Numerical Comparison of Momentum Model and Vorticity Model

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Abstract

This is a numerical study of Dennis Sullivan's lattice hydrodynamical models. The momentum model and vorticity model is computed on a lattice size of 128^3 with various initial conditions, and with zero viscosity. Computations show that momentum model blows up while vorticity models are stable.

1 Introduction

We present here the comparison of numerical computations of fourth author's model of hydrodynamics of an incompressible fluid - namely the momentum model and the vorticity model. The formulas for the two models are

$$\frac{\partial\{V\}}{\partial t} = \{*\delta(V_F \cdot v_F)\} + \delta P - \nu \triangle \{V\}, \quad \partial\{V\} = 0.$$

and

$$\frac{\partial}{\partial t}\{V\} = *(\{V\} \wedge *\delta\{V\}) + \delta P - \nu \triangle \{V\}, \quad \partial\{V\} = 0.$$

The topological objects and operators involved in the above ODE is explained in Sullivan's "Lattice Hydrodynamics" paper [1]. The first equation is derived from conservation of momentum while the second equation is derived from preservation of vorticity. The vorticity model has an additional property that energy is conserved when kinematic viscosity $\nu = 0$. Preserving energy means that the L_2 norm of V is preserved. When the kinematic viscosity $\nu = 0$, many computations of the momentum model has L_2 norm of the solution diverging to infinity. This report aims to compare the two models to see if they behave similarly before the momentum model starts to become unstable.

2 Description of Numerical Scheme

The numerical calculation is performed on a 3-Torus (i.e. periodic boundary conditions). Let us define, for the momentum model,

$$F(V) = \{*\delta(V_F \cdot v_F)\} + \delta P - \nu \triangle \{V\}$$

or for the vorticity model,

$$F(V) * (\{V\} \wedge *\delta\{V\}) + \delta P - \nu \triangle \{V\}.$$

Note that though the right side seems to also depend on P, δP is computed so that $\partial F(V) = 0$, and hence it only depends on V. Suppose V_i is the velocity vector field at i-th time step (which is at time t). Then the V_{i+1} (which is at time $t + dt$) is obtained through the standard RK4 time-stepping:

$$\begin{aligned}
R_1 &= dt * F(V_i) \\
R_2 &= dt * F(V_i + \frac{1}{2} R_1) \\
R_3 &= dt * F(V_i + \frac{1}{2} R_2) \\
R_4 &= dt * F(V_i + R_3) \\
V_{i+1} &= V_i + \frac{1}{6}(R_1 + 2R_2 + 2R_3 + R_4)
\end{aligned}$$

The most time-consuming part is computing the pressure P, which under some transformation is a Poisson type problem. While this is a common feature in many numerical methods incompressible fluids, the vorticity model has an interesting structure that enables the pressure equations to be split into 8 independent systems which can be solved efficiently in parallel. We used parallel multigrid solver to solve this system of equations.

3 Restriction on dt

The time step dt cannot be too big. There are two main inequalities it must satisfy. One is the von Neumann stability condition

$$dt \leq \frac{h^2}{2\nu},$$

for viscosity $\nu \neq 0$. Here, h is the spacing between lattice points. This inequality guarantees that the diffusion term $\nu \triangle V$ does not cause instability. Second inequality is

$$dt \leq C \frac{h}{|V|_\infty},$$

which we call the CFL condition (Courant-Friedrichs-Lewy condition). The above inequality is derived by assuming that V does not change rapidly, and treating the ODE as a hyperbolic system. The value of C depends on the choice of numerical scheme. It is customary to take $C = 1$ for explicit time stepping schemes like RK4. Note that if higher value of C can be chosen, then one can increase the value of dt so that the calculation can be done faster.

Note that $|V|_\infty$ will change as computation proceeds, so CFL condition is computed at every time step. We use this to control dt as follows - if CFL decreases, we reduce the value of dt as needed. If CFL condition becomes large enough (i.e. twice the value of dt), then we increase the dt to be 90% of the CFL condition. In this second case, we still have to check other conditions, such as the hard upper bound for dt and the von Neumann stability condition, so that those are not violated.

4 Comparison of Momentum and Vorticity Models

In general, the vector field $\{V\}$ is represented by a $1 - chain$

$$\{V\} = \sum V_x^{ijk}[X]_{ijk} + V_y^{ijk}[Y]_{ijk} + V_z^{ijk}[Z]_{ijk},$$

where $[X]_{ijk}$ is a directed edge of size $2h$ whose center is located at lattice coordinate (i, j, k) and directed in the positive x direction. $[Y]_{ijk}$ and $[Z]_{ijk}$ are defined similarly. V_x^{ijk} is the x component of the vector at (i, j, k) and so on.

We proceed to describe 4 kinds of initial conditions used in the numerical calculation and show graphs to compare the momentum model and vorticiy model calculations.

4.1 Initial Condition #0

Initial condition #0 is a single vortex created by taking the curl (which is $*\delta$ in the formula) of a vector field whose x component is a bump function. In explicit form:

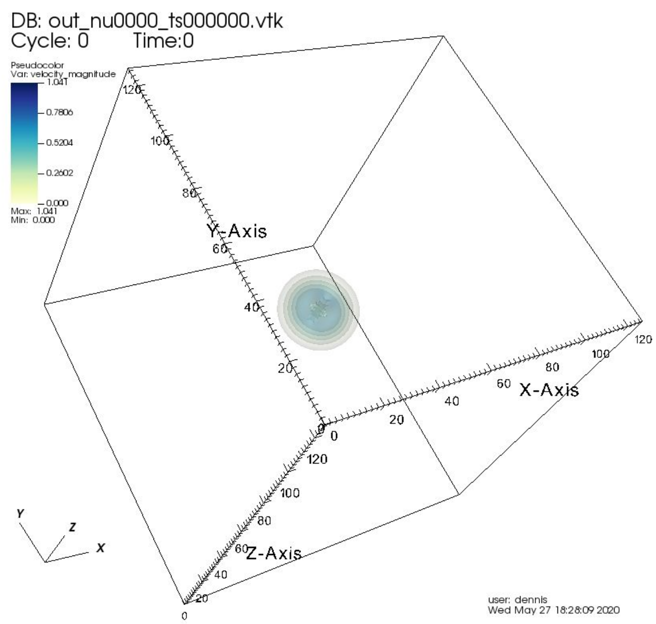
$$\{V_0\} = *\delta \sum C^{ijk}[X]_{ijk}$$

where

$$C^{ijk} = \mu e^{-\eta h^2((i-i_0)^2+(j-j_0)^2+(k-k_0)^2)/4}$$

(i_0, j_0, k_0) is the center of the domain. The default value for μ is 2.5 and for η is 100.0. (In the description for other initial conditions, we use the same default values.) Since divergence of curl is zero, the above initial condition satisfies $\partial\{V\} = 0$.

Figure 1: Initial Condition #0. Norm of the velocity is shown. The actual vectors are rotating around the axis.



