# The Cumulant Bijection and Differential Forms

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## Description

Given a graded commutative algebra $A$ there is a canonical (tautologous) map $\tau : SA \to A$ where $SA$ is the graded commutative algebra generated by $A$. The map $\tau$ is given by the formula

$$\tau(x_1 \wedge x_2 \wedge \dots x_n) = x_1 x_2 \dots x_n$$

$SA = A \oplus A \wedge A \oplus \dots$ also has a coproduct structure $\Delta : SA \to SA \wedge SA$ given by the formula

$$\Delta(x_1 \wedge x_2 \wedge \dots x_n) = \sum \epsilon \cdot x_{\pi_1} \wedge x_{\pi_2}$$

where $\pi_1 \sqcup \pi_2$ is a partition of the indices into two arbitrary subsets and $\epsilon$ is the sign that arises when the first subset is moved all the way to the left. One knows that any map like $\tau : SA \to A$ lifts to a canonical coalgebra mapping $\tilde{\tau} : SA \to SA$ so that $\pi \circ \tilde{\tau} = \tau$ where $\pi : SA \to A$ is projection onto $A$, the linear summand of $SA$.

We verify that $\tilde{\tau}$ is given by the formula

$$\tilde{\tau}(v) = \tau(v) + \tau \wedge \tau \circ \Delta(v) + \tau^{\wedge 3} \circ \Delta^2(v) + \dots \quad (1)$$

Note that $\tilde{\tau}(v) = v +$ lower order terms. This implies that it is a bijection. We call this coalgebra isomorphism $\tilde{\tau}$ the *cumulant bijection*.

The relation of $\tilde{\tau}$ to cumulants appears when it is used to change coordinates in $SA$, in particular when it is used to conjugate extensions to $SA$ of a linear mapping of $A$ to either a coderivation of $SA$ or to a coalgebra mapping of $SA$.

General coderivations $D : SA \to SA$ have canonical decompositions by orders $D = D_1 + D_2 + \dots$ where $D_k$ is determined by "Taylor coefficients" $D^k$;

$$D^1 : A \to A$$
$$D^2 : A \wedge A \to A$$
$$D^3 : A \wedge A \wedge A \to A$$

and so on.

The explicit formulae relating $D_k$ to $D^k$ are

$$D_k(x_1 \wedge \dots \wedge x_n) = \sum \epsilon \cdot D^k(x_I) \wedge \dots \hat{x_I} \dots$$

where the sum is taken over all subsets $I$ of size $k$ of the indexing set.

A coalgebra mapping $G : SA \to SA$ is also determined by independent maps (also called the Taylor coefficients)

$$G^1 : A \to A$$

$$G^2 : A \wedge A \to A$$

$$G^3 : A \wedge A \wedge A \to A$$

and so on by the formula

$$G(x_1 \wedge x_2 \wedge \ldots) = \sum_{\text{partitions}} \epsilon \cdot G^{j_1}(x_{\pi_{j_1}}) \wedge G^{j_2}(x_{\pi_{j_2}}) \ldots$$

The canonical extension of a map $A \to A$ to either a coderivation or a coalgebra mapping, only has a non-zero leading order coefficient. After conjugating the canonical extensions by $\tilde{\tau}$ they have in general Taylor coefficients of all orders.

In the case of a coalgebra extension $\tilde{f}$ of $f : A \to A$ the conjugated $g = \tilde{\tau}^{-1}\tilde{f}\tilde{\tau}$ has coefficients which measure the deviation of $f$ from being an algebra homomorphism of $A$.

