Final Exam Practice Problems

Problem 1. Let the sequence (x_n) be defined as follows: $x_1 = 1, x_2 = 2$ and $x_{n+2} = \frac{1}{2}(x_n + x_{n+1})$ for any $n \in \mathbb{N}$. Prove that $1 \leq x_n \leq 2$ for any $n \in \mathbb{N}$.

Problem 2. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that $\sup S = -\inf\{-s : s \in S\}$.

Problem 3. Find the infimum of the set $A = \{1 + \frac{(\sin n)^2}{\sqrt{n}} \mid n \in \mathbb{N}\}$.

Problem 4. Prove

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right) = 1.$$

Problem 5. Let (a_n) be a positive sequence such that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$. Prove that (a_n) is unbounded.

Problem 6. Assume that $\lim_{n\to\infty} x_n = +\infty$. Prove that

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = +\infty.$$

Problem 7. Suppose $f(x)$ is a strictly increasing function on $[a, b]$ and $(x_n) \subset [a, b]$ is a sequence such that $\lim_{n\to\infty} f(x_n) = f(a)$. Prove that $\lim_{n\to\infty} x_n = a$.

Problem 8. * Let f be a function defined on $(0, 1)$ such that for any $c \in (0, 1)$, $\lim_{n\to\infty} f(\frac{c}{n}) = 0$. Can we conclude that $\lim_{x\to 0^+} f(x) = 0$?

Problem 9. Assume that the function f is continuous at 0 and $f(0) > 0$. Prove that there exists a $\delta > 0$ such that $f(x) > 0$ for any $|x| < \delta$.

Problem 10. For any function f, we define $w_a(\delta) = \sup\{|f(x) - f(y)| \mid |x - a| < \delta \text{ and } |y - a| < \delta\}$. Prove that f is continuous at a if and only if $\lim_{\delta\to 0^+} w_a(\delta) = 0$.

Problem 11. Suppose there exists a constant $L > 0$ such that for any $x, y \in [a, \infty)$ we have

$$|f(x) - f(y)| \leq L|x - y|.$$

If $a > 0$, prove that $\frac{f(x)}{x}$ is uniformly continuous on $[a, \infty)$.

Problem 12.

Let the function f be defined as

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that f is differentiable at 0.

Problem 13. Suppose $|f(x)|$ is differentiable at a and $f(a) = 0$, prove that $f'(a) = 0$.

Problem 14. Assume there exist constants M and $a > 1$ such that for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq M|x - y|^a.$$

Prove that f is a constant.

Problem 15. Let $f(x)$ and $g(x)$ be convex functions and f is increasing. Prove that $f(g(x))$ is convex.

Problem 16. If f defined on $[0, 1]$ is a continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$. Prove that $f(x) = 0$ for any $x \in [0, 1]$.

Problem 1

We prove by induction on n.

① Base case : $n=1$, $x_1 = 1$ and $1 \leq x_1 \leq 2$

$n=2$, $x_2 = 2$ and $1 \leq x_2 \leq 2$

② Inductive step : assume it is true for any $1 \leq k \leq n$

then $1 \leq x_{n-1} \leq 2$ and $1 \leq x_n \leq 2$.

Hence $1 \leq x_{n+1} = \frac{x_n + x_{n-1}}{2} \leq 2$.

Therefore it also holds for $n+1$. #

Problem 2

Denote $A = \sup S$ and $B = \inf \{-s : s \in S\}$. We need to show that $A = -B$.