The following graphs are the graphs of L_∞ norm of velocity, L_2 of the velocity, and L_2 norm of gradient of the velocity for both models plotted together for comparison.

Figure 2: L_∞ norm for initial condition #0. Blue is the momentum model, and red is the vorticity model.

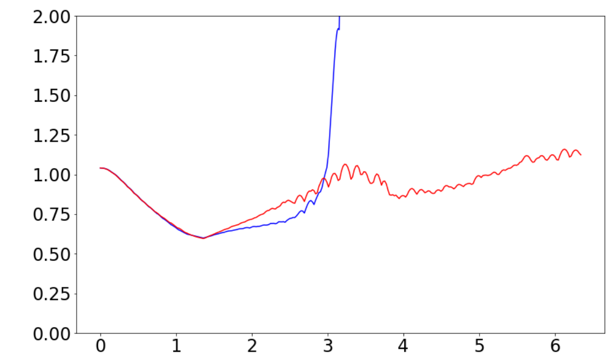


Figure 3: L_2 norm for initial condition #0. Blue is the momentum model, and red is the vorticity model. Because vorticity model preserves energy, we see a horizontal line.

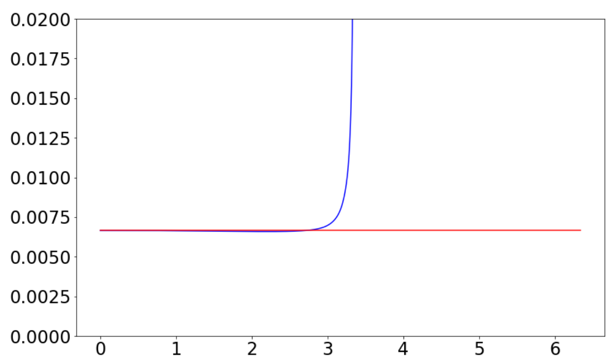
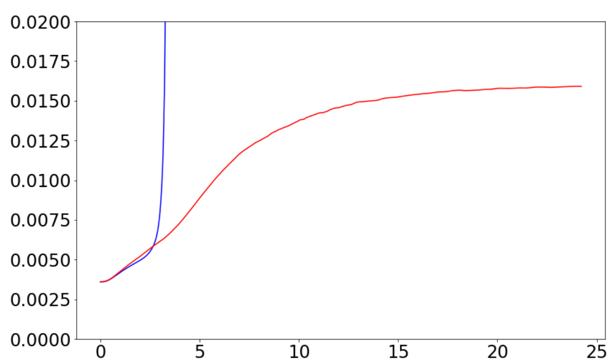
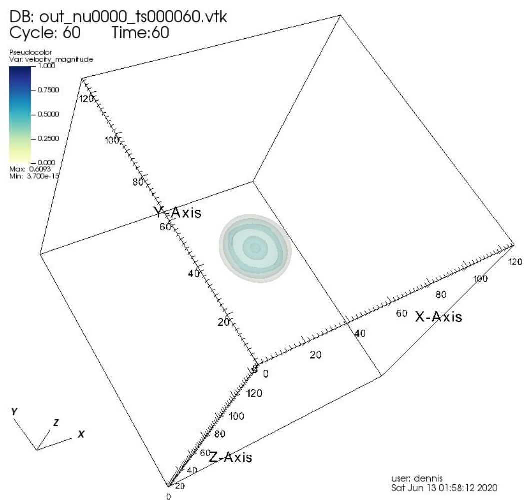
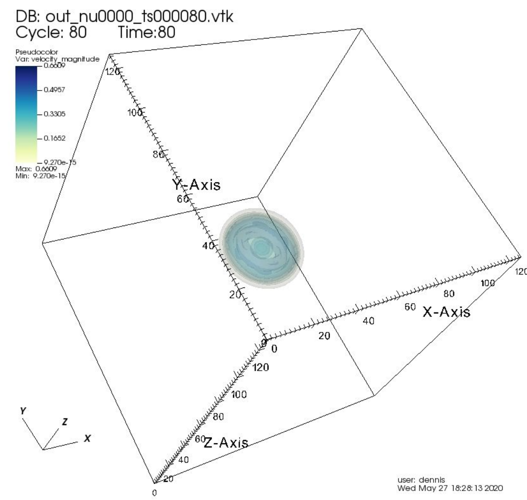
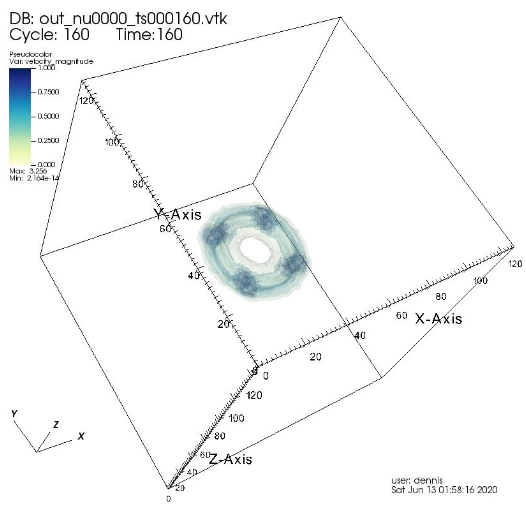
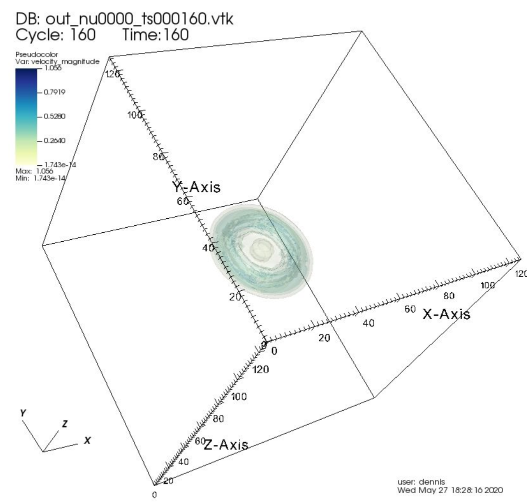
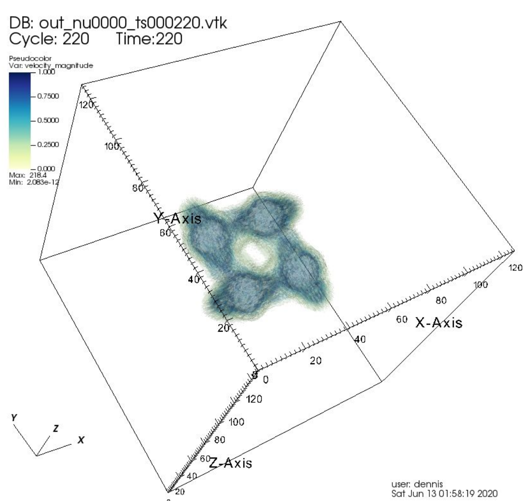
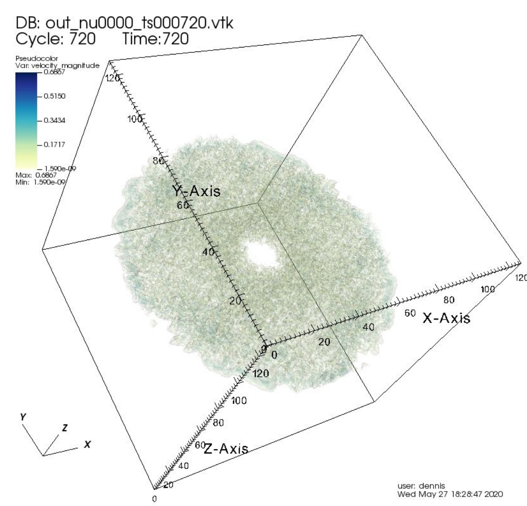


Figure 4: L_2 norm of the gradient of velocity for initial condition #0. Blue is the momentum model, and red is the vorticity model.