The first few Taylor coefficients of $g$ are

$$g^1(x) = f(x)$$

$$g^2(x, y) = f(xy) - f(x)f(y)$$

$$g^3(x, y, z) = f(xyz) - f(xy)f(z) - f(yz)f(x) - f(zx)f(y) + 2f(x)f(y)f(z)$$

In the case of a coderivation extension $\hat{f}$ of $f$, the first few coefficients of the conjugate $h = \tilde{\tau}^{-1}\hat{f}\tilde{\tau}$ are

$$h^1(x) = f(x)$$

$$h^2(x, y) = f(xy) - f(x)y - xf(y)$$

$$h^3(x, y, z) = f(xyz) - f(xy)z + xyf(z) - f(yz)x + yxf(x) - f(zx)y + zxf(y)$$

In the first case all the higher Taylor coefficients vanish iff $f(xy)-f(x)f(y) = 0$, that is $f$ is an algebra map.

In the second case all higher Taylor coefficients vanish iff $f(xy) - f(x)y - xf(y) = 0$, that is $f$ is a derivation.

# 1 First Application

Let us now consider the case where $A$ is a chain complex with a differential $\partial$ which is not necessarily a derivation of the product structure. Suppose $i : (C, \partial_C) \to (A, \partial)$ is a sub-complex of $(A, \partial)$ such that $A$ deformation retracts to $C$. That is, there is $I : (A, \partial) \to (C, \partial_C)$ so that $I \circ i = Id_C$ and there is a degree +1 mapping $s : A \to A$ so that $\partial s + s\partial = i \circ I - Id_A$. Now let $\tilde{I}$ and $\tilde{i}$ be the canonical lifts of $I$ and $i$ to coalgebra maps between $SA$ and $SC$.

$$\begin{array}{ccc} SA & \xrightarrow[\pi]{} & A \\ \tilde{i}\uparrow\downarrow\tilde{I} & & i\uparrow\downarrow I \\ SC & \xrightarrow[\pi]{} & C \end{array}$$

Let $d$ be the extension of $\partial$ to $SA$ and $\tilde{d} = \tilde{\tau}^{-1} d\tilde{\tau}$, Then by our previous discussion the cumulant bijection $\tilde{\tau}$ gives an isomorphism between $(SA, d)$ and $(SA, \tilde{d})$.

**Theorem 1.** Let $C$ be as above. Suppose there is a coderivation $\partial_\infty$ on $SC$ and a differential graded coalgebra map $\iota$ from $(SC, \partial_\infty)$ to $(SA, \tilde{d})$ which extends $i$. Then there is an induced isomorphism, "the induced cumulant bijection" $\tilde{\tau}_C$ between $(SC, \partial_\infty)$ and $(SC, d_C)$ where $d_C$ is the canonical coderivation extension of $\partial_C$. $\tilde{\tau}_C$ is uniquely characterized by the commutativity of the following diagram.

$$\begin{array}{ccc} (SA, \tilde{d}) & \xrightarrow{\tilde{\tau}} & (SA, d) \\ \iota\uparrow & & \downarrow\tilde{I} \\ (SC, \partial_\infty) & \xrightarrow[\tilde{\tau}_C]{} & (SC, d_C) \end{array}$$

*Proof.* As $\iota$ is an injection and $\tilde{I}$ is a surjection, $\tilde{\tau}_C$ must be defined to be equal to $\tilde{I} \circ \tilde{\tau} \circ \iota$. The fact that $\tilde{\tau}_C$ is an isomorphism follows as in Lemma 2, from the fact that since $\iota$ agrees with $i$ on the linear terms and $I \circ i$ is identity, $\tilde{\tau}_C$ induces the identity map on linear terms. Then since $\tilde{\tau}_C$ is a coalgebra map the inductive step of Lemma 2 holds. For more details see section 3.

□

*Remark* 1. Note that $SC$ receives the induced cumulant bijection from the map $\iota$ and not from a commutative algebra structure on $C$.

