Differential Characters for K-theory

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Dedicated to Jeff Cheeger for his 65th birthday

Abstract. We describe a sequence of results that begins with the introduction of differential characters on singular cycles in the seventies motivated by the search for invariants of geometry or more generally bundles with connections. The sequence passes through an Eilenberg-Steenrod type uniqueness result for ordinary differential cohomology using these characters and a construction of a differential K-theory using Grothendieck's construction on classes of complex bundles with connection. The last element of the sequence returns full circle with a differential character definition of differential K-theory. The cycles in this definition of characters for differential K-theory are closed smooth manifolds provided with complex structures and hermitian connections on their stable tangent bundles.

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Background

Maps from smooth singular cycles in a manifold X to $\mathbb{R}/\mathbb{Z}$ satisfying a differential form homology variation property, termed Differential Characters and denoted by $\hat{H}^k(M, \mathbb{R}/\mathbb{Z})$, were introduced and studied by Jeff Cheeger and one of us in the early 70s [1]. Precisely,

$$1) \qquad \hat{H}^i(M, \mathbb{R}/\mathbb{Z}) = \left\{ f \in \operatorname{Hom}(Y_{i-1}(X), \mathbb{R}/\mathbb{Z}) \,|\, f(\partial a) = \int_a \omega_f \mod \mathbb{Z} \right\}$$

where $Y_{i-1}(X)$ denotes smooth cycles of dim $i-1$. It follows from the definition that ω_f is uniquely determined by f and is contained in $\wedge^i_{\mathbb{Z}}$, closed k-forms with integral periods. $\hat{H}(\mathbb{R}/\mathbb{Z})$ comprises a functor from the smooth category into $\mathbb{Z}$-graded rings and may be shown to satisfy the following commutative diagram of natural transformations:

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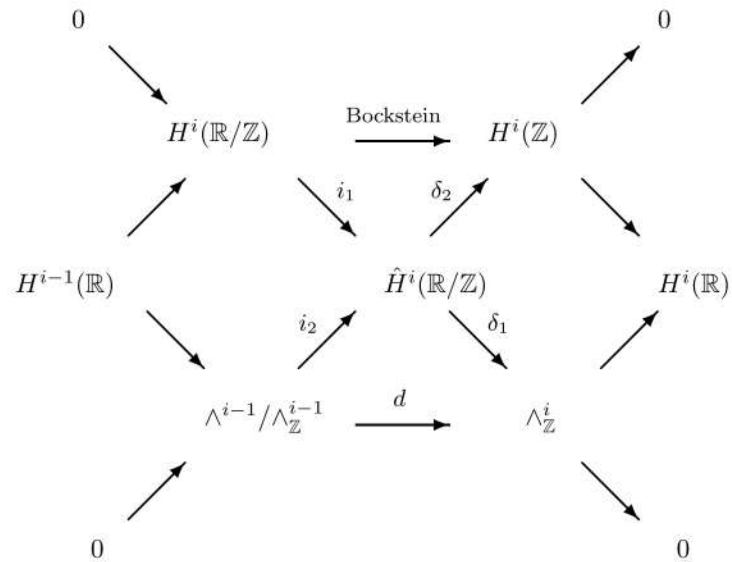
2)



where the upper outside sequence is the Bockstein exact sequence, and the lower outside sequence is easily derived from the deRham theorem.

In one application it was shown that in a principal G-bundle with connection, the pair consisting of an invariant polynomial on G whose Chern-Weil form defines a real class with integral periods, together with the choice of an integral characteristic class with the same real image determines a differential character on the base manifold. δ_1 and δ_2 map the character into the form and the integral class respectively. The diagram in 2) shows these data to determine the character modulo an element in a torus of dimension equal to the one lower odd Betti number. Should both the form and the class vanish the character lies in this torus and may be non-zero even if the bundle is locally flat.

Other constructions arising in different contexts have been made and then shown equivalent to differential characters in the smooth context. See [20] for a discussion of Deligne cohomology somewhat in the spirit of the summary here. Other examples related to Harvey-Lawson spark complexes and Lawson homology can be found in the references and discussion of [21]. A third set of examples appears in the circle of ideas combining algebraic geometry, arithmetic and analysis discussed in [22] and its references. In the smooth category all these functors satisfied 2) and are naturally equivalent to $\hat{H}(\mathbb{R}/\mathbb{Z})$. This observation inspired the uniqueness result in [2], discussed below.

