Uniformization of metric surfaces of finite area

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Quasiworld Workshop
University of Helsinki

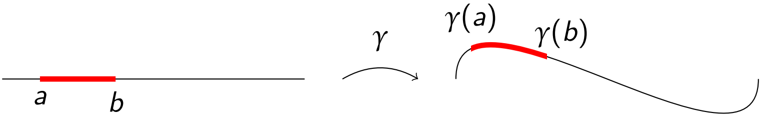
August 14 - 18, 2023

The uniformization problem

Question

*How can we parametrize a **curve** of finite **length** in a natural way?*

Arclength parametrization



Lipschitz property: $|\gamma(a) - \gamma(b)| \leq |a - b|$

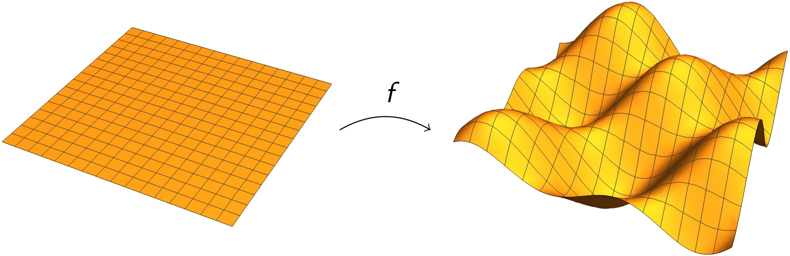
Problem

*How can we parametrize a **surface** of finite **area** in a natural way?*

Smooth surfaces

Theorem (Uniformization Theorem, Koebe, Poincaré 1907)

Every simply connected Riemannian surface can be conformally uniformized by the complex ***plane*** *or the unit* ***disk*** *or the Riemann* ***sphere****.*

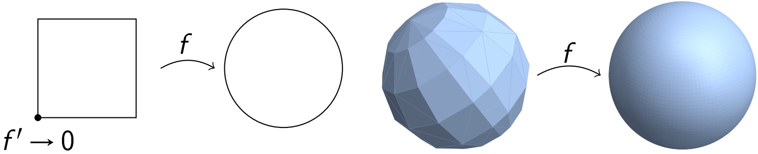


f **conformal:** balls $\longrightarrow$ balls (or squares $\to$ squares) in infinitesimal scale

$\Longrightarrow$ f locally bi-Lipschitz $C^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq C\ell(\gamma)$

Bi-Lipschitz parametrization

⚠ In non-smooth surfaces conformal parametrizations are *not bi-Lipschitz*!



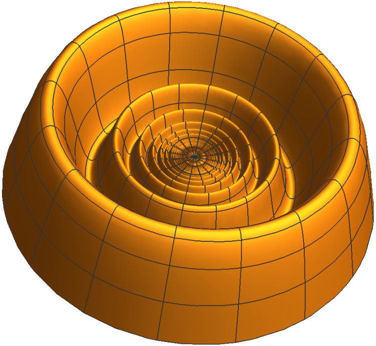
Existence of local bi-Lipschitz parametrizations:

- Bounds on flatness (Toro, David,...)
- Existence of flat forms (Heinonen, Sullivan, Keith,..)
- Curvature bounds (Fu, Bonk, Lang,...)

Surfaces with singularities

Lipschitz parametrization $f : \mathbb{C} \to X \implies \ell(f \circ \gamma) \leq C\ell(\gamma)$

$\implies$ Every two points can be joined with a curve of **finite length**



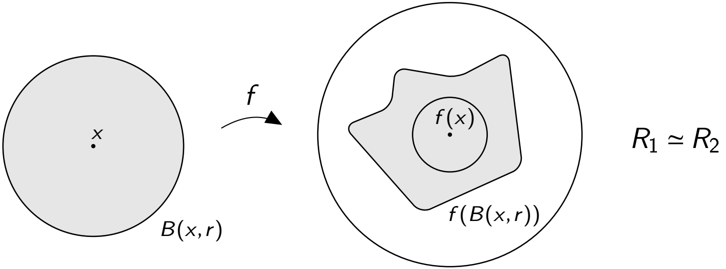
1. Finite area
2. Smooth except for one point P
3. Every curve passing through P has infinite length

$\Downarrow$

No Lipschitz parametrization

$f : X \to Y$ homeomorphism between metric spaces

Quasiconformal: preserves shapes infinitesimally:



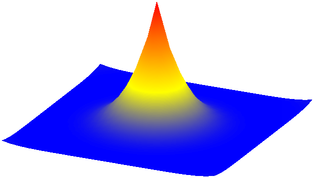
Quasisymmetric: preserves shapes in all scales.