Some 3D pictures to compare their shapes:

Initial Condition #0

Momentum Model	Vorticity Model
 t=1.70	 t=1.70
 t=3.24	 t=3.39
 t=3.34	 t=15.17

4.2 Initial Condition #1

Initial condition #1 is what is known as the Taylor-Green vortex. The formula we use is:

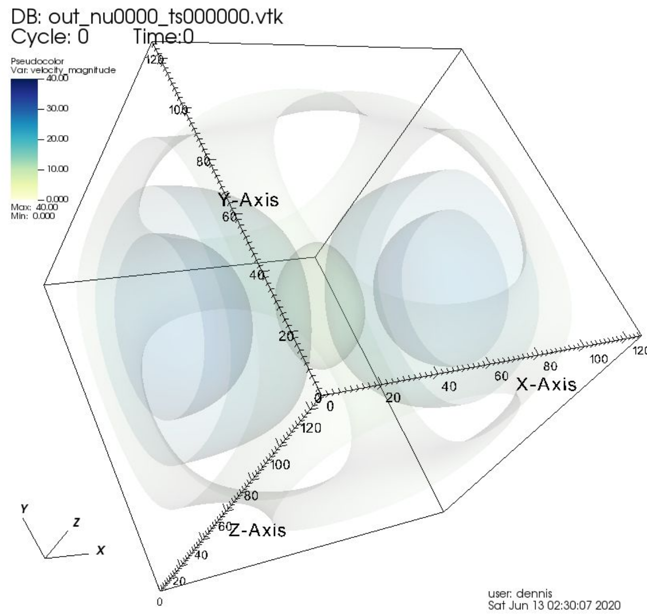
$$\{V_0\} = \sum V_x^{ijk}[X]_{ijk} + V_y^{ijk}[Y]_{ijk} + V_z^{ijk}[Z]_{ijk},$$

where

$$\begin{aligned} V_x^{ijk} &= 4\mu \cos(ih)\sin(jh)\sin(kh) \\ V_y^{ijk} &= 4\mu \sin(ih)\cos(jh)\sin(kh) \\ V_z^{ijk} &= 4\mu \sin(ih)\sin(jh)\cos(kh) \end{aligned}$$

Now, though the continuum version of this vector field is divergence free, the discrete version may not be. So we perform a Leray projection of this initial condition before proceeding.

Figure 5: Initial Condition #1. Norm of the velocity is shown. The actual vectors are rotating around the axis.



The following graphs are the graphs of L_∞ norm of velocity, L_2 of the velocity, and L_2 norm of gradient of the velocity for both models plotted together for comparison.

Figure 6: L_∞ norm for initial condition #1. Blue is the momentum model, and red is the vorticity model.

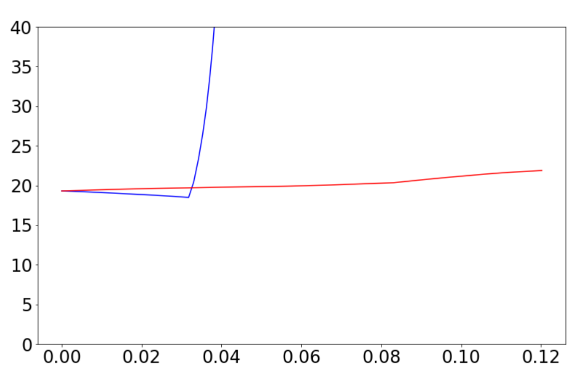


Figure 7: L_2 norm for initial condition #1. Blue is the momentum model, and red is the vorticity model. Because vorticity model preserves energy, we see a horizontal line.