Differential cohomology

In the modern viewpoint differential characters are one instantiation of a contravariant functor called ordinary differential cohomology, a fibre product functor that combines differential forms with ordinary integer cohomology defined, say, using singular cochains in a manner essentially as depicted in 2). Exotic or generalized differential cohomology theory also makes sense as a fibre product functor combining differential forms with any exotic or generalized cohomology theory [1] defined, say, using spectra [19]. This possibility is based on two neat facts: the first is that any finite type generalized cohomology theory tensor a field k of characteristic zero is canonically isomorphic to ordinary cohomology theory with graded coefficients in the exotic theory of a point tensor k. The second is the well-known fact that real cohomology for manifolds is canonically described by the complex of differential forms. See Hopkins and Singer [5] for the general construction using spectra of the fibre product of differential forms with any finite type generalized cohomology theory.

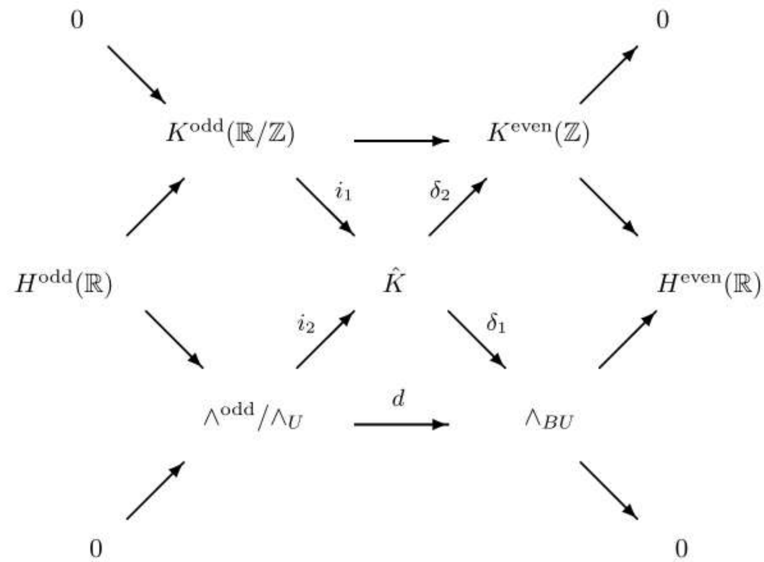
In a second paper [2] the present authors verified the current viewpoint on differential characters by showing ordinary differential cohomology as a contravariant functor on the smooth category is uniquely characterized by the diagram in 2). Specifically it is shown that any abstract theory satisfying 2) is canonically isomorphic to differential characters.

Differential K-theory via structured bundles

In a third paper [3] the present authors constructed a differential geometric instantiation of differential K-theory in even degrees. This was defined abstractly in [5] by combining total even-dimensional differential forms with the spectrum of the generalized cohomology theory associated to K-theory. The geometric construction of [3] was made using stable equivalence classes of complex hermitian bundles with unitary connections. The equivalence relation in this construction combines strict isomorphism together with stabilization and a further equivalence (termed CS) whereby one connection can be changed to another connection if and only if the Chern-Simons difference form is exact. Such objects are called Structured Bundles, and their isomorphism classes over a given manifold naturally form a commutative semi-ring. Applying the Grothedieck construction yields a commutative ring-valued functor $\hat{K}$. This construction should be contrasted with the first reference in [22] and the fibre product definition of [11]. $\hat{K}$ satisfies the following commutative diagram of natural transformations:

[1] A contravariant functor satisfying three out of the four Eilenberg-Steenrod axioms: a homotopy functor on pairs with the exact sequence and excision but omitting the point axiom.

3)



In the above, $\wedge_{BU}$ is the ring of total even closed forms cohomologous to the Chern characters of complex vector bundles over X. $\wedge_U = \{g^*(\Theta)\} + \wedge^{\text{odd}}_{exact}$, where Θ is the total odd bi-invariant form on U representing the universal transgression of the Chern character and $g : X \to U$ runs through all smooth maps. δ_2 is the map which forgets the connection, and δ_1 maps an element of $\hat{K}(X)$ into its Chern character form (well defined for a CS equivalence class). That $\text{Im}(i_1)$ is the kernel of δ_1 follows fairly easily from work in [23], and the construction and properties of i_2 are exposed in [3]. The upper exact sequence is the Bockstein in K-theory, and the lower exact sequence is easily derived from the deRham theorem. Again, should both the Chern character form and the element of $K(X)$ vanish, the element of $\hat{K}(X)$ lies in a torus of dimension the sum of the odd Betti numbers. Here, the lattice defining the torus is the cohomological image of $\wedge_U$ in $H^{\text{odd}}(\mathbb{R})$, commensurate but distinct from the lattice in the ordinary case. Changing lattices and torsion is the essential difference between differential K-theory and the appropriate sum of ordinary differential cohomologies.

K-characters

In a fourth paper [4] the present authors construct a character instantiation of differential K-theory, analogous to that of differential characters in ordinary differential cohomology. In place of singular cycles one uses representatives of complex bordism endowed with unitary connections on their stable tangent bundles.