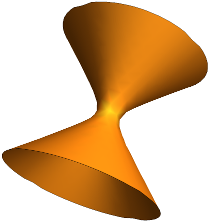
Quasisymmetric uniformization

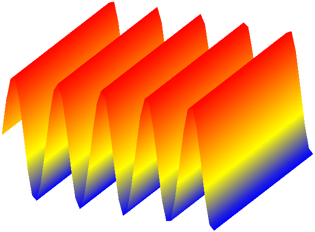
Theorem (Bonk–Kleiner 2002)

If a metric sphere X is ***Ahlfors*** *2-****regular*** *and* ***LLC****, then there exists a* ***quasisymmetric*** *map $f:\widehat{\mathbb{C}}\to X$.*

- **Ahlfors** 2-**regular:** $C^{-1}r^2 \leq \mu(B(x,r)) \leq Cr^2$
- **LLC (Linearly Locally Connected):** no *cusps*, *thin bottlenecks*, *dense wrinkles*







Quasisymmetric uniformization

Methods of proof:

- Through circle packings (Bonk–Kleiner)
- Through quasiconformal uniformization (Rajala)
- Through solution to Plateau's problem (Lytchak–Wenger)

Generalizations to other surfaces:

- Plane, disk, half-plane (Wildrick)
- Compact surfaces (Geyer–Wildrick, Ikonen, Fitzi–Meier)
- Domains (Merenkov–Wildrick, Rajala–Rasimus, Rehmert)

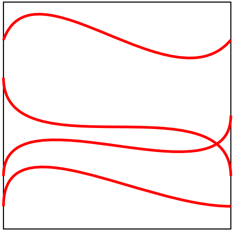
Geometric definition of quasiconformality

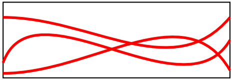
X metric surface of locally finite area (Hausdorff 2-measure)

Γ family of curves in X

$\rho : X \to [0,\infty]$ is *admissible* for Γ if $\int_\gamma \rho \, ds \geq 1$ for all $\gamma \in \Gamma$

$\operatorname{Mod}\Gamma = \inf_\rho \int_X \rho^2 \, d\mathcal{H}^2 \longrightarrow$ Outer measure on curve families

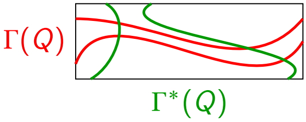




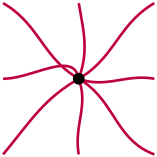
f **conformal:** $\operatorname{Mod}\Gamma = \operatorname{Mod} f(\Gamma)$

f **quasiconformal:** $K^{-1}\operatorname{Mod}\Gamma \leq \operatorname{Mod} f(\Gamma) \leq K \operatorname{Mod}\Gamma$

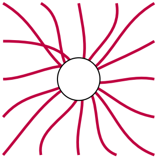
Properties of modulus in the plane



$$\operatorname{Mod}\Gamma(Q)\cdot\operatorname{Mod}\Gamma^*(Q)=1$$



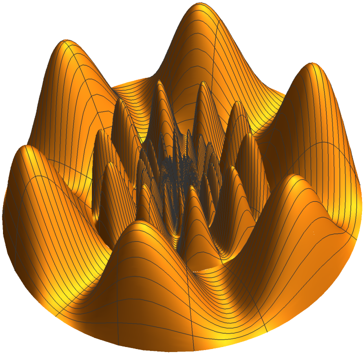
$$\operatorname{Mod}\Gamma=0$$



$$\operatorname{Mod}\Gamma>0$$

(Quasi)conformal parametrization $f : \mathbb{C} \to X$

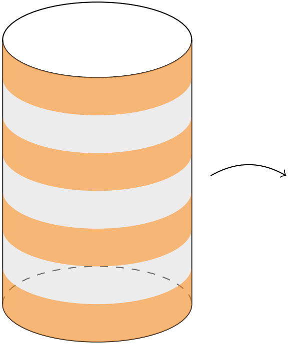
$\Longrightarrow$ The family of (non-constant) curves passing through each point has **modulus zero**



1. Finite area
2. Smooth except for one point P
3. The family of curves passing through P has positive modulus.

$\Downarrow$

No quasiconformal parametrization



Magic Ball
Designed by:
Yuri Shumakov
Presented by:
Jo Nakashima

1. Length-isometric to cylinder outside poles
2. The family of curves through poles has positive modulus
3. Not quasiconformal to sphere

Question

Is this the only enemy?