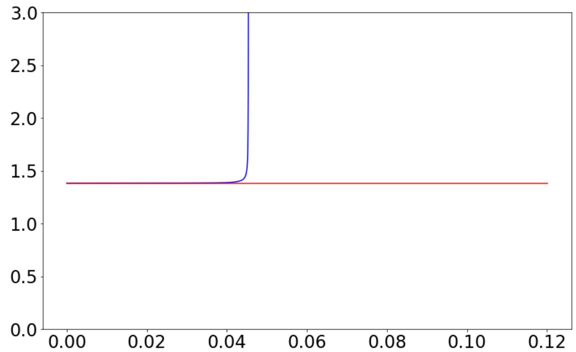
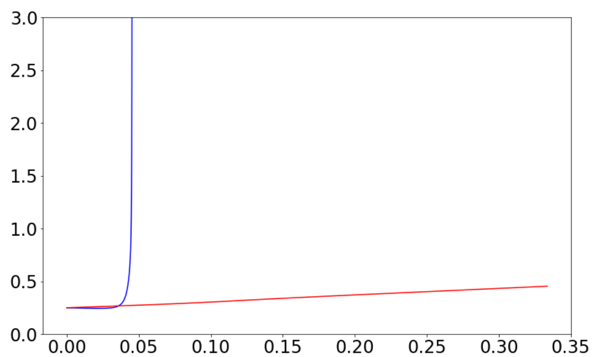
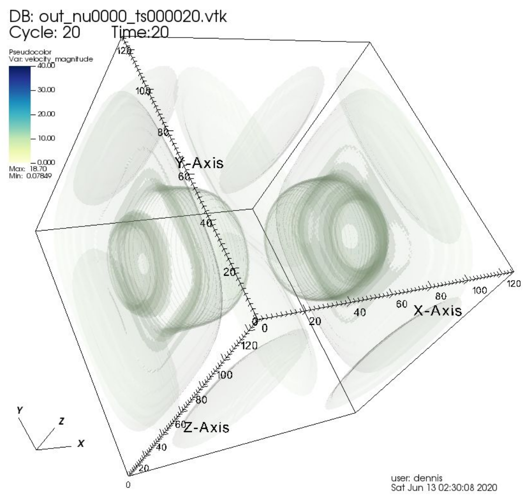
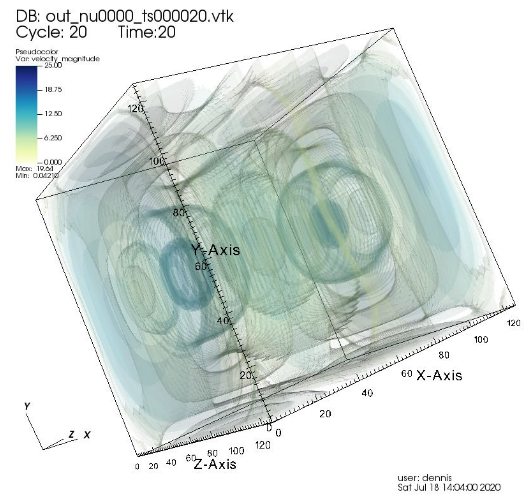
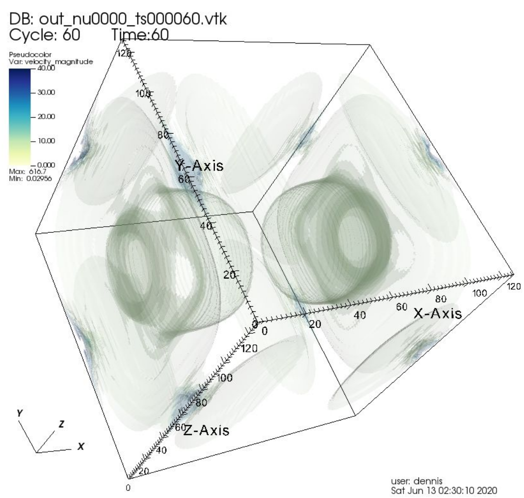
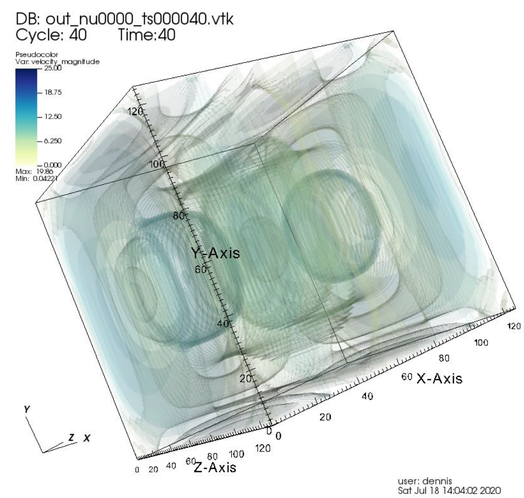
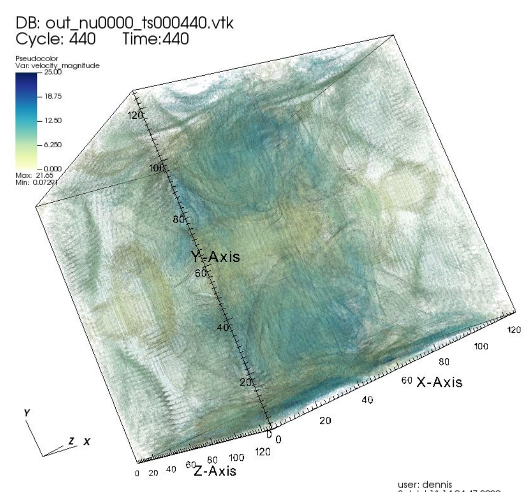


Figure 8: L_2 norm of the gradient of velocity for initial condition #1. Blue is the momentum model, and red is the vorticity model.



Some 3D pictures to compare their shapes:

Initial Condition #1

Momentum Model	Vorticity Model
 t=0.025	 t=0.025
 t=0.0452	 t=0.050
	 t=0.480

4.3 Initial Condition #2

Initial condition #2 is made of two vortex on the same axis rotating in opposite direction, created by taking the curl of a vector field whose x component is difference of two bump functions. In explicit form:

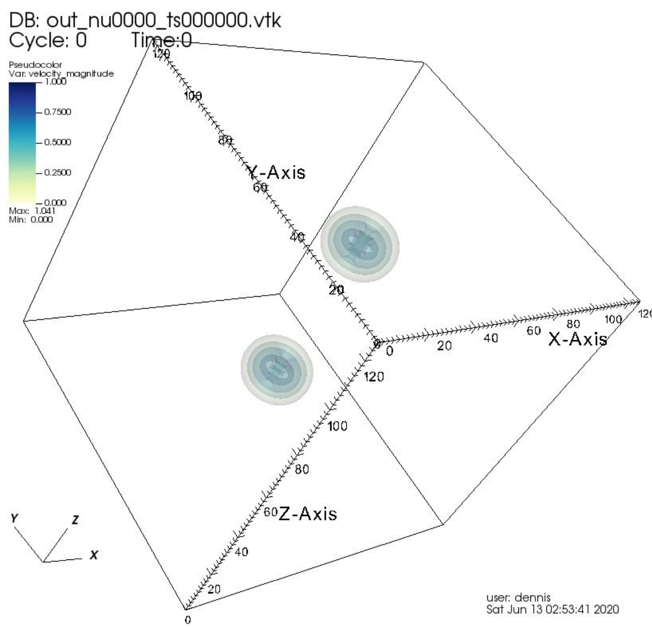
$$\{V_0\} = *\delta \sum \left(C_1^{ijk} - C_2^{ijk} \right) [X]_{ijk}$$

where

$$C_1^{ijk} = \mu e^{-\eta h^2((i-i_1)^2+(j-j_1)^2+(k-k_1)^2)/4}$$
$$C_2^{ijk} = \mu e^{-\eta h^2((i-i_2)^2+(j-j_2)^2+(k-k_2)^2)/4}$$

(i_1, j_1, k_1) and (i_2, j_2, k_2) lie on the x-axis and their distance is half the length of the domain.

Figure 9: Initial Condition #2. Norm of the velocity is shown. The actual vectors are rotating around the axis.



The following graphs are the graphs of L_∞ norm of velocity, L_2 of the velocity, and L_2 norm of gradient of the velocity for both models plotted together for comparison.

Figure 10: L_∞ norm for initial condition #2. Blue is the momentum model, and red is the vorticity model.

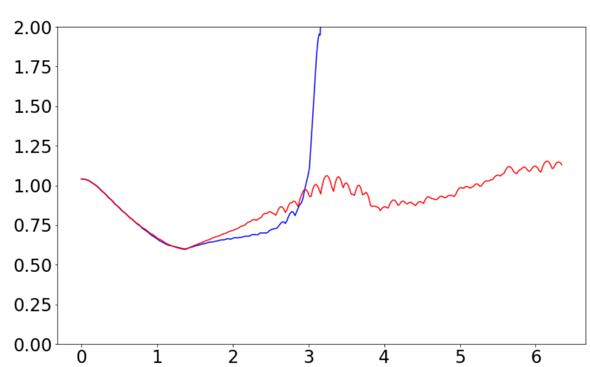


Figure 11: L_2 norm for initial condition #2. Blue is the momentum model, and red is the vorticity model. Because vorticity model preserves energy, we see a horizontal line.

