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3D INCOMPRESSIBLE FLUIDS: COMBINATORIAL MODELS, EIGENSPACE MODELS, AND A CONJECTURE ABOUT WELL-POSEDNESS OF THE 3D ZERO VISCOSITY LIMIT

Dennis Sullivan

Dedicated to Fritz Hirzebruch,
who exemplified diligence and beauty in mathematics

Part 1. Combinatorial Models for Computation

1. Introduction

We make combinatorial models of spatial calculus with special regard for nonlinear structures. We apply this to incompressible fluid motion in the zero viscosity limit in the 3-dimensional space made periodic. We take advantage of a special duality property of the cubical partitions of 3-dimensional space. The nonlinear structure comes from the evolution PDE of fluids

$$\dot{Y} = [X, Y],$$

where X is a divergence free vector field (i.e., incompressible), $Y = \mathrm{curl} X$, and $[\ ,\ ]$ is the Lie bracket which is our nonlinear structure. This equation states that *the vorticity of the fluid motion is transported by the motion of the fluid*.

We use the powerful tools of algebraic topology to somewhat open up the structure of the nonlinear term. In order to use these tools it is necessary to embed vector fields on a smooth manifold into the chain complex of multivector fields with natural monomial grading and boundary operator of degree -1. This structure appears by regarding multivector fields as linear functionals on smooth differential forms. To do this it is enough to choose any smooth probability measure that charges every open set. The point is that the Lie bracket is now intertwined with this chain complex structure as explained below.

2. The homotopy category of chain complexes

By a *chain complex* C we mean a family $C = \{C_i\}_{i\in\mathbb{Z}}$ of real vector spaces together with linear maps $\partial = \partial_i : C_i \to C_{i-1}$ such that the

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composition $\partial \circ \partial : C_i \to C_{i-2}$ is zero. The *i-th homology* of the chain complex C is defined to be quotient vector space $H_i(C) = \mathrm{Ker}\partial_i / \mathrm{Im}\partial_{i+1}$.

A *morphism of chain complexes* (or a *chain map*) $f : (C_i, \partial_C) \to (D_i, \partial_D)$ is a family of linear maps $f_i : C_i \to D_i$ commuting with ∂ in the sense that $f_{i-1} \circ \partial_C = \partial_D \circ f_i$. We say that two chain maps $h_1, h_2 : (C_i, \partial_C) \to (D_i, \partial_D)$ are *chain homotopic* and write $h_1 \sim h_2$ if there exists a family of linear maps $S = S_i : C_i \to D_{i+1}$ such that $h_1 - h_2 = \partial_D S + S \partial_C$.

Chain maps between chain complexes induce linear maps on homology. In particular, note that two chain homotopic chain maps induce the same map on homology. If a chain map induces an isomorphism on homology we call it a *quasi-isomorphism*. We have the following

Proposition 2.1. *A chain map $f : (C_i, \partial_C) \to (D_i, \partial_D)$ is a quasi-isomorphism if and only if there exists a chain map $g : (D_i, \partial_D) \to (C_i, \partial_C)$ so that $fg \sim Id_D$ and $gf \sim Id_C$.*

The *homotopy category of chain complexes* is the category whose objects are chain complexes and whose morphisms are classes of chain maps modulo the equivalence relation $\sim$.

3. Example of a nonlinear structure on a chain complex

Let M be a smooth manifold, and consider the graded vector space V of *multivector fields* on M. More precisely, V is the vector space of smooth sections of the exterior algebra bundle $\wedge^*(TM)$, where TM is the tangent bundle of M. V has a natural grading given by the grading of $\wedge^*(TM)$. In particular, elements of degree 0 on V are smooth functions, and elements of degree 1 are smooth vector fields.

In the presence of a volume measure we have an isomorphism of vector spaces $\mu : V \cong \Omega^*(M)$, where $\Omega^*(M)$ denotes the vector space of differential forms on M. We can transport the exterior derivative on differential forms by the isomorphism μ to obtain a linear map $\partial : V \to V$ of degree -1 that satisfies $\partial \circ \partial = 0$ and thus equips V with the structure of a chain complex. In particular, if X is a vector field, ∂X is the function that describes the distortion of the volume measure by the vector field X, i.e., the divergence of X relative to the volume measure.