By definition of sup, $s \leq A$, $\forall s \in S$

$\Rightarrow$ $-s \geq -A$, $\forall s \in S$

$\Rightarrow$ $B = \inf \{-s : s \in S\} \geq -A$ $\Rightarrow$ $A \geq -B$ ①

By definition of inf

$-s \geq B$ $\forall s \in S$

$\Rightarrow$ $s \leq -B$ $\forall s \in S$

$\Rightarrow$ $A = \sup S \leq -B$ ②

Combining ① and ② we get $A = -B$. #

Problem 3

First we notice that $\lim\limits_{n\to+\infty} \frac{(\sin n)^2}{\sqrt{n}} = 0$

This is because

$$0 \leqslant \frac{(\sin n)^2}{\sqrt{n}} \leqslant \frac{1}{\sqrt{n}},$$

Since $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, by squeeze theorem, $\lim \frac{(\sin n)^2}{\sqrt{n}} = 0$.

$\forall n \in \mathbb{N}$, $1+\frac{(\sin n)^2}{\sqrt{n}} \geq 1$. Therefore 1 is a lower bound of the set A. We claim that 1 is the greatest lower bound.

Suppose there exists $\varepsilon_0 > 0$ such that

$$1 + \frac{(\sin n)^2}{\sqrt{n}} \geq 1+\varepsilon_0 \qquad \forall n \in \mathbb{N}$$

By the comparison property of limits,

$$1 = \lim_{n\to\infty} \left(1 + \frac{(\sin n)^2}{\sqrt{n}}\right) \geq 1+\varepsilon_0$$

which is a contradiction! Therefore $\inf A = 1$. #

[Alternatively, one can argue $\forall \varepsilon > 0$, $\exists N \geq 1$ such that

$$1 \leqslant \frac{(\sin N)^2}{\sqrt{N}} + 1 < \varepsilon + 1$$

therefore by Lemma 2.3.4 (for inf), $1 = \inf A$.]

Problem 4

Notice that for each $1 \leq i \leq n$

$$\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+i}} \leq \frac{1}{n},$$

therefore $\frac{n}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq \frac{n}{n} = 1$.

Since $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = \frac{1}{\sqrt{1+0}} = 1$,

by squeeze theorem, we have

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right) = 1.$$

#

Problem 5

Since $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$, for $\varepsilon = \frac{1}{2}$, there exists N such that for $n > N$, we have

$$\left| \frac{a_n}{a_{n+1}} \right| < \frac{1}{2}.$$

Since $a_n > 0$ for each n, we have

$$a_{n+1} > 2a_n \quad \text{for } n > N.$$

Therefore $a_n > 2^{n-N} \cdot a_N$ for $n > N$

We can conclude that a_n is unbounded because

$$\lim_{n\to\infty} 2^{n-N} \cdot a_N = +\infty.$$

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Problem 6   First proof

To show $\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n} = +\infty$, we need to show that for any $A>0$, there exists $N \geq 1$ such that when $n>N$,

$$\frac{a_1+\cdots+a_n}{n} > A.$$

For any $A>0$, because $\lim_{n\to\infty} a_n = +\infty$, there exists $N_1 \geq 1$ such that $a_n > 2A$ for $n > N_1$.

Therefore
$$\frac{a_1+\cdots+a_n}{n} = \frac{a_1+\cdots+a_{N_1}+a_{N_1+1}+\cdots+a_n}{n}$$
$$> \frac{a_1+\cdots+a_{N_1}+(n-N_1)\cdot 2A}{n} \quad \text{if } n>N_1.$$

Notice that
$$\lim_{n\to\infty} \frac{a_1+\cdots+a_{N_1}+(n-N_1)\cdot 2A}{n}$$
$$= \lim_{n\to\infty} 2A + \frac{a_1+\cdots+a_{N_1}-2N_1A}{n}$$
$$= 2A > A$$

Therefore by the property of limit, there exists $N_2 \geq 1$ such that when $n>N_2$,

$$\frac{a_1+\cdots+a_{N_1}+(n-N_1)2A}{n} > A$$

Hence if we choose $N = \max\{N_1, N_2\}$, for $n>N$ we have

$$\frac{a_1+\cdots+a_n}{n} \underset{n>N_1}{>} \frac{a_1+\cdots+a_{N_1}+(n-N_1)2A}{n} \underset{n>N_2}{>} A. \qquad \#$$

Problem 6 Second proof

$\forall A > 0$, we need to choose N such that when $n > N$, $\dfrac{x_1 + x_2 + \cdots + x_n}{n} > A$

① Since $\lim x_n = +\infty$, there exists N_1 such that when $n > N_1$, $x_n > 2A$.