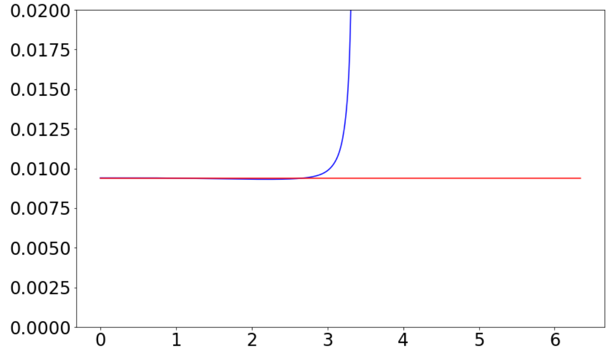
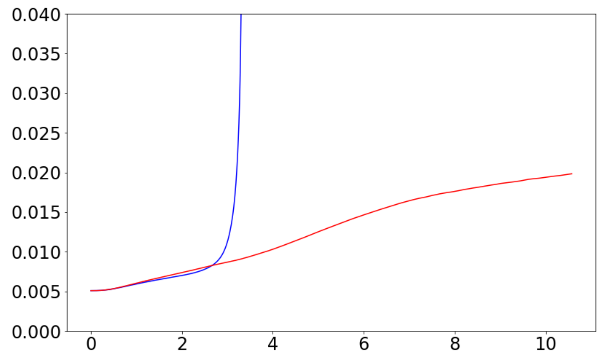
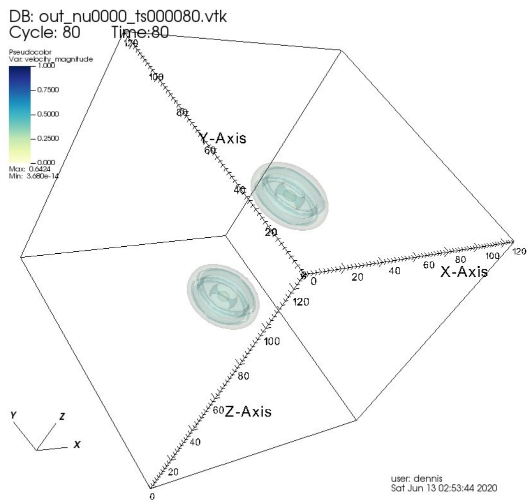
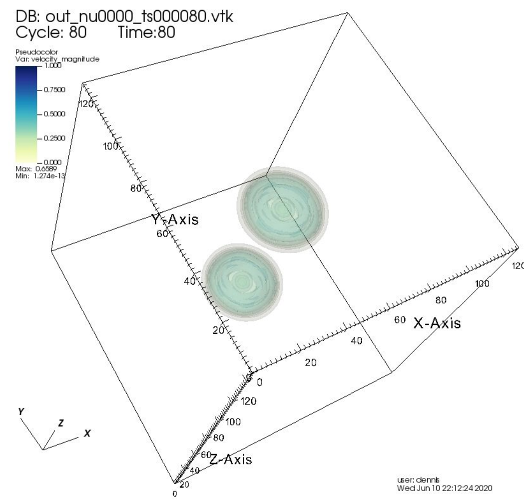
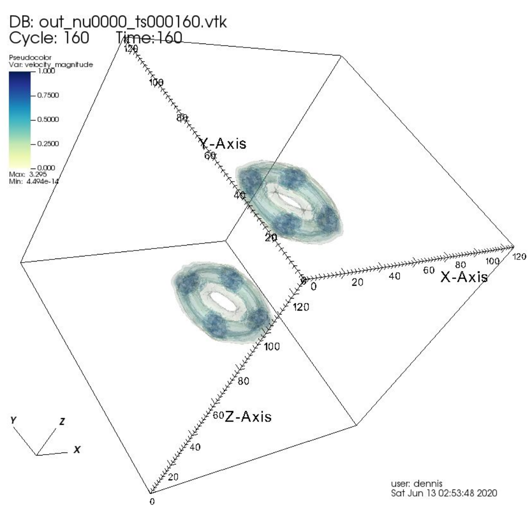
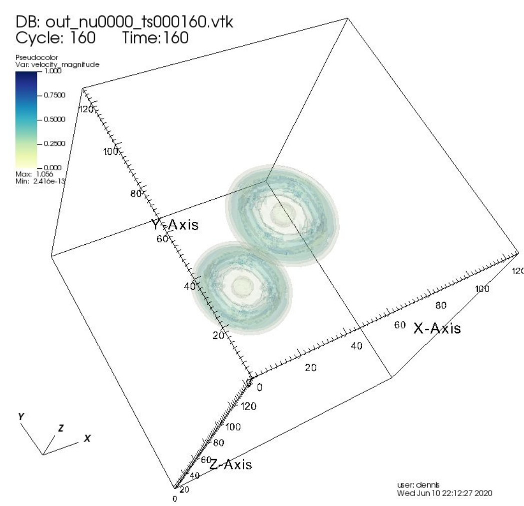
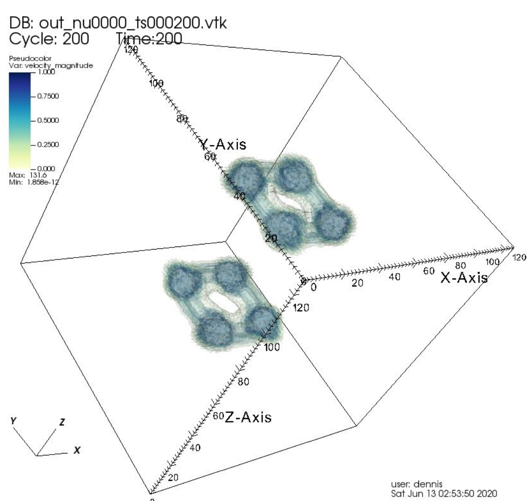
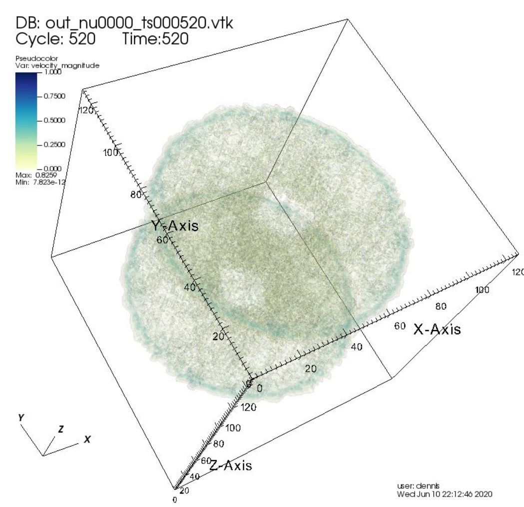


Figure 12: L_2 norm of the gradient of velocity for initial condition #2. Blue is the momentum model, and red is the vorticity model.