To be precise, in complex bordism a Cycle in a smooth manifold X is a closed compact stably almost complex manifold M (termed an SAC) together with a smooth map of M into X. A compact SAC, S, with boundary diffeomorphic to M

induces a SAC structure on M. Should a map of M into X be extendable to a map of such an S into X, M is termed a Boundary. Of course all boundaries are cycles. A cycle together with a complex hermitian structure with unitary connection on a stabilized version of its tangent bundle will be termed an Enriched Cycle. A boundary M is called an Enriched Boundary (of S) if the hermitian structure and unitary connection on its stabilized tangent bundle is induced from similar data on the stabilized tangent bundle of S which data is of product form [2] in a collar neighborhood of M. Under disjoint union, isomorphism classes over X of enriched cycles and boundaries mapping into X form semi-groups, $EC(X)$ and $EB(X)$.

We also note that if M is an enriched cycle or boundary in X, and Q is an enriched cycle mapping to a point, then $Q \times M$ is an enriched cycle or boundary in X. Finally, $ES(X)$ and $EB(X)$ are graded by dimension of M, and it is important to distinguish between the even and odd cases. Thus ES^{even} and ES^{odd} have the obvious meaning. We define K-characters to be the abelian groups

$$\widehat{Kch}^{\text{even}}(X) = \{f \in \text{Hom}(EC^{\text{odd}}(X), \mathbb{R}/\mathbb{Z})\}$$

satisfying:

1. $f(\partial S) = \int_S \omega_f \wedge \text{Todd}(S) \bmod Z,$ where Todd(S) is its total Todd form with respect to the connection on its stabilized tangent bundle, and ω_f is a total even form on X pulled back to S.
2. $f(Q \times M) = \text{Todd}(Q) f(M),$ where $Q \in EC^{\text{even}}(\text{point})$.

$\widehat{Kch}^{\text{odd}}$ has the same definition but the cycles are even and ω_f is odd. In either case it is straightforward to show that

a) ω_f is closed, and uniquely determined by f, and
b) the pulled back cohomology class of ω_f cupped with the Todd class of an even cycle (even case) or odd cycle (odd case) takes integral value on the fundamental homology class of the cycle.

One may associate a K-character to a bundle E with unitary connection ∇. The value of the character on an odd cycle M in the base is the reduction mod 1 of the integral as computed in b) where ω_f is $ch(E)$, the Chern form of the bundle E with unitary connection ∇ and the manifold S with boundary M is constructed abstractly outside X via algebraic topology considerations. The integrality of the integral over the closed manifold V constructed by glueing two such choices of S together,

$$\int_V ch(E)\,\text{Todd}(V) \in Z$$

means the character associated to (E, ∇) by this construction is well defined.

[2]By product form we mean a combing of the collar neighborhood such that parallel transport along the strands is an isomorphism (including hermitian structure and unitary connection) between the stabilized version of the tangent bundle of S along its boundary with that over each interior slice of the collar neighborhood.

From the character of a bundle with connection one may compute invariants of the bundle E itself. These are homomorphisms of the complex bordism with $\mathbb{Q}/\mathbb{Z}$ coefficients into $\mathbb{Q}/\mathbb{Z}$ which assign values in $\mathbb{Z}/k$ to $\mathbb{Z}/k$-manifolds.

In more detail if ϕ denotes the character and if k times a cycle M in X bounds S in X, one forms the quantity $\phi(M) - [1/k \int_S ch(E)\,\mathrm{Todd}(S)]$ in $\mathbb{R}/\mathbb{Z}$. This actually lies in $\mathbb{Q}/\mathbb{Z}$ and is well defined on $\mathbb{Z}/k$-bordism, cf. [7].

Theorem T (topology). *The $\mathbb{Q}/\mathbb{Z}$ periods and the $\mathbb{Q}$ periods: $\int_V ch(E)\,\mathrm{Todd}(V)$ for even cycles V in X satisfy the* Todd *product rule (second property in the description of cycle above), they are compatible in that the complex bordism diagram commutes*

$$\begin{array}{ccc} \Omega(X,\mathbb{Q}) & \rightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \Omega(X,\mathbb{Q}/\mathbb{Z}) & \rightarrow & \mathbb{Q}/\mathbb{Z} \end{array}$$

and the invariants determine the complex bundle E up to stable isomorphism.

Moreover, any such compatible system of bordism periods satisfying the Todd *product rule comes from a complex bundle, cf.* [7].

Corollary. *The ω_f in the definition of K-character lies in $\wedge_{BU}$ in the even case and $\wedge_U$ in the odd case.*

Remark. Thus just as the original differential characters of a bundle with connection determine integral characteristic classes, the K-characters determine the bundle in integral K-theory itself.