The vector space $\Gamma(TM)$ of smooth vector fields on M has the structure of a Lie algebra. This means there exists a linear map $[,] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ which is bilinear, skew symmetric, and satisfies the Jacobi identity. Intuitively, $[X, Y]$ is a vector field which describes the infinitesimal change of Y along X. The bracket $[,]$ can be extended to the the space V of multivector fields by the Leibniz rule to obtain a Lie bracket $[,] : V \times V \to V$. This Lie bracket is known in the literature as the Schouten–Nijenhuis bracket. The boundary map ∂ is a derivation of the bracket $[\ ,\ ]$.

Now we describe a construction that completely encodes the data of $(V, \partial, [\ ,\ ])$ and its algebraic constraints into an algebraic structure satisfying a single equation. Consider the graded vector space $SV = V \oplus S^2V \oplus \cdots \oplus S^kV \oplus \cdots$, where S^kV is the vector space generated by length k alternating monomials of elements of V. Namely, $a \cdot b = -b \cdot a$ if both have odd degree. Otherwise $a \cdot b = b \cdot a$.

SV has the property that any linear map $l^k \colon S^kV \longrightarrow V$ can be extended to a map $l_k : SV \to SV$ by setting

$$l_k(a_1 \wedge \cdots \wedge a_n) = \sum (-1)^{\epsilon} l^k(a_{i_1} \wedge \cdots \wedge a_{i_k}) \\ \wedge a_1 \wedge \cdots \wedge \hat{a}_{i_1} \wedge \cdots \wedge \hat{a}_{i_k} \wedge \cdots \wedge a_n$$

where the above sum runs through all k-tuples of indices $(i_1, \ldots, i_k)$ satisfying $1 \leq i_1 < \cdots < i_k \leq n$, the hat on top of an element means you omit that element in the monomial, and $(-1)^{\epsilon}$ is a sign determined by the permutation moving the chosen to the left.

Applying this construction to the example of interest, we have that the linear maps $\partial : S^1V = V \to V$ and $[,] : S^2V \to V$ can be extended to linear maps $\partial_1 : SV \to SV$ and $\partial_2 : SV \to SV$, respectively. Both maps ∂_1 and ∂_2 are infinitesimally compatible with the coproduct on SV defined by taking the sum of all possible splittings of a monomial into two monomials. A map that is infinitesimally compatible with a coproduct is called a *coderivation*.

Fact 3.1. $\partial_\infty = \partial_1 + \partial_2 : SV \to SV$ is a coderivation and $\partial_\infty \circ \partial_\infty = 0$.

∂_∞ is a coderivation since it is the sum of two coderivations. The second property follows from the following three equations:

1) $\partial_1 \circ \partial_1 = 0$. This follows from the fact that the $d \circ d = 0$, where d is the exterior derivative on differential forms.
2) $\partial_1 \circ \partial_2 + \partial_2 \circ \partial_1 = 0$. This follows because ∂ is a derivation of the Lie bracket of vector fields $[,]$
3) $\partial_2 \circ \partial_2 = 0$. This follows from the Jacobi identity for $[,]$.

The following fact motivates the main definition in the next section.

Fact 3.2. We can reconstruct $(V, \partial, [,])$ and its good properties from the map $\partial_\infty : SV \to SV$ satisfying the above.

4. Theory of Lie infinity structures on a chain complex

Definition 4.1. An L_∞-*algebra structure* on a chain complex (V, ∂) is a coderivation $\partial_\infty = \partial_1 + \partial_2 + \partial_3 + \cdots : SV \to SV$ satisfying $\partial_\infty \circ \partial_\infty = 0$ and $\partial_1 = \partial$. The maps $\partial^k \colon S^kV \longrightarrow V$ determining ∂_k are called the *Taylor coefficients* of ∂_∞.

Fact 4.2. If ∂_∞ is an L_∞-algebra structure on (V, ∂), then $\partial_2 : S^2V \to V$ is a chain mapping for ∂_1 and $\partial_3 : S^3V \to S$ is a chain homotopy to provide ∂_2 with its missing "Jacobi" identity.