Some 3D pictures to compare their shapes:

Initial Condition #2

Momentum Model	Vorticity Model
	
t=1.70	t=1.70
	
t=3.24	t=3.40
	
t=3.34	t=11.01

4.4 Initial Condition #3

Initial condition #3 is made of one big vortex at the center and two small vortex around it. This is created by taking the curl of a vector field whose x component is sum of three bump functions. In explicit form:

$$\{V_0\} = *\delta \sum \left(C_1^{ijk} + C_2^{ijk} + C_3^{ijk} \right) [X]_{ijk}$$

where

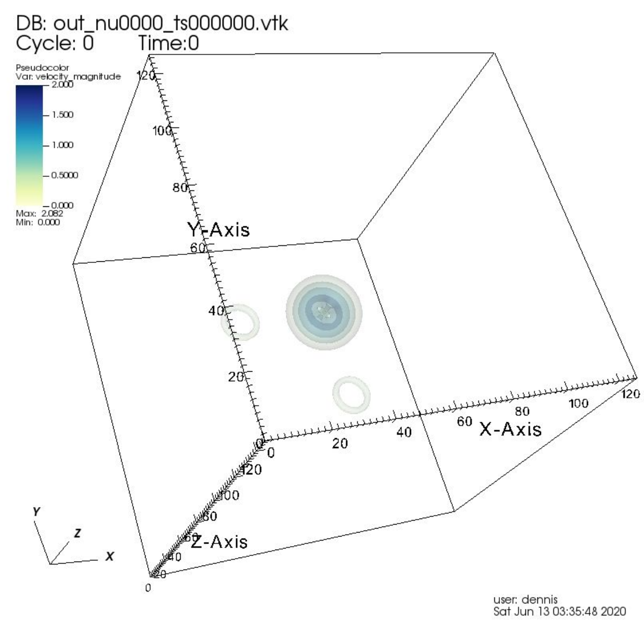
$$C_1^{ijk} = 2\mu e^{-\eta h^2((i-i_1)^2+(j-j_1)^2+(k-k_1)^2)/4}$$

$$C_2^{ijk} = \frac{1}{2}\mu e^{-\eta h^2((i-i_2)^2+(j-j_2)^2+(k-k_2)^2)/4}$$

$$C_3^{ijk} = \frac{1}{2}\mu e^{-\eta h^2((i-i_3)^2+(j-j_3)^2+(k-k_3)^2)/4}$$

(i_1, j_1, k_1), (i_2, j_2, k_2) and (i_2, j_2, k_2) all lie on the same yz plane. Let us call l the 25% of the length of the domain. Then (i_2, j_2, k_2) is distance l in the positive y direction from (i_1, j_1, k_1), and (i_3, j_3, k_3) is distance l in the positive z direction from (i_1, j_1, k_1).

Figure 13: Initial Condition #3. Norm of the velocity is shown. The actual vectors are rotating around the axis.



The following graphs are the graphs of L_∞ norm of velocity, L_2 of the velocity, and L_2 norm of gradient of the velocity for both models plotted together for comparison.

Figure 14: L_∞ norm for initial condition #3. Blue is the momentum model, and red is the vorticity model.

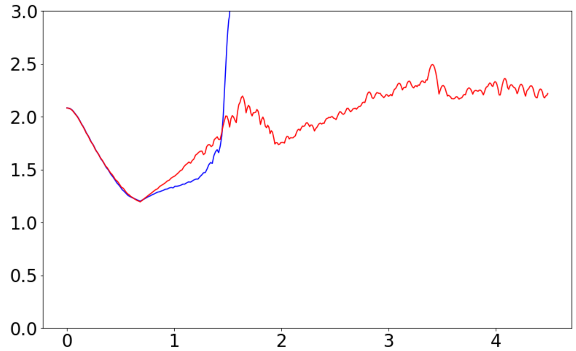


Figure 15: L_2 norm for initial condition #3. Blue is the momentum model, and red is the vorticity model. Because vorticity model preserves energy, we see a horizontal line.

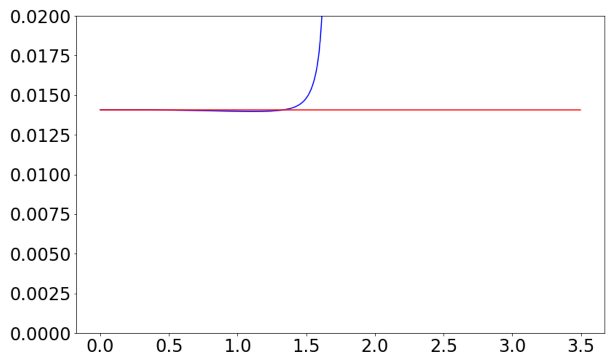
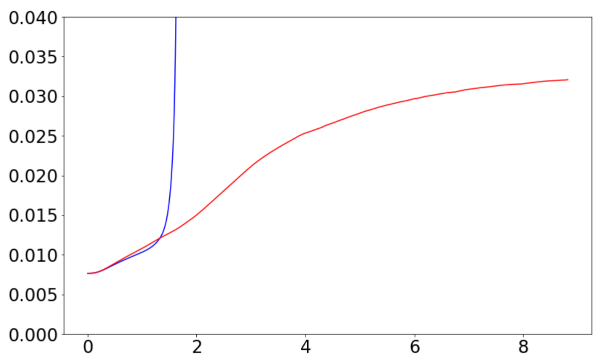


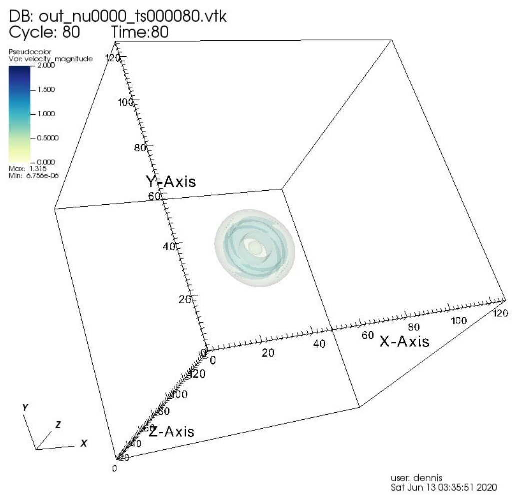
Figure 16: L_2 norm of the gradient of velocity for initial condition #3. Blue is the momentum model, and red is the vorticity model.



