Uniformization of Sierpiński carpets by square carpets

Dimitrios Ntalampekos
dimitrisnt@math.ucla.edu

UCLA

Quasisymmetric uniformization

Uniformization is the problem of transforming a given metric space X to a *canonical space* with a map that preserves the geometry. In the metric space setting we are interested in *quasisymmetric maps*.

A homeomorphism $f : X \to Y$ between metric spaces X, Y is a **quasisymmetry** if there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ called the *distortion function* such that for every triple $x, y, z \in X$:

$$\frac{d(x,y)}{d(x,z)} \leq t \quad \text{implies} \quad \frac{d(f(x),f(y))}{d(f(x),f(z))} \leq \eta(t).$$

"A quasisymmetry quasi-preserves *relative distances*, instead of absolute distances".

Sierpiński carpets

Construction of a planar Sierpiński carpet S:

Let $\Omega \subset \mathbb{C}$ be a Jordan region, and $Q_i \subset \Omega, i \in \mathbb{N}$ be Jordan regions such that:

- $\overline{Q}_i \cap \overline{Q}_j = \emptyset$ and $\overline{Q}_i \cap \partial\Omega = \emptyset$
- $\operatorname{diam}(Q_i) \to 0$
- $S := \overline{\Omega} \setminus \bigcup_{i=1}^{\infty} Q_i$ has empty interior.

$\partial Q_i, \partial\Omega =:$ peripheral circles of the carpet S.

Fact: All Sierpiński carpets are homeomorphic to each other [5].

Geometric assumptions

A Jordan curve J is a k-**quasicircle** if for every two points $x, y \in J$ there exists an arc $\gamma \subset J$ connecting x and y such that

$$\operatorname{diam}(\gamma) \leq k|x-y|.$$

Two continua E, F are δ-**relatively separated** if

$$\frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}} \geq \delta.$$

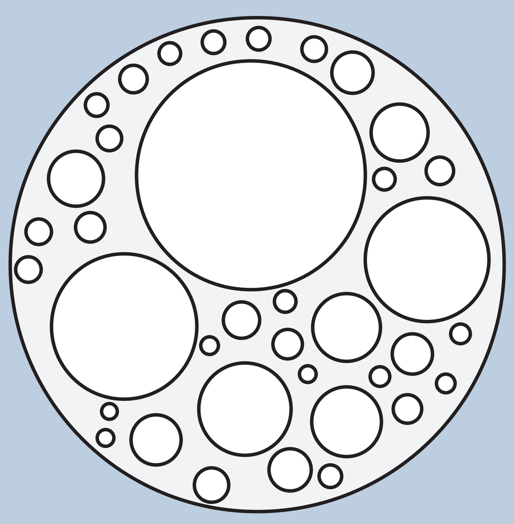
The peripheral circles of a carpet S are *uniform quasicircles* if they all are k-quasicircles for some $k > 0$.

The peripheral circles are *uniformly relatively separated* if every pair of them is δ-relatively separated for some $\delta > 0$.

Round carpets

Bonk [1] proved the following uniformization result for carpets:

Theorem. *Let $S \subset \mathbb{C}$ be a Sierpiński carpet whose peripheral circles are uniformly relatively separated, uniform quasicircles. Then there exists a quasisymmetry from S onto a round carpet.*



Main Theorem

A **square carpet** is a carpet whose peripheral circles are squares, except possibly for $\partial\Omega$, which is a rectangle, and all their sides are parallel to the coordinate axes. The main result in [3] is the following:

Theorem. *Let $S \subset \mathbb{C}$ be a Sierpiński carpet whose peripheral circles are uniformly relatively separated, uniform quasicircles. Furthermore, assume that Area$(S) = 0$. Then there exists a quasisymmetry from S onto a square carpet.*

Why square carpets? Square carpets arise naturally as *extremal domains* for a *minimizing problem*.

Remark. If we remove the assumption of uniform relative separation, and weaken the assumption of uniform quasicircles to (e.g.) uniform *John domains*, then the uniformizing map is *not* quasisymmetric in general, but it is "quasiconformal" in a discrete sense.

Idea of proof

We develop a theory of harmonic functions on Sierpiński carpets in [2], and then follow the steps of Rajala [4] to prove the uniformization theorem.