② Choose N_2 such that $N_2 > \dfrac{2N_1 A - (x_1 + \cdots + x_{N_1})}{A}$.

Then we choose $N = \max\{N_1, N_2\}$.

If $n > N$, then $\dfrac{x_1 + \cdots + x_n}{n} = \dfrac{(x_1 + \cdots + x_{N_1}) + (x_{N_1+1} + \cdots + x_n)}{n}$

$$> \frac{x_1 + \cdots + x_{N_1} + (n - N_1)2A}{n}$$

$$= A + \frac{nA - (2N_1A - (x_1 + \cdots + x_{N_1}))}{n}$$

$$> A + \frac{N_2A - (2N_1A - (x_1 + \cdots + x_{N_1}))}{n} > A + 0 = A \qquad \#$$

[less informal proof: $\dfrac{x_1 + \cdots + x_{N_1} + (n - N_1)2A}{n} > A$

$\Leftrightarrow \quad x_1 + \cdots + x_{N_1} + (n - N_1)2A > nA$

$\Leftrightarrow \quad nA > 2N_1A - (x_1 + \cdots + x_{N_1})$

$\Leftrightarrow \quad n > \dfrac{2N_1A - (x_1 + \cdots + x_{N_1})}{A}$]

Problem 7

We will prove by contradiction.

Suppose $\lim_{n\to\infty} x_n \neq a$, then there exists $\varepsilon_0 > 0$

and a subsequence $(x_{n_k}) \subset [a,b]$ such that $|x_{n_k} - a| > \varepsilon_0$,

in particular $x_{n_k} > a + \varepsilon_0$.

Since f is a strictly increasing function,

$$(*) \qquad f(x_{n_k}) > f(a+\varepsilon_0) \quad \forall\, k \geq 1.$$

Because $\lim_{n\to\infty} f(x_n)$ exists,

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = f(a).$$

From $(*)$ we know that

$$\lim_{k\to\infty} f(x_{n_k}) \geq f(a+\varepsilon_0).$$

In particular this implies that

$$f(a) \geq f(a+\varepsilon_0).$$

But this contradicts with the fact that f is strictly increasing. Therefore our assumption is wrong and we must have $\lim_{n\to\infty} x_n = a$. #

Problem 8.

We cannot conclude that $\lim_{x \to 0^+} f(x) = 0$.

Here is the construction of a counter-example.

Because $(0,1)$ is uncountable, we can choose a number $b \in (0,1)$

such that $\forall k \geq 1$, $b^k \notin \mathbb{Q}$.

(For example b can be chosen to be any transcendental

number like $\frac{1}{e}$ or $\frac{1}{\pi}$).

Let $f(x) = \begin{cases} 1 & \text{if } x = b^k \text{ for some } k \geq 1 \\ 0 & \text{else.} \end{cases}$

We will show that $f(x)$ satisfies

① $\forall c \in (0,1)$, $\lim_{n \to \infty} f(\frac{c}{n}) = 0$

② $\lim_{x \to 0^+} f(x) \neq 0$

① Let $c \in (0,1)$ be any real number, then there is at

most one integer n such that $\frac{c}{n} = b^k$ for some $k \geq 1$.

Suppose this is not true and there exists n_1, n_2 such that

$\frac{c}{n_1} = b^{k_1}$, $\frac{c}{n_2} = b^{k_2}$ and $k_2 > k_1$.

$\Rightarrow$ $b^{k_1} \cdot n_1 = b^{k_2} \cdot n_2$ $\Rightarrow$ $b^{k_2 - k_1} = \frac{n_1}{n_2} \in \mathbb{Q}$

This contradicts the choice of b.