A second application of K-characters is differential geometric. One may observe that the K-character of a bundle with connection is unchanged by strict isomorphism, stabilization, or by changing the connection by CS equivalence. Thus there is a well-defined map from differential theory $\hat{K}$ as defined in [3] to the set of K-characters denoted $\widehat{Kch}$.

Theorem G (geometry). *The K-character of a complex bundle with unitary connection determines its position in $\hat{K}$. Moreover any K-character comes from a complex bundle with unitary connection. In other words we have a canonical equivalence $\hat{K} \rightarrow \widehat{Kch}$.*

The proofs of Theorem T and Theorem G use two remarkable properties related to bordism and duality of complex K-theory itself. The first is the sixties result of Conner-Floyd [6] expressing K-homology of a space X (the homology theory Alexander dual to K-theory) by stably almost complex bordism of X tensored with the integers regarded as a module over the bordism of a point using the Todd genus ring homomorphism. The second property, also noted in the sixties [7], is that the homology theory constructed algebraically from K cohomology theory via Pontryagin duality by applying the functor $\mathrm{Hom}(\,,\mathbb{R}/\mathbb{Z})$ agrees with the Alexander dual homology theory. As a consequence one may show that the subgroup of even K-characters with $\omega_f = 0$ may be identified with $K(\mathbb{R}/\mathbb{Z})$. This leads to the exact sequence

$$4)\quad 0 \to K^{\text{odd}}(\mathbb{R}/\mathbb{Z}) \to \widehat{Kch}^{\text{even}} \to \wedge_{BU} \to 0$$

and the theorem follows from the Five Lemma using part of diagram 3).

The argument in the proof of Theorem G, together with a proof of the Mayer-Vietoris property for any abstract functor fitting into 3), (see arXiv 2010) immediately yields the K^{even} theory analog of the uniqueness theorem in [2]. Namely that in the even case any functor satisfying 3) is canonically isomorphic to K-characters[3].

Theorem G shows that $\widehat{Kch}^{\text{even}}$ must fit the diagram in 3), and indeed i_1, i_2 and δ_1 are obviously and intrinsically defined.

As an immediate consequence of the map from $\hat{K}$ to $\widehat{Kch}^{\text{even}}$ and the Atiyah-Patodi-Singer theorem we get

Theorem A (analysis). *Let V be a complex hermitian vector bundle over X with unitary connection, and let M be an enriched odd cycle in X, with the property that the unitary connection on its stabilized tangent bundle is consistent with the Levi-Civita connection on the tangent bundle itself associated with the Riemannian metric induced by the stabilized hermitian structure in the sense that they define the same $\hat{A}$ forms. Then the value on M of the K-character associated to V is equal to the η invariant of the spin $\mathbb{C}$ Dirac operator on M with coefficients in the pull back of V defined using the spin^c structure and line bundle connection coming from the SAC structure on M.*

A consequence of the character instantiation of even differential K-theory is a simple definition of the **wrong way map** for families. Namely, let $Y \to X$ be a projection, the fibres of which are compact even-dimensional SAC manifolds provided with a smooth family of hermitian structures and unitary connections on their stabilized tangent bundles. If M is an enriched odd cycle in X, one can show that the pre-image of M may itself be regarded as an enriched odd cycle in Y. This map from enriched cycles in X to enriched cycles in Y immediately provides a map from $\widehat{Kch}^{\text{even}}(Y)$ to $\widehat{Kch}^{\text{even}}(X)$. With this map in hand we have

Question (families index theorem for $\hat{K}$). *Can the wrong way map in $\hat{K}^{\text{even}}$ for a metrized family of stably almost complex manifolds of even dimension defined above using topology and differential geometry be computed analytically as follows: for each complex bundle with unitary connection (E, ∇) in the total space and for each odd enriched cycle M in the base form the pulled back cycle $\bar{M}$ in the total space and restrict (E, ∇) to $\bar{M}$ to obtain $(\bar{E}, \bar{\nabla})$. The value of the character of the push forward of (E, ∇) on the cycle M is the eta invariant mod 1 of the spin $\mathbb{C}$ Dirac operator of $\bar{M}$ with coefficients in $(\bar{E}, \bar{\nabla})$.*

[3]At the abstract level of spectra this last uniqueness result was achieved independently, more generally and somewhat earlier by Bunke and Schick [9].

Remark. Thanks to Bismut, this is true whenever the connection on the tangent bundle used to define the pull back cycle is equivalent as structured bundles to a Bismut metric connection: a metric connection for the direct sum metric on the pullback cycle so that the three form obtained by cyclic symmetrization of the torsion is closed. We are in the process of analyzing this possibility.

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