# 2 The missing details, the cumulant bijection in the associative context and more applications

**Definition 1.** A *graded coassociative conilpotent coalgebra* is a graded vector space $C$ together with a degree zero coassociative coproduct $\Delta : C \to C \otimes C$ with the property that for all $v$ in $C$ there exists an integer $n$ such that $\Delta^n(v) = 0$. If the image of the coproduct is in the graded symmetric tensors then $C$ is called the *graded symmetric coassociative conilpotent coalgebra*

**Definition 2.** The *free coassociative conilpotent coalgebra without co-unit* generated by a graded vector space $V$ is a nilpotent coalgebra $T^cV$ with a projection map onto $V$ with the following universal property: Given a linear map from a graded nilpotent coassociative coalgebra to $V$ there exists a unique coalgebra map $\tilde{f}$ from $C$ to $T^cV$ such that the following diagram commutes.

$$\begin{array}{ccc} & & C \\ & \overset{\exists \tilde{f}}{\swarrow} & \downarrow f \\ T^cV & \xrightarrow{\pi} & V \end{array} \tag{2}$$

On the space $TV = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \ldots$ consider the coproduct $\Delta$ given by the following formula.

$$\Delta(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = \sum_{i=1}^{n-1} x_1 \otimes \ldots x_i \bigotimes \ldots \otimes x_n$$

$TV$ with the coproduct $\Delta$ is the universal free conilpotent coalgebra associated to $V$. The formula for the lift is given later in Lemma 1.

**Definition 3.** The *free graded coassociative cocommutative nilpotent coalgebra* generated by $V$ is a sub-coalgebra $S^cV$ of $T^cV$ such that if $C$ taken as above is also graded cocommutative then the image of $\tilde{f}$ lies in $S^cV$, that is the following diagram commutes.

$$\begin{array}{ccc} & & C \\ & \overset{\exists \tilde{f}}{\swarrow} & \downarrow f \\ S^cV & \xrightarrow{\pi} & V \end{array} \tag{3}$$

$$S^cV = \bigoplus_{n=1}^{\infty} S^nV$$

where $S^nV$ is the subspace of $V^{\otimes n}$ generated by

$$\sum_{\sigma \in S_n} \pm x_{\sigma(1)} \otimes x_{\sigma(2)} \ldots x_{\sigma(n)}$$

The signs in the summation are given by the degrees of $x_i$, for example in $S^2V$ is generated by elements of the form $x \otimes y + (-1)^{|x||y|} y \otimes x$. The coproduct $\Delta$ restricts to a coproduct on $S^cV$. $S^cV$ is linearly isomorphic to $SV = V \oplus V \wedge V \oplus V \wedge V \wedge V \ldots$ by the map which sends $x_1 \wedge x_2 \wedge \ldots x_n$ in $SV$ to $\sum_{\sigma \in S_n} \pm x_{\sigma(1)} \otimes x_{\sigma(2)} \ldots x_{\sigma(n)}$ in $S^cV$. Note the product that $T^cV$ acquires from it's isomorphism to $TV$ does not restrict to the product $S^cV$ acquires from it's isomorphism to $SV$. Here onwards we will use $SV$ to denote both, the algebra with the usual product structure and the coalgebra with the induced coproduct structure.

**Lemma 1.** *The map $\tilde{f}$ in (2) is given by the formula,*

$$\tilde{f}(x) = f(x) + f \otimes f \circ \Delta_C(x) + f^{\otimes 3} \circ \Delta_C^2(x) + \dots$$

*where $\Delta_C$ is the reduced coproduct on $C$. Since $C$ is nilpotent this sum is finite.*

*Proof.* Note that $\pi \circ \tilde{f}(x) = f(x)$. To prove the lemma we need to show that

$$\Delta_T \circ \tilde{f} = \tilde{f} \otimes \tilde{f} \circ \Delta_C$$

The Sweedler notation for higher powers of $\Delta_C$ applied to $x$ is

$$\Delta_C^n(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \dots \otimes x_{(n+1)}$$