Therefore for n large enough, $\frac{c}{n}$ is not equal to any b^k.

By the definition of f, we have $f(\frac{c}{n}) = 0$.

Thus $\lim_{n\to\infty} f(\frac{c}{n}) = 0$.

② We will prove $\lim_{x\to 0^+} f(x) \neq 0$ by contradiction.

Suppose $\lim_{x\to 0^+} f(x) = 0$, then for any sequence x_n

such that $\lim_{n\to\infty} x_n = 0$, we must have $\lim_{n\to\infty} f(x_n) = 0$.

Let $x_n = b^n$, since $b < 1$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} b^n = 0$.

But $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(b^n) = \lim_{n\to\infty} 1 = 1 \neq 0$

Therefore $\lim_{x\to 0^+} f(x) \neq 0$.

The conclusion is that f is a counterexample. #

9) f continuous at 0, $f(0)>0$. Want δ s.t. $f(x)>0$ $\forall$ $|x|<\delta$

Say $f(0)=a>0$. Note that $\varepsilon := \frac{a}{2} > 0$. Since f is continuous at 0, $\exists$ $\delta>0$ s.t. $\forall$ $|x-0|<\delta$, we have $|f(x)-a|<\varepsilon$. That is, $\forall$ $|x|<\delta$, $|f(x)-a|<\frac{a}{2}$

$\Rightarrow$ $-\frac{a}{2} < f(x)-a < \frac{a}{2}$ so $\underbrace{0<\frac{a}{2}<f(x)}$ $< \frac{3a}{2}$.

So $f(x)>0$ $\forall$ $|x|<\delta$ as desired

(we will reference this argument later in a more general setting. The content is the same, just with "0" replaced by "c")

10. $w_a(\delta) := \sup\{|f(x)-f(y)| : |x-a|\leq\delta, |y-a|<\delta\}$. want f cts at a $\Leftrightarrow$ $\lim_{\delta\to 0^+} w_a(\delta)=0$.

($\Rightarrow$): Suppose f is cts at a. Want to show that $\forall$ $\varepsilon>0$, $\exists$ $\gamma>0$ s.t. $\forall$ $0<\delta-0<\gamma$, we have $w_a(\delta)<\varepsilon$. Since $w_a(\delta)$ is the sup of a set, that means we want γ s.t. $\forall$ $0<\delta<\gamma$, ε is an <u>upper bound</u> for $\{|f(x)-f(y)| : |x-a|\leq\delta, |y-a|<\delta\} =: S_a(\delta)$.

Since f is cts at a, $\exists$ $\gamma>0$ s.t. $\forall$ z s.t. $|z-a|<\gamma$, we have $|f(z)-f(a)|<\varepsilon/2$. Now, for $\delta<\gamma$, x, y s.t. $|x-a|\leq\delta$, $|y-a|<\delta$. Then $|x-a|\leq\delta<\gamma$, $|y-a|<\delta<\gamma$

so $|f(x)-f(a)|<\varepsilon/2$ and $|f(y)-f(a)|<\varepsilon/2$.

$\Rightarrow$ $|f(x)-f(y)| = |f(x)-f(a)+f(a)-f(y)| \leq |f(x)-f(a)|+|f(y)-f(a)|$ $< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. So ε is a strict upper bound for $S_a(\delta)$ and so $w_a(\delta)<\varepsilon$.

($\Leftarrow$): Suppose $\lim_{\delta\to 0^+} w_a(\delta)=0$. Want: $\forall$ $\varepsilon>0$ $\exists$ $\gamma>0$ s.t. whenever $|z-a|<\gamma$, have $|f(z)-f(a)|<\varepsilon$. Fix $\varepsilon>0$. We know (hypothesis) $\exists$ $\gamma>0$ s.t. whenever $0<\delta<\gamma$ we have

ε is a strict upper bound for $S_a(\delta)$. Now for any z s.t. $\delta := |z-a| < \gamma$, we have (since $|a-a| = 0 < \delta$) that $|f(z) - f(a)| \in S_a(\delta)$, so $|f(z) - f(a)| < \varepsilon$.