Question

Is this the only enemy?

Let $C \subset \mathbb{R}^2$ Cantor set. Set $\omega = \chi_{\mathbb{R}^2 \setminus C}$.

$$d_\omega(x,y) = \inf_\gamma \int_\gamma \omega \, ds$$

$(\mathbb{R}^2, d_\omega)$ is homeomorphic to $\mathbb{R}^2$

If $|C| > 0$ then $(\mathbb{R}^2, d_\omega)$ is not quasiconformal to $\mathbb{R}^2$

Near density points

$$\operatorname{Mod}\Gamma(Q)\operatorname{Mod}\Gamma^*(Q) \to \infty$$

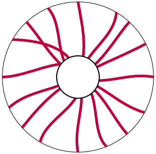
Quasiconformal uniformization

Theorem (Rajala 2017)

*Let X be a metric sphere of **finite area**. There exists a **quasiconformal** map $f:\widehat{\mathbb{C}}\to X$ if and only if X is **reciprocal**.*

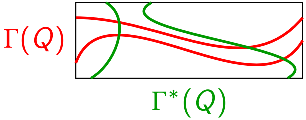
Reciprocity conditions:

① The family of non-constant curves passing through each point x has modulus zero.



$$\lim_{r\to 0}\operatorname{Mod}\Gamma(B(x,r),X\setminus B(x,R))=0$$

② For each topological quadrilateral Q:



$$\kappa^{-1}\leq \operatorname{Mod}\Gamma(Q)\cdot\operatorname{Mod}\Gamma^*(Q)\leq\kappa$$

Quasiconformal uniformization

- If X is reciprocal, there exists f with $\frac{\pi}{4}\operatorname{Mod}\Gamma \leq \operatorname{Mod} f(\Gamma) \leq \frac{\pi}{2}\operatorname{Mod}\Gamma$ (Rajala, Romney)
  Optimal constants attained by $id : \mathbb{R}^2 \to X = (\mathbb{R}^2, \ell^\infty)$
- X Ahlfors 2-regular and LLC
  $\Longrightarrow$ Quasiconformal maps are quasisymmetric
  $\Longrightarrow$ Bonk–Kleiner Theorem
- For **every** surface
  $\kappa^{-1} \leq \operatorname{Mod}\Gamma(Q) \cdot \operatorname{Mod}\Gamma^*(Q)$ (Rajala–Romney)
  $\kappa^{-1} = (\pi/4)^2$ (Eriksson-Bique–Poggi-Corradini)
- X is reciprocal if and only if
  $\operatorname{Mod}\Gamma(Q) \cdot \operatorname{Mod}\Gamma^*(Q) \leq \kappa$ (N.–Romney)
- If the modulus of curves passing through each point is zero, then X is not necessarily reciprocal. (N.–Romney)

Uniformization of general surfaces

Problem (Rajala–Wenger)

*Let X be a metric sphere of **finite area**. Does there exist a **weakly quasiconformal** map $f : \widehat{\mathbb{C}} \to X$?*

Weakly quasiconformal map:

1. Uniform limit of homeomorphisms
2. $\operatorname{Mod} \Gamma \leq K \operatorname{Mod} f(\Gamma)$

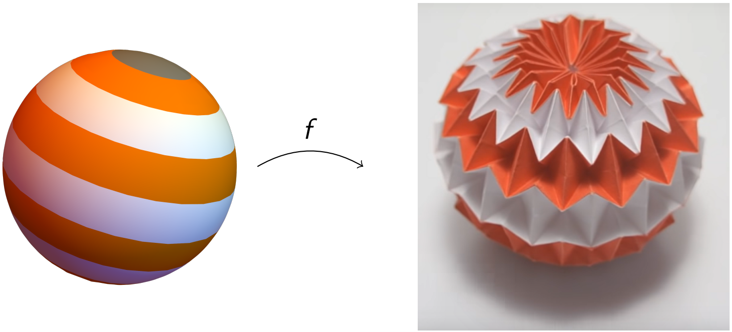
Theorem (N.–Romney 2021, Meier–Wenger 2021)

Yes for length surfaces.

Theorem (N.–Romney 2022)

Yes for all surfaces.

Example



1. f is weakly quasiconformal
2. f is **not injective** in black balls around poles
3. f is **conformal** outside black balls

Weakly quasiconformal uniformization

Theorem (N.–Romney 2022)

Let X be a metric surface of locally finite area.

