xt}\nu_1-\frac{a}{2}u_{xt}^2\nu_2+ \\
auu_{xt}\nu_1-\frac{a}{2}u_x^2\nu_2-au_{tt}u_{txxx}\nu_1+au_{xtt}u_{txx}\nu_1-\frac{a}{2}u_{xxt}^2\nu_2+ \\
au_{xx}u_{txxx}\nu_1-au_{xxx}^2\nu_2-auu_{xxxt}\nu_1+au_xu_{xxt}\nu_1-au_{xx}^2\nu_2\Big) \\
\geq C\parallel u\parallel_{(3,1)}^2
\end{gathered}$$

since all boundary terms vanish or are non-negavite for $u\in C_B^\infty(\Omega)$. Furthermore, as in the elliptic case

$$\parallel v\parallel_{(-s,0)}\leq C\sup_{w\in H^{(s,0)}(\Omega)}\frac{|(w,u_t)|}{\parallel w\parallel_{(s,0)}}\leq C\parallel u\parallel_{(s,1)}$$

For example

$$\begin{aligned}
-(w,v)&=-(\ w\ ,\ a(t)(-u_{txxxx}+u_{txx}-u_t)\ )=(w,a(t)u_t)_{(2,0)} \\
&+\int_{\partial\Omega}(awu_{txxx}\nu_1-aw_xu_{txx}\nu_1-awu_{tx}\nu_1)=(w,a(t)u_t)_{(2,0)}
\end{aligned}$$

since we can assume that $w(\pm1,t)=0$ as before, so $|(w,v)|\leq C\parallel w\parallel_{(2,0)}\parallel u\parallel_{(2,1)}$ which gives the desired result.

We now have

$$\parallel u\parallel_{(s+1,1)}\parallel L^*v\parallel_{(-s-1,-1)}\geq(u,L^*v)=(Lu,v)=(Mu,Lu)\geq C\parallel u\parallel_{(s+1,1)}^2\quad\text{for all } v\in C_{B^*}^\infty(\Omega)$$

$((u,L^*v)=(Lu,v)$ is justified because all boundary terms vanish). Hence

$$\parallel v\parallel_{(-s,0)}\leq C\parallel u\parallel_{(s+1,1)}\leq C\parallel L^*v\parallel_{(-s-1,-1)}$$

This proves

Theorem 7. *For each $f\in H^{(s,0)}(\Omega)$ there exists a weak solution $u\in H^{(s+1,1)}(\Omega)$ of $Lu=f$, i.e., $(u,L^*v)=(f,v)$ for all $v\in C_{B^*}^\infty(\Omega)$.*

Remark 8. *Higher regularity may be proven in the same way as for the elliptic case. Then* u *solves* $Lu = f$ *pointwise and because* $v \in C^\infty_{B^*}(\Omega)$ *we have that* $u \in C^\infty_B(\Omega)$ *has the correct boundary values. Again this method does not necessarily gives the best result.*

Exercise: Prove the same result for the parabolic heat equation

$$Lu = u_t - u_{xx} + u = f$$
$$u(x,0),\ u(\pm 1, t) = 0 \text{ or } u_x(\pm 1, t) = 0$$

with adjoint boundary value problem

$$L^* v = v_t - v_{xx} + v = g$$
$$v(x,1),\ v(\pm 1, t) = 0 \text{ or } v_x(\pm 1, t) = 0$$