11) $\exists$ $L > 0$ s.t. $\forall$ $x, y \in [a, \infty)$, we have

$$|f(x) - f(y)| \leq L|x-y| \quad , \quad w/ \; a > 0.$$

want: $\forall$ $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. whenever $|x-y| < \delta$, $\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| < \varepsilon$.

Fix $\varepsilon > 0$. Let $\delta = \frac{\varepsilon a^2}{2La + |f(a)|}$ (Note: this is > 0).

Now,

$$\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| = \left|\frac{y f(x) - x f(y)}{xy}\right| = \frac{|y f(x) - y f(y) + y f(y) - x f(y)|}{|xy|}$$

$$\leq \frac{|y(f(x) - f(y))| + |f(y)(x-y)|}{|xy|} = \frac{|y||f(x) - f(y)| + |f(y)||x-y|}{|xy|}$$

$$\leq \frac{|y| L|x-y| + |f(y)||x-y|}{|xy|} = |x-y|\left(\frac{L|\cancel{y}|}{|x\cancel{y}|} + \frac{|f(y)|}{|xy|}\right)$$

Need to do something about $f(y)$: See how far f can move you from a.

$$|f(y)| = |f(y) - f(a) + f(a)| \leq |f(y) - f(a)| + |f(a)| \leq L|y-a| + |f(a)|$$

$$\underset{(y \geq a)}{=} L(y-a) + |f(a)| \underset{(a \geq 0)}{\leq} Ly + |f(a)|$$

$$\text{So } \left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| \leq |x-y|\left(\frac{L}{x} + \frac{Ly}{xy} + \frac{|f(a)|}{xy}\right)$$

$x, y \geq 0$ so $|\cdot|$ can be dropped

$$= |x-y|\left(\frac{2L}{x} + \frac{|f(a)|}{xy}\right) \underset{x,y \geq a}{\leq} |x-y|\left(\frac{2L}{a} + \frac{|f(a)|}{a^2}\right) = |x-y|\left(\frac{2La + |f(a)|}{a^2}\right)$$

$$< \delta\left(\frac{2La + |f(a)|}{a^2}\right) = \varepsilon. \quad \square$$

12) $f(x) = \begin{cases} x^2 & x \in Q \\ 0 & x \notin Q \end{cases}$

Need to examine $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)-0}{x} = \lim_{x\to 0} \frac{f(x)}{x}$

$\frac{f(x)}{x} \underset{\text{for } x\neq 0}{=} \begin{cases} x & x \in Q \\ 0 & x \notin Q \end{cases}$

This is a piecewise function whose pieces have the same limit, (0), at 0,

so $\lim_{x\to 0} \frac{f(x)}{x} = 0.$

13) Suppose $|f(x)|$ is differentiable at a. with $f(a)=0$. Show that $f'(a)=0$.

suppose $f'(a)\neq 0$, then $|f'(a)| = L > 0$. $f(a)=0$ $|\cdot|$ cts

Since $|\cdot|$ is cts, $|f'(a)| = \left|\lim_{x\to a} \frac{f(x)}{x-a}\right| = \lim_{x\to a} \left|\frac{f(x)}{x-a}\right|$

we will show that $|f|$ is not diff'ble at a. well,

$\lim_{x\to a} \frac{|f(x)|-|f(a)|}{x-a} = \lim_{x\to a} \frac{|f(x)|}{x-a}$ Consider sequences $(x_n):=(a+\frac{1}{n})$ $(y_n):=(a-\frac{1}{n})$

If the above limit exists, both sequences must give the same result.