- *There exists a complete Riemannian surface Z of constant curvature.*
- *Z is homeomorphic to X.*
- *There exists a $\frac{4}{\pi}$-WQC map $f\colon Z \to X$.*

- f is QC if and only if X is reciprocal $\implies$ Rajala's Theorem
- X is Ahlfors 2-regular and LLC sphere
  $\implies$ f is quasisymmetric
  $\implies$ Bonk–Kleiner Theorem

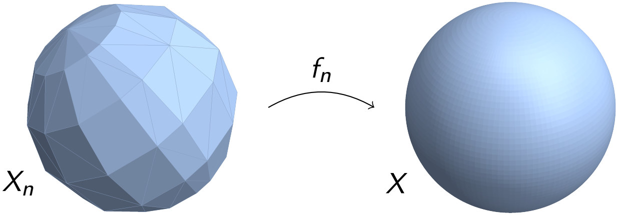
Approximation by polyhedral surfaces

Theorem (N.–Romney 2021, 2022)

Let X be a metric sphere of finite area. There exists a sequence X_n of polyhedral spheres and approximately isometric homeomorphisms $f_n\colon X_n \to X$ such that

$$\limsup_{n\to\infty} \left|f_n^{-1}(A)\right| \leq K|A|$$

for each compact set $A \subset X$, where K is a uniform constant.

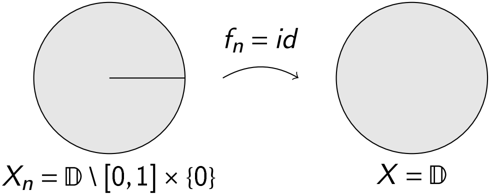


Proof of WQC uniformization

- Consider polyhedral **spheres** $X_n \to X$
- Orientable polyhedral surfaces are Riemann surfaces
- Classical uniformization theorem
  $\implies$ There exist conformal parametrizations $g_n \colon \widehat{\mathbb{C}} \to X_n$.
- Area bounds on X_n
  - g_n is equicontinuous
  - $|Dg_n|$ bounded in L^2
- The maps g_n (sub)converge to a WQC map $g \colon \widehat{\mathbb{C}} \to X$.

Proof of WQC uniformization

⚠ Proof scheme fails for general surfaces!



$g_n \colon \mathbb{D} \to X_n$ conformal maps
do not converge to WQC map $g \colon \mathbb{D} \to X$

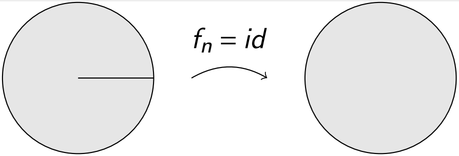
Approximation by polyhedral surfaces

Theorem (N.–Romney 2021, 2022)

Let X be a metric surface of locally finite area. There exists a sequence X_n of polyhedral surfaces and approximately isometric embeddings $f_n\colon X_n \to X$ such that

$$\limsup_{n\to\infty} \left|f_n^{-1}(A)\right| \leq K|A|$$

for each compact set $A \subset X$, where K is a uniform constant. Moreover, there exist approximately isometric ***retractions*** *$R_n\colon X \to f_n(X_n)$.*



$f_n = id$

$X_n = \mathbb{D} \setminus [0,1]\times\{0\}$ $\qquad$ $X = \mathbb{D}$

There exists no retraction $R\colon X \to X_n$

Conjecture: Optimal constant $K = 4/\pi$ attained by $X = (\mathbb{R}^2, \ell^\infty)$.

Proof of polyhedral approximation

For simplicity assume that X has a **length metric**:

$$d(x,y) = \inf_{\gamma} \ell(\gamma)$$

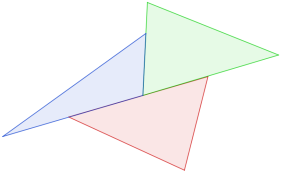
Step 1: Triangulate X

Theorem (Creutz–Romney 2022)

Let X be a length surface with polygonal boundary. For each $\varepsilon > 0$ there exists a convex triangulation of X with mesh $< \varepsilon$.