A function $u : S \to \mathbb{R}$ lies in the *carpet-Sobolev space* $\mathcal{W}^{1,2}(S)$ if it satisfies the L^2-integrability conditions

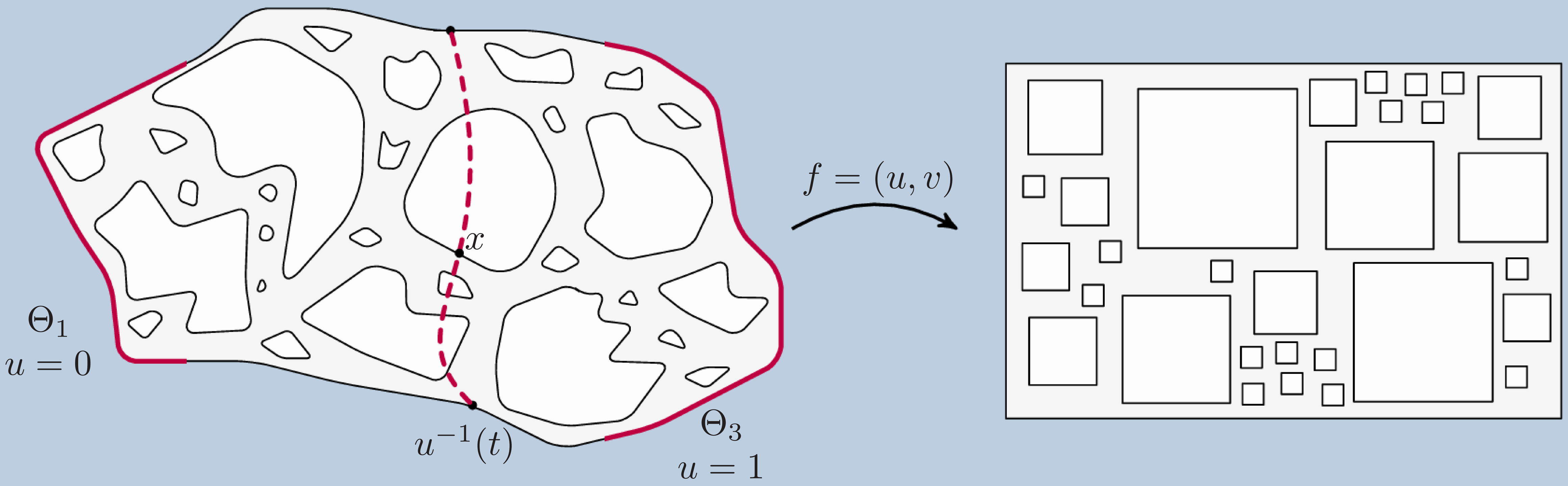
$$\sum_{i=1}^{\infty} \sup_{\partial Q_i}(u)^2 \operatorname{diam}(Q_i)^2 < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \operatorname*{osc}_{\partial Q_i}(u)^2 < \infty,$$

and $\{\operatorname{osc}_{\partial Q_i}(u)\}_{i \in \mathbb{N}}$ is an *upper gradient* of u, i.e. for *almost every* path $\gamma \subset \overline{\Omega}$ and points $x, y \in \gamma \cap S$ we have

$$|u(x) - u(y)| \leq \sum_{i: \gamma \cap Q_i \neq \emptyset} \operatorname*{osc}_{\partial Q_i}(u).$$

The *Dirichlet Energy* of u is $D(u) := \sum_{i=1}^{\infty} \operatorname{osc}_{\partial Q_i}(u)^2$.

We mark two sides Θ_1, Θ_3 on $\partial\Omega$ as in the figure. Consider the problem of *minimizing* the Dirichlet Energy $D(u)$ among all functions $u \in \mathcal{W}^{1,2}(S)$ with $u \equiv 0$ on Θ_1 and $u \equiv 1$ on Θ_3. A solution to this problem exists, is *carpet-harmonic*, and it is continuous on S. This is the **real part** of the uniformizing function f.



To define the **harmonic conjugate** v of u we need to "integrate ∇u over the level sets of u". If $x \in u^{-1}(t)$ and γ_x is the subpath of $u^{-1}(t)$ from the "bottom" to x, then

$$v(x) := \sum_{i: \gamma_x \cap Q_i \neq \emptyset} \operatorname*{osc}_{\partial Q_i}(u)$$

This definition works for a.e. level t, and gives a continuous function v in $\mathcal{W}^{1,2}(S)$.

For the pair $f = (u, v)$ one has:

(i) It is a **homeomorphism** onto a square carpet $\mathcal{R}$, contained in $[0,1] \times [0, D(u)]$.

(ii) It is **"quasiconformal"** in a discrete sense, namely it preserves a notion of *carpet-modulus*. This is because f is "absolutely continuous on lines" and maps the peripheral circles of S to the peripheral circles of $\mathcal{R}$.

(iii) It is **quasisymmetric**. This follows from the principle that "a quasiconformal map from a Loewner space onto a LLC space is quasisymmetric". That principle also holds in our discrete setting, and uses the geometric assumptions of uniform relative separation and uniform quasicircles.

References

[1] M. Bonk, *Uniformization of Sierpiński carpets in the plane*, Invent. Math.

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[4] K. Rajala, *Uniformization of two-dimensional metric surfaces*, Invent. Math.

[5] G.T. Whyburn, *Topological characterization of the Sierpiński curve*, Fund. Math.