So the formula for $\tilde{f}$ as defined becomes

$$\tilde{f}(x) = \sum_n \sum_{(x)} f(x_{(1)}) \otimes f(x_{(2)}) \otimes \dots \otimes f(x_{(n+1)})$$

and when we apply the coproduct to this we get

$$\Delta_T \circ \tilde{f}(x) = \sum_n \sum_{(x)} \sum_i f(x_{(1)}) \otimes \dots \otimes f(x_{(i)}) \bigotimes \dots \otimes f(x_{(n+1)})$$

On the other hand we have

$$\tilde{f} \otimes \tilde{f} \circ \Delta_C(x) = \sum_{(x)} \tilde{f}(x_{(1)}) \otimes \tilde{f}(x_{(2)})$$

$$= \sum_{(x)} \sum_{i,j} f(x_{(1)}) \otimes \dots \otimes f(x_{(i)}) \bigotimes \dots \otimes f(x_{(i+j)})$$

$$= \sum_n \sum_{(x)} \sum_i f(x_{(1)}) \otimes \dots \otimes f(x_{(i)}) \bigotimes \dots \otimes f(x_{(n+1)})$$

which is what is required. □

If $C$ is graded cocommutative then the image of $\tilde{f}$ is in $S^cV$ which we have identified with $SV$. So the formula in Lemma 1 becomes

$$\tilde{f}(x) = f(x) + f \wedge f \circ \Delta_C(x) + f^{\wedge 3} \circ \Delta_C^2(x) + \dots$$

Suppose $A$ is a graded commutative associative algebra. Then consider the map $\tau : SA \to A$ which takes $x_1 \wedge x_2 \wedge \dots x_n$ to $x_1 x_2 \dots x_n$. We call this map the tautologous map. Since this is a map from a graded cocommutative coalgebra we can lift it to a map $\tilde{\tau} : SA \to SA$ and from Lemma 1 the formula for $\tilde{\tau}$ is given by the equation 1.

**Lemma 2.** *$\tilde{\tau} : SA \to SA$ is a coalgebra isomorphism.*

*Proof.* Let $F_n = \bigoplus_{i=1}^{n} \wedge^i A$ where $\wedge^i A$ is the graded symmetric product of $i$ copies of $A$. Then $F_n$ defines an increasing filtration of $SA$ and $SA$ is the direct limit of this sequence of sub-coalgebras.

$$A = F_1 \hookrightarrow F_2 \hookrightarrow \ldots F_n \hookrightarrow \ldots$$

For $v \in \wedge^n A$, $\tilde{\tau}$ is given by the formula

$$\tilde{\tau}(v) = \tau(v) + \tau \wedge \tau(\Delta(v)) + \ldots + \tau^{\wedge n}(\Delta^{n-1}(v))$$

Note that the highest degree term in the formula is $v$ and so this formula preserves the filtration $F_n$. We will show by induction that $\tilde{\tau}$ restricted to $F_n$ is an isomorphism for each $n$. Now on $F_1$, $\tilde{\tau}$ is identity hence an isomorphism. Now consider the following short exact sequence

$$0 \to F_{n-1} \hookrightarrow F_n \to \wedge^n A \to 0$$

On $F_{n-1}$, $\tilde{\tau}$ is an isomorphism by induction hypothesis and on $\wedge^n A$ the map induced is the identity map. So $\tilde{\tau}$ is an isomorphism on $F_n$. Since it's restriction to each $F_n$ is an isomorphism, $\tilde{\tau}$ is an isomorphism from $SA$ to $SA$. □

A linear map from a graded commutative algebras $A$ to itself can be extended to either a coderivation or a coalgebra map from $SA$ to $SA$. These can then be conjugated by $\tilde{\tau}$ as it is an isomorphism. The linear map itself doesn't have to be an algebra map or a derivation.

**Theorem 2.** i) Suppose $d$ is any linear map from a graded commutative algebra $A$ to $A$. Then there exists a unique coderivation $\tilde{d}$ from $SA$ to $SA$ such that