$\lim_{n\to\infty} \frac{|f(x_n)|}{x_n-a} = \lim_{n\to\infty} \frac{|f(x_n)|}{1/n} = \left|\lim_{n\to\infty} \frac{f(x_n)}{1/n}\right| = |f'(a)| = L > 0$

$\lim_{n\to\infty} \frac{|f(y_n)|}{y_n-a} = \lim_{n\to\infty} \frac{|f(y_n)|}{-\frac{1}{n}} = -\left|\lim_{n\to\infty} \frac{f(y_n)}{1/n}\right| = -|f'(a)| = -L < 0.$

So $|f|$ is not differentiable at a.

14) $\exists$ M, $a>1$ s.t $\forall$ $x,y\in\mathbb{R}$
$|f(x)-f(y)|\leq M|x-y|^a$
Show f is const.

we will show that f is diff'ble at every $c\in\mathbb{R}$ and that $f'(c)=0$. Let $c\in\mathbb{R}$.

$$\left|\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right| = \lim_{x\to c}\left|\frac{f(x)-f(c)}{x-c}\right| \leq \lim_{x\to c}\frac{M|x-c|^a}{|x-c|} = M\lim_{x\to c}\frac{|x-c|^a}{|x-c|}$$

Since $a>1$, $\lim_{x\to c}\frac{|x-c|^a}{|x-c|}=0$ so $\left|\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right|\leq M\cdot 0=0$

so f is differentiable at c and $f'(c)=0$. This holds $\forall$ $c\in\mathbb{R}$, so f is constant.

15) Let $f(x)$, $g(x)$ convex, f increasing. Prove $f(g(x))$ is convex.

Let $x_1, x_2\in\mathbb{R}$, $t\in[0,1]$ want
$f(g((1-t)x_1+tx_2))\leq(1-t)f(g(x_1))+f(g(x_2))$

well, g is convex, so $g((1-t)x_1+tx_2))\leq(1-t)g(x_1)+tg(x_2)$

f is increasing so $f(g((1-t)x_1+tx_2))\leq f((1-t)\underbrace{g(x_1)}_{y_1}+t\underbrace{g(x_2)}_{y_2})$

f is convex, so $f((1-t)y_1+ty_2)\leq(1-t)f(y_1)+tf(y_2)$
$=(1-t)f(g(x_1))+tf(g(x_2))$ as desired.

16) $f:[0,1]\to\mathbb{R}$ is cts, $\int_0^x f=\int_x^1 f$ $\forall$ $x\in[0,1]$ want $f(x)=0$ $\forall$ $x\in[0,1]$

Note that $\forall$ $x\in[0,1]$, $\int_0^1 f=\int_0^x f+\int_x^1 f$ (Thm 7.2.13)
$=2\int_0^x f$. Setting $x=0\Rightarrow\int_0^1 f=0$.
($\int_0^0 f=0$ by def)

Now, $\int_0^x f+\int_x^1 f=0$, so
$\int_0^x f=\int_x^1 f=-\int_0^x f\Rightarrow\int_0^x f=0$ $\forall x\in[0,1]$ ...

For $x, y \in [0,1]$, $x < y$, we have ${}_0\int^x f = 0 = {}_0\int^y f$
and ${}_x\int^y f = {}_0\int^y f - {}_0\int^x f = 0$, Finally, for $c \in [0,1]$ if

$f(c) > 0$, by problem 9, $\exists$ $a, b \in [0,1]$ s.t. $f(x) > 0$ $\forall x \in [a,b]$
$\Rightarrow$ ${}_a\int^b f > 0$, but this is a contradiction.

if $f(c) < 0$, then $-f(c) > 0$ so $-{}_a\int^b f > 0$ (where a, b are as above)
so ${}_a\int^b f < 0$, but this is a contradiction.

So $f(c) = 0$. This holds $\forall$ $c \in [0,1]$, since c was arbitrary.

$\square$