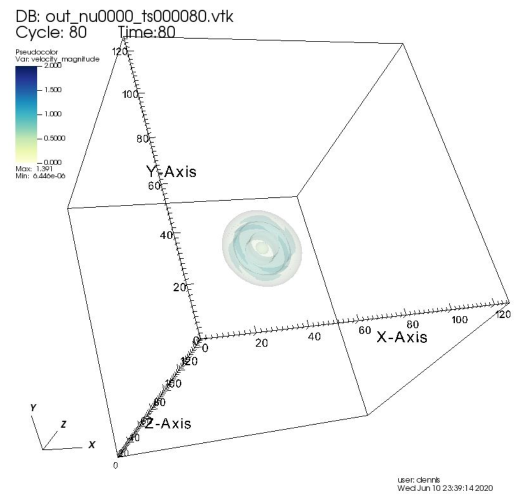
Some 3D pictures to compare their shapes:

Initial Condition #3

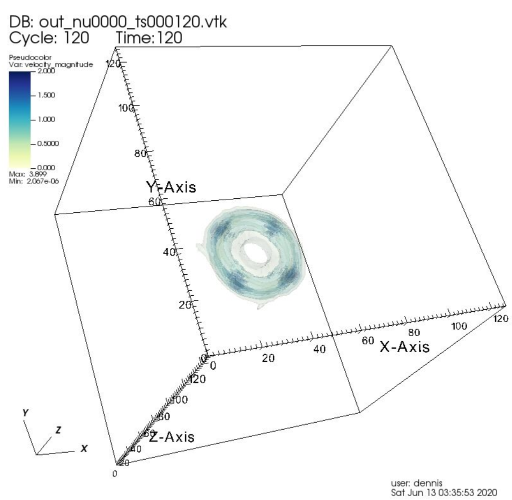
Momentum Model	Vorticity Model



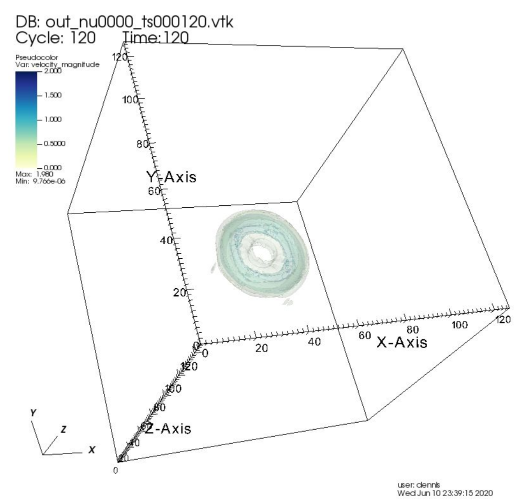
t=0.937



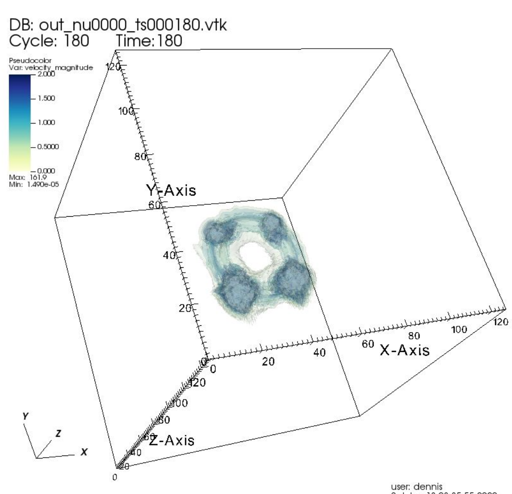
t=0.937



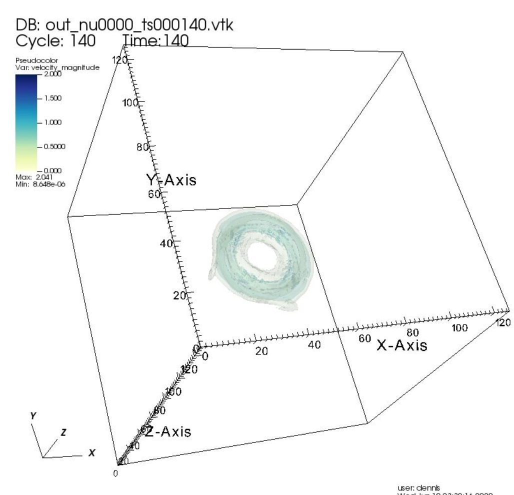
t=1.54



t=1.53



t=1.66



t=1.76

4.5 Initial Condition #4

Initial condition #4 is same as initial condition #2, but the two vortex are rotating in the same direction

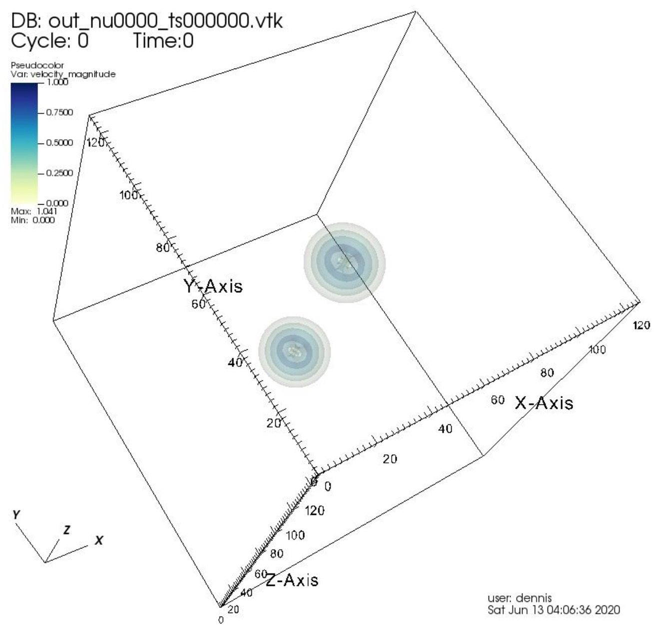
$$\{V_0\} = *\delta \sum \left(C_1^{ijk} + C_2^{ijk}\right)[X]_{ijk}$$

where

$$C_1^{ijk} = \mu e^{-\eta h^2((i-i_1)^2+(j-j_1)^2+(k-k_1)^2)/4}$$
$$C_2^{ijk} = \mu e^{-\eta h^2((i-i_2)^2+(j-j_2)^2+(k-k_2)^2)/4}$$

(i_1, j_1, k_1) and (i_2, j_2, k_2) lie on the x-axis and their distance is half the length of the domain.

Figure 17: Initial Condition #4. Norm of the velocity is shown. The actual vectors are rotating around the axis.



The following graphs are the graphs of L_∞ norm of velocity, L_2 of the velocity, and L_2 norm of gradient of the velocity for both models plotted together for comparison.

Figure 18: L_∞ norm for initial condition #4. Blue is the momentum model, and red is the vorticity model.

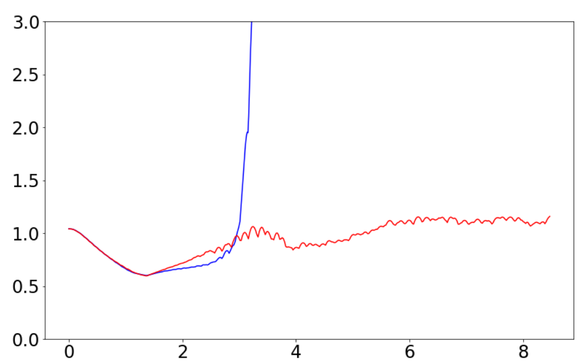


Figure 19: L_2 norm for initial condition #4. Blue is the momentum model, and red is the vorticity model. Because vorticity model preserves energy, we see a horizontal line.