In particular, the above fact implies that ∂_2 satisfies the Jacobi identity on the homology vector space $H_*(V, \partial)$.

Chain complexes with L_∞-algebra structure form a category whose morphisms are defined by mappings $F = f_1 + f_2 + \cdots : SV \to SW$ respecting the ∂_∞'s and the coproducts (in other words, maps of differential graded cocomutative coalgebras). A morphism of chain complexes with L_∞ structures is also called an L_∞-*morphism*. An L_∞-morphism $F = f_1 + f_2 + \cdots : SV \to SW$ is determined by its components $f^k : S^kV \longrightarrow W$, also called the *Taylor coefficients* of F.

We have the following two fundamental theorems.

Theorem 4.3. *Let (V, ∂_V) and (W, ∂_W) be two chain complexes with L_∞-algebra structures. If $F = f_1 + f_2 + \cdots : SV \to SW$ is an L_∞-morphism such that $f_1 : (V, \partial_V) \to (W, \partial_W)$ is a quasi-isomorphism then there exists an L_∞-morphism $G = g_1 + g_2 + \cdots : SW \to SV$ such that $g_1 : (W, \partial_W) \to (V, \partial_V)$ is a quasi-isomorphism and the induced maps $(f_1 \circ g_1)_* : H_*(W, \partial_W) \to H_*(W, \partial_W)$ and $(g_1 \circ f_1)_* : H_*(V, \partial_V) \to H_*(V, \partial_V)$ are identity maps.*

Remark 4.4. Actually, g is unique up to a nonlinear notion of chain homotopy. This is proven (and formulated) in [**4**]. Also the map in the next theorem is also well defined up to this same notion of nonlinear chain homotopy.

Theorem 4.5. *Let $f : (C, \partial_C) \to (V, \partial_V)$ be a quasi-isomorphism of chain complexes. Given an L_∞-algebra structure ∂_∞^V on (V, ∂_V), there exists an L_∞-algebra structure ∂_∞^C on (C, ∂_C) together with an L_∞-morphism $F = f_1 + f_2 + \cdots : SC \to SV$ such that $f_1 = f$.*

The first theorem says that L_∞-morphisms that extend quasi-isomorphisms can be inverted up to equivalence. The second says that we can transport our nonlinear structure when considered as L_∞-algebra structure through quasi-isomorphisms of chain complexes.

5. Combinatorial nonlinear structure

Choose a scale and divide periodic 3-dimensional space into cubes at that scale. A smoothing operator at that scale embeds the cellular chain complex (C, ∂_C) associated to the cubical decomposition into the chain complex of multivector fields (V, ∂). Integration gives a map in the opposite direction, and one composition is the identity. Such smoothing also gives a chain homotopy S for the other composition. By Theorem 2, the L_∞-algebra structure $\partial_\infty = \partial_1 + \partial_2$ on (V, ∂) induces an L_∞-algebra structure ∂_∞^C on (C, ∂_C). There are algorithms to describe the Taylor coefficients of ∂_∞^C in terms of the chain homotopy S that are sums over trees of operators constructed from the trees where interior edges are decorated by S.

6. Duality of the cubical decomposition and the combinatorial curl

The Poincarè dual decomposition of the cubical decomposition of 3-dimensional space is also a cubical decomposition of 3-dimensional space. These are related by a translation.

The combinatorial curl of a 1-cycle in (C, ∂_C) is defined on generators (edges) e by

1) forming the four square plaques that abut the edge e,
2) then taking the Poincaré dual to these and translating back. (See Figure 1.)

The composition of the Poincaré dual cell construction [defined in general for cell structures on manifolds] composed with the translation back to the original decomposition [a possibility for the cubical decomposition] defines the combinatorial analogue of the continuum Hodge star on forms and currents . It has good algebraic properties for our cubical decomposition. We will work with the operator D, which is the commutator of our boundary operator with this star operator. In the continuum case this operator in dimension 1 is the sum of the divergence operator and the curl operator. In our combinatorial setting it agrees with the curl pictured when applied to 1-cycles.