Triangulation:

- $X = \bigcup_{T \in \mathscr{T}} T$, non-overlapping, locally finite
- T Jordan region, ∂T union of three geodesics
- Edges and vertices do not match exactly



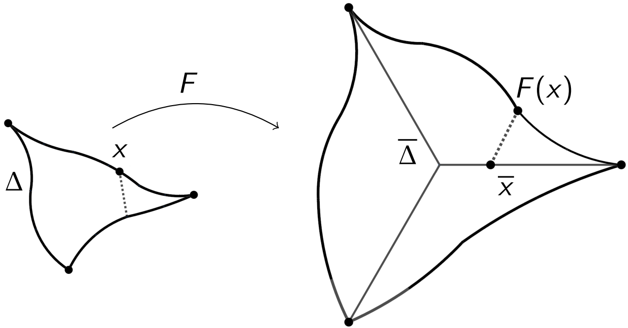
Idea: Replace each triangular region T with a polyhedral surface S such that $|S| \leq C|T|$ and $\mathrm{diam}(S) \leq C \mathrm{diam}(T)$

Step 2: Bi-Lipschitz embedding of triangles into the plane

Metric triangle $\Delta = \partial T$: homeomorphic to $\mathbb{S}^1$, union of three non-overlapping geodesics

Proposition (N.–Romney)

Every metric triangle is 4-bi-Lipschitz embeddable into $\mathbb{R}^2$.



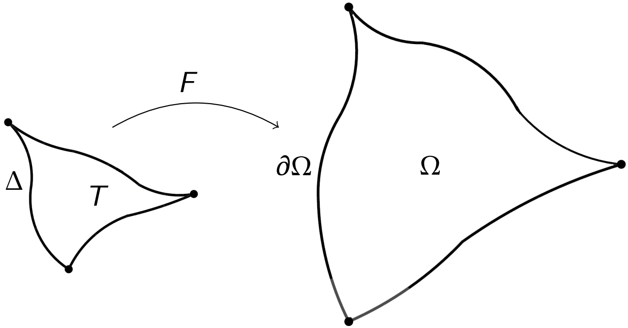
Idea: Construct polyhedral surface S in the plane and glue it to the surface X via F

Step 3: Area estimate

Theorem

Let T be a metric closed disk with $\Delta = \partial T$. If $F : \Delta \to \partial\Omega \subset \mathbb{R}^2$ is an L-Lipschitz homeomorphism, then

$$|\Omega| \leq \frac{4L^2}{\pi}|T|.$$



Non-length surfaces

We define the **extended length metric** $\overline{d}\colon X \times X \to [0,\infty]$

$$\overline{d}(x,y) = \inf_{\gamma} \ell_d(\gamma)$$

- $d \leq \overline{d} \leq \infty$
- If X has locally finite area, then $\overline{d}(x,y) < \infty$ for a dense set of $x,y \in X$
- $\overline{d}$ might not be continuous with respect to d

Idea: Apply previous proof strategy to the "length metric" $\overline{d}$

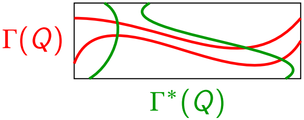
① Simplification of definition of reciprocal surfaces (N.–Romney)

Theorem (N.–Romney)

*A metric surface of locally finite area is **reciprocal** if and only if there exists $\kappa > 0$ such that*

$$\operatorname{Mod}\Gamma(Q)\cdot\operatorname{Mod}\Gamma^*(Q)\leq\kappa$$

for each quadrilateral Q.



(2) Coarea inequality on surfaces without assumptions (Esmayli–Ikonen–Rajala, Meier–N.)

Theorem (Esmayli–Ikonen–Rajala 2022)

*Let X be a metric surface of locally finite area and $u\colon X \to \mathbb{R}$ be a **monotone** function with weak upper gradient $\rho \in L^p_{\mathrm{loc}}(X)$, $p \in [1,\infty]$. Then*

$$\int \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq C \int g \rho \, d\mathcal{H}^2$$

for each Borel function $g\colon X \to [0,\infty]$.

⚠ Can fail for Lipschitz functions! (True for smooth X)

Applications of uniformization

③ Lipschitz-Volume rigidity (Meier–N.)

Theorem (Folklore)

Let X, Y be closed Riemannian n-manifolds with $|X| = |Y|$. Then every 1-Lipschitz map from X onto Y is an isometry.

Theorem (Meier–N. 2023)

*Let X be a closed **metric** surface and Y be a closed Riemannian surface with $|X| = |Y|$. Then every 1-Lipschitz map from X onto Y is an isometry.*

Open problems

Problem

Classify metric surfaces of locally finite area up to QC maps.

Is there a Riemannian surface Z and a **degenerate conformal weight** ω such that (Z, d_ω) is QC to X?

$$d_\omega(x,y) = \inf_\gamma \int_\gamma \omega \, ds$$

Problem (Le Donne)

If X is a length surface, is there a length-isometric/BLD embedding into $\mathbb{R}^N$?

Yes for Heisenberg group (Le Donne)

Thank you!

Happy birthday Mario!