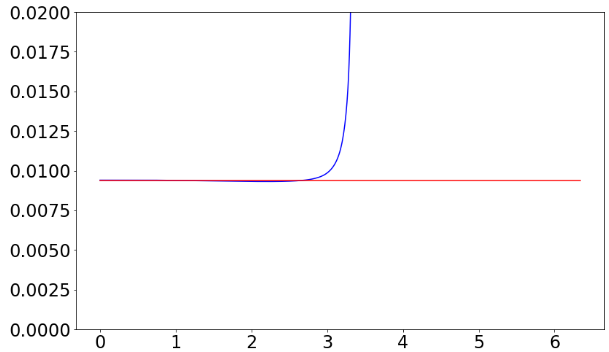
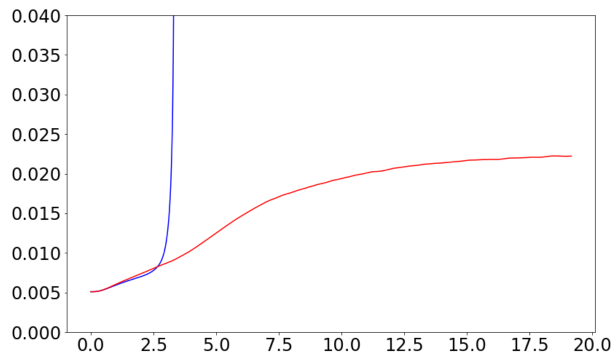


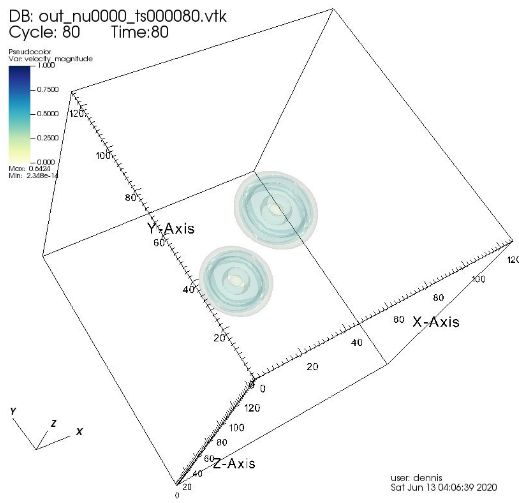
Figure 20: L_2 norm of the gradient of velocity for initial condition #4. Blue is the momentum model, and red is the vorticity model.



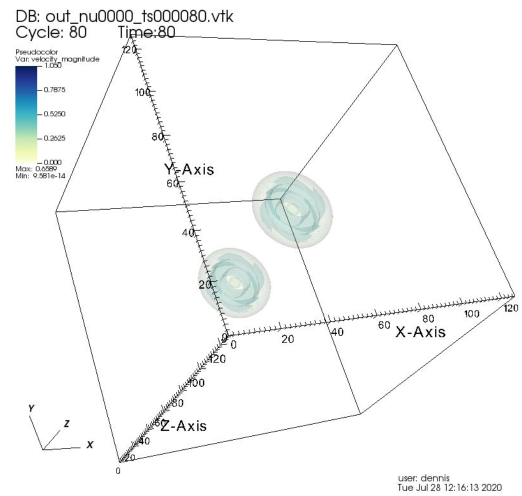
Some 3D pictures to compare their shapes:

Initial Condition #2

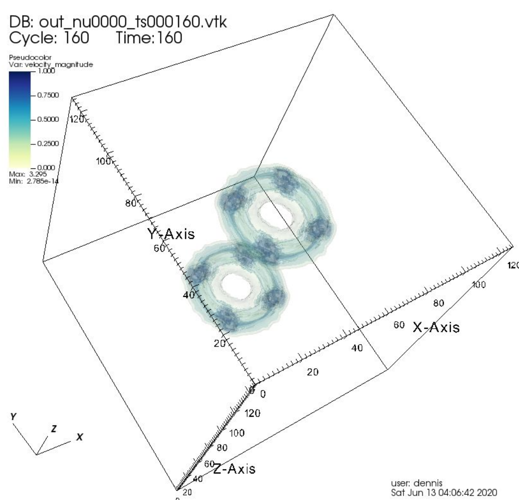
Momentum Model	Vorticity Model



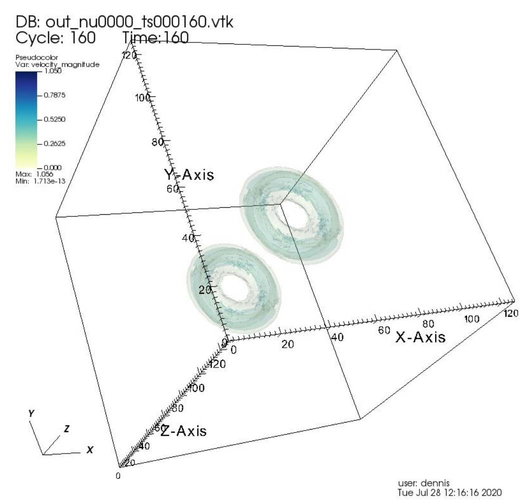
t=1.70



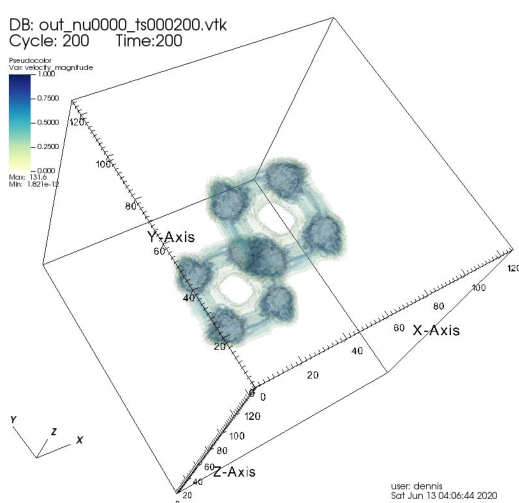
t=1.70



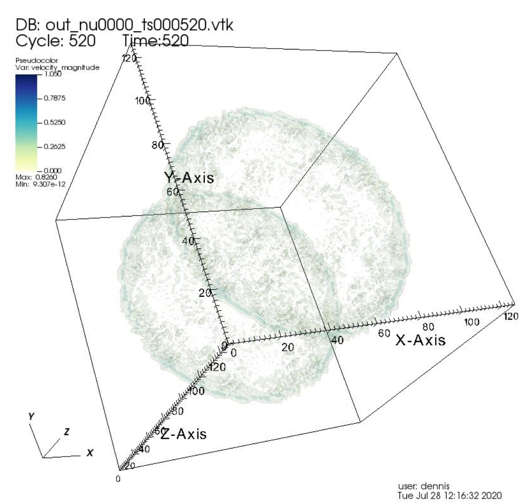
t=3.24



t=3.40



t=3.34



t=11.01

5 Conclusion

In all of these runs, the two models closely resemble each other in terms of the norms in the beginning. Then the momentum model becomes unstable and eventually blows up.

References

[1] D. Sullivan, "Lattice hydrodynamics," 2018.