7. Back to fluids

Extend the combinatorial curl to all of $SC = C \oplus S^2C \cdots$ as follows. First, extend the combinatorial star to a derivation and coderivation mapping. Second, extend the boundary as a derivation and coderivation mapping. The commutator of these two is the first extension of the combinatorial curl D. Then, we conjugate by a coalgebra automorphism to get the desired extension, still denoted D (see Remark 7.1). Denote by Ad the generalization of $\mathrm{ad}_X : Y \mapsto [X, Y]$ for Lie algebras to L_∞-algebras(see Remark 7.1). The continuum fluid equation

$$\mathrm{div} X = 0, \dot{Y} = \mathrm{ad}_X Y, \mathrm{curl} X = Y$$

is compressed to the combinatorial setting on SC:

$$\partial_\infty X = 0, \dot{Y} = \mathrm{Ad}_X Y, DX = Y.$$

Note that there are two parameters in the discretization process: the monomial weight or number of terms in the Taylor expansion of ∂_∞^C, and the scale or mesh of the cubical decomposition.

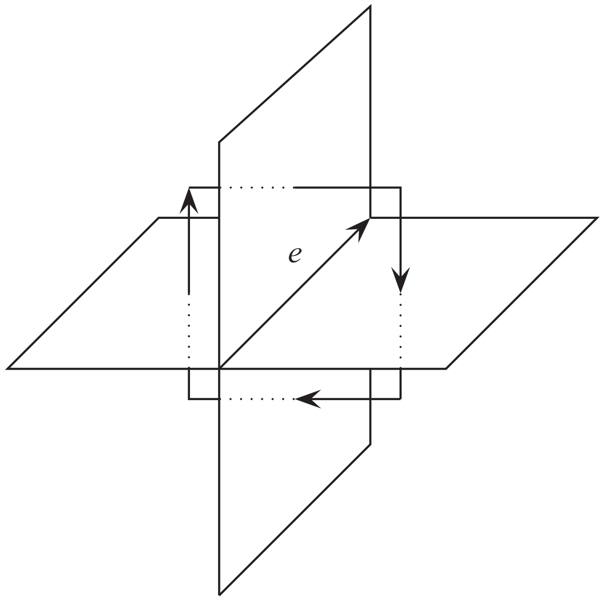

Figure 1. Curl of e before translation

Remark 7.1. 1) To describe $\mathrm{Ad}_X Y$ for X, Y in SC one works inside sums of finite order differential and codifferential operators on SC. An example of a first-order codifferential operator is multiplication by an element from C. Multiplication by an element of SC of monomial weight m is an mth order codifferential operator. Then for X in SC and Y in SC, $\mathrm{Ad}_X Y :\equiv \partial_\infty(X \cdot Y) + X \cdot \partial_\infty Y$. This, when X has odd degree; otherwise, the negative sign. In our case, D on C is an isomorphism on image ∂ inside C. In order to invert the relation $DX = Y$ on SC we assume that X has "mean zero" as in Part 2. This means X belongs to image ∂_∞, not just to kernel ∂_∞.

2) We define D to be an isomorphism on image ∂_∞ by first extending D to SC as mentioned above. Secondly, we conjugate it by the "cumulant" bijection [**2**] $(SC, \partial_1) \sim (SC, \partial_\infty)$. Then D commutes with ∂_∞. By carefully extending the Hodge decomposition on C to SC, we find that D is an isomorphism on image $\partial\infty$. So the combinatorial evolution equations have the same form as the continuum equations and make complete sense.

Now we need to compute!

Part 2. Eigenspace models for computation and for theory

Let $\mathcal{V}_\infty$ denote the space of divergence free mean flow zero smooth vector fields on $\mathbb{R}^3/\mathbb{Z}^3$ with L^2 inner product. A quadratic ODE $\dot{x} =$

$Q(x)$ on a finite-dimensional subspace $\mathcal{V}_n$ of $\mathcal{V}_\infty$ is called *Eulerian* if there is given

1) a self-adjoint operator D_n on $\mathcal{V}_n$ that is invertible and
2) a totally skew-symmetric 3-form $\{\ ,\ ,\ \}_n$ on $\mathcal{V}_n$ so that for all z in $\mathcal{V}_n$ the inner product $(Q(x), z)$ is given by $\{x, D_n x, z\}_n$.

