MAT 319

Midterm II. November 5, 2013

This is a closed notes/ closed book/ electronics off exam.

You are allowed and encouraged to motivate your reasoning, but at the end your proofs should be formal logical derivations, whether proving that something holds for all, or proving that your example works.

You can use any theorem or statement proven in the book; please refer to it in an identifiable way, eg. "by the completeness axiom", "by the definition of the limit", etc.

You should attempt Problem 1 and three of the remaining four questions. If you attempt all four questions, your total score will be made up of the score for Problem 1 and your best three scores on the remaining questions.

Please write legibly and cross out anything that you do not want us to read.

Each problem is worth 25 points.

Name:

Problem	1	2	3	4	5	Total
Grade						

Problem 1.

a) Define what it means to say that the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

b) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges. (Hint: show $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$).

c) Define what it means for a function $f\colon \mathrm{dom}(f) \to \mathbb{R}$ to be continuous at a point $x_0 \in \mathrm{dom}(f)$.

d) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at x_0 and $f(x_0) > 0$. Show that there exists $\delta > 0$ such that $f(x) > 0$ on $(x_0 - \delta, x_0 + \delta)$.

Answer 1.

a) Define the nth partial sum of the series to be

$$s_n = \sum_{k=1}^{n} a_k.$$

Then we say the series converges if s_n converges to a real number.

b) It is easy to show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. So now we get

$$\begin{aligned} s_n = \sum_{k=1}^{n} \frac{1}{n(n+1)} &= \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \left(\frac{1}{2} - \frac{1}{2}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Hence $s_n \to 1$ so the series converges by the definition in part (a). (*Remark: Quite a few people used the p-test and comparison test to prove this, which to be honest I had not considered when I wrote the question. I graded it correct as long as all the relevant tests were stated and used correctly.*)

c) The function $f\colon \mathrm{dom}(f) \to \mathbb{R}$ is continuous at x_0 if given $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathrm{dom}(f)$ and

$$|x - x_0| < \delta \quad \text{then} \quad |f(x) - f(x_0)| < \varepsilon.$$

d) Let $\varepsilon = \frac{f(x_0)}{2} > 0$. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$\begin{aligned} x \in (x_0 - \delta, x_0 + \delta) &\Longrightarrow |x - x_0| < \delta \\ &\Longrightarrow |f(x) - f(x_0)| < \varepsilon = \frac{f(x_0)}{2} \\ &\Longrightarrow 0 < \frac{f(x_0)}{2} < f(x) < \frac{3f(x_0)}{2} \\ &\Longrightarrow f(x) > 0. \end{aligned}$$

Remark: No-one got this completely correct. The important thing to realise is that if we know f is continuous, then we are at liberty to choose any $\varepsilon > 0$ we want, and continuity guarantees us the existence of some $\delta > 0$ satisfying the definition. So here we were able to choose $\varepsilon = \frac{f(x_0)}{2}$, for example. No-one actually stated a particular choice for ε, which is what the question requires.

Problem 2.

a) State the *Alternating Series Test.*

b) Prove that the series
$$\sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}n}{n^2}$$
converges.

c) Prove or provide a counterexample to the following statements.

  i) If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

  ii) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n^2$ converges.

Answer 2.

a) Let a_n be a decreasing sequence with $\lim_{n\to\infty} a_n = 0$. Then the series
$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$
converges.

b) We can rewrite the series as
$$\sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$
Now, we know that $\sum \frac{1}{n^2}$ converges by the p-test, and for the second series, we can apply the Alternating Series Test from part (a), since $\frac{1}{n}$ is a decreasing sequence which converges to 0. Hence our original series is the sum of two convergent series, and hence is convergent.

c) i) This is false. Consider the sequence $a_n = \frac{1}{n}$. Then $a_n \to 0$ but $\sum \frac{1}{n}$ diverges.

  ii) This is false. Consider the sequence $a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$. Then $\sum a_n$ converges by the Alternating Series Test but $\sum a_n^2 = \sum \frac{1}{n}$ diverges.

Remark: Everyone attempted this question. Generally the understanding was good, though c) ii) caused a lot of difficulty, as I thought it might (hence I put the counterexample on the practice midterm so it would be fresh in your minds!). However c) i) was generally well done.

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is continuous at 0 and not continuous at any x in $\mathbb{R} \setminus \{0\}$.

Answer 3.

- *Continuity at* 0. Let $\varepsilon > 0$ and set $\delta = \varepsilon$. Then if $|x| = |x - 0| < \delta$ we have

  $$|f(x) - f(0)| = |x - 0| = |x| < \delta = \varepsilon \quad \text{if } x \in \mathbb{Q}$$
  $$|f(x) - f(0)| = |0 - 0| = 0 < \varepsilon \quad \text{if } x \in \mathbb{R} \setminus \mathbb{Q}.$$

  So we see that $|x - 0| < \delta \implies |f(x) - f(0)| < \varepsilon$.
- *Not continuous on* $\mathbb{R} \setminus \mathbb{Q}$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. By denseness of $\mathbb{Q}$, we can take a sequence (x_n) such that $x_n \in \mathbb{Q}$ for all n and $x_n \to x$. But then we see that since $x_n \in \mathbb{Q}$, we must have $f(x_n) = x_n$ for all n. So we have

  $$\lim_{n\to\infty} x_n = x \quad \text{but} \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x \neq 0 = f(x)$$

  and so f is not (sequentially) continuous at x.
- *Not continuous on* $\mathbb{Q} \setminus \{0\}$. Let $x \in \mathbb{Q}$ with $x \neq 0$. Consider the sequence $x_n = x + \frac{\sqrt{2}}{n}$. Then $x_n \in \mathbb{R} \setminus \mathbb{Q}$ for all n and $x_n \to x$. By the definition of f, we must have $f(x_n) = 0$ for all n, and so

  $$\lim_{n\to\infty} x_n = x \quad \text{but} \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 \neq x = f(x).$$

  Hence f is not (sequentially) continuous at x.

Remark: A lot of people attempted this - probably because this was an example done in class. Indeed, a couple got very high scores on this question. I think the main difficulty here was not so much the understanding of the problem, but more on how to write out the proof - this was particularly apparent when proving continuity at 0*. Remember, when proving continuity, it is almost always good to start with the statement "Let* $\varepsilon > 0$*." and then proceed from there. When proving discontinuity, I was looking for a justification of why a sequence* x_n *existed, either by giving a formula or by using density of* $\mathbb{Q}$ *(or* $\mathbb{R} \setminus \mathbb{Q}$*).*

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

a) Prove $f(0) = 0$.
b) Prove $f(-x) = -f(x)$.
c) Suppose f is continuous at 0. Show that f is continuous at all $x \in \mathbb{R}$.

Answer 4.

a) We write
$$f(0) = f(0+0) = f(0) + f(0) = 2f(0),$$
from which it follows that $f(0) = 0$.

b) Now we write
$$0 = f(0) = f(x + (-x)) = f(x) + f(-x).$$
Rearranging this gives $f(-x) = -f(x)$.

c) Let $x_0 \in \mathbb{R}$ be arbitrary and let $\varepsilon > 0$. By continuity of f at 0, we know that for (this) $\epsilon > 0$ there exists $\delta > 0$ such that
$$|x| < \delta \Longrightarrow |f(x)| < \varepsilon.$$
From this, it follows that
$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| = |f(x) + f(-x_0)| = |f(x - x_0)| < \varepsilon$$
and so f is continuous at x_0.

Remark: Of course, the crux of this problem is in part (c); the first two parts were to set up the proof in (c). I don't think anyone got a full answer to this question, though I must admit this is a question I like, and the proof once you see it is very short. For part (c), it is perhaps helpful to notice that f behaves like a linear function $f(x) = kx$ (indeed, you can prove these are the only such continuous functions that have the additive property in the question, but that's another story), and so given $\varepsilon > 0$ the choice of δ at 0 will be the same as the one you require at an an arbitrary $x_0 \in \mathbb{R}$.

Problem 5.

a) State the *Intermediate Value Theorem*
b) Let $f\colon \mathbb{R} \to \mathbb{Z}$ be continuous. Show that f must be constant.
c) Let $f\colon [0,1] \to [0,1]$ be continuous. Show that there exists $c \in [0,1]$ such that $f(c) = c$. Hint: consider the function $h(x) = f(x) - x$.

Answer 5. a) Let $f\colon I \to \mathbb{R}$ be a continuous function on an interval I, and let $a, b \in I$ with $a < b$. Then for all $y \in (f(a), f(b))$ (or for all $y \in (f(b), f(a))$) there exists $c \in (a, b)$ such that $f(c) = y$.

b) Suppose f is not constant. Then there exists $n < m \in \mathbb{Z}$ and $x, y \in \mathbb{R}$ such that $f(x) = m$ and $f(y) = n$. Let $y = n - \frac{1}{2}$ (other choices of y are available); then $y \in (m, n) = (f(x), f(y))$. By the Intermediate Value Theorem, since f is continuous by assumption, there must exist $c \in (x, y)$ (or in (y, x) if $y < x$) such that $f(c) = y$. But $y \notin \mathbb{Z}$, so this is a contradiction, hence f must be constant.

c) Following the hint, let $h(x) = f(x) - x$. Then h is continuous since it is the difference of two continuous functions. Furthermore, since $f(0) \in [0,1]$, we must have

$$h(0) = f(0) - 0 = f(0) \geq 0 \tag{1}$$

and since $f(1) \in [0,1]$ we have

$$h(1) = f(1) - 1 \leq 0. \tag{2}$$

If we have equality in either (1) or (2) then we have found our point such that $f(c) = c$, so suppose $h(0) > 0$ and $h(1) < 0$. Then by the Intermediate Value Theorem there exists $c \in (0, 1)$ such that $h(c) = 0$. Therefore $f(c) - c = 0$ and so $f(c) = c$ as required.

Remark: Most people were able to state the Intermediate Value Theorem, but not many tackled the applications in parts (b) and (c) - perhaps this was down to time constraints. A number of people gave heuristic arguments as to how the proofs would work, but a few details were general lacking - this was especially true in (b) where I don't think anyone invoked the Intermediate Value Theorem directly in their solution (though many perhaps were trying to use it implicitly). Part (c) is a classical result about continuity that again makes use of the Intermediate Value Theorem at a vital moment. Remember that when trying to show two functions have a common point on their graph (as here), it is often easier to consider their difference and use the Intermediate Value Theorem to show that this difference must be equal to 0 *somewhere. Note also that part (c) is fairly similar to the example I did in class where I showed that an odd degree polynomial has an real root.*