Remark 7.2. *In another work we have shown the Eulerian ODEs on finite dimensional inner product spaces preserve circulation along transported elements and that this property characterizes this class of quadratics ODEs* [**3**]. *A similar result (both ways) holds for the transport of vorticity, where vorticity is defined to be the Dx where x is the velocity* [**3**]. *The transport defined there is skew symmetric but need not satisfy a Jacobi identity. The generalized Lie structures in Part 1 were brought in to treat that discrepancy. The eigenspace models below are amenable to that treatment. We plan to do this elsewhere.*

Proposition 7.3. *Eulerian ODEs preserve the norm $(x, x) = |x|^2$ and the Gaussian measure $\mathrm{e}^{-|x|^2/2}dx$.*

Proof. 1) $\frac{d(x,x)}{dt} = 2(\dot{x}, x) = 2\{x, D_n x, x\}_n = 0$.

2) The differential of the mapping $x \mapsto Q(x)$ is the associated bilinear mapping $B(x, y) = Q(x + y) - Q(x) - Q(y)$. In the basis of eigenvectors of D_n, one computes the matrix for $y \mapsto B(x, y)$ has zero diagonal entries for each eigenvector x.

Thus, the trace of $y \mapsto B(x, y)$ is zero for each x and the vector field $x \mapsto Q(x)$ is both tangent to the L^2 spheres and Euclidean volume preserving on spherical shells that are preserved. It follows that the Gaussian meassure is preserved as well. q.e.d.

Remark 7.4. *Identifying vector fields with 1-forms we have the alternating 3-form $\{\ ,\ ,\ \}_\infty = \int \nu \wedge \nu' \wedge \nu''$ that is defined on $\mathcal{V}_\infty$ and extends continuously to L^3 $(\mathbb{R}^3/\mathbb{Q}^3)$.*

We also have the self-adjoint operator curl denoted D that is defined and invertible on $\mathcal{V}_\infty$. As a bilinear form (x, Dy) extends continuously to $1/2$ derivative on L^2. In 3D this space is contained in L^3. The continuum Euler ODE, $\dot{X} = QX$, describing 3D incompressible fluid motion in the zero viscosity limit is the Eulerian flow associated to this 3-form $\{\ ,\ ,\ \}_\infty$ and operator D on the entire space $\mathcal{V}_\infty$ with respect to the L^2 inner product on the infinite-dimensional space.

Theorem 7.5. 1) *There is an exhausting sequence of finite dimensional subspaces $\mathcal{V}_n$ of $\mathcal{V}$ with consistent alternating forms $\{\ ,\ ,\ \}_n$ and self-adjoint invertible operators $D_n : \mathcal{V}_n \mapsto \mathcal{V}_n$ so that the corresponding Eulerian ODEs converge to the Euler ODE at infinity.*

2) *The Gaussian measure on L^2 defined by $\prod \mathrm{e}^{-|x_i|^2/2}dx_i$ over a basis is "formally" preserved by the Euler ODE at infinity.*

Proof. Choose the eigenbasis for the curl operator D. Define $\mathcal{V}_n$ by restricting to eigenspaces of eigenvalues at most n in absolute value. Let D_n be the restriction of curl $= D$ to $\mathcal{V}_n$. Let $\{\ ,\ ,\ \}_n$ be the restriction of the form $\{\ ,\ \}_\infty$ from Remark 7.4 (which is checked by easy calculation). The rest "follows" by the proposition. One must use normalized measures in the infinite product. q.e.d.

Conjecture *Using Poincaré recurrence for the approximating Eulerian flows above, it is possible to prove that for almost all initial conditions with respect to the Gaussian measure* $3D$ *incompressible fluid motion with zero or positive viscosity can be uniquely defined for all time.*

Readers familiar with Bourgain's work as in [**1**] will appreciate the conjecture.

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Mathematics Departments
CUNY Graduate Center
365 Fifth Ave
New York, NY 10016
and
Stony Brook University
New York 11794
E-mail address: dennis@math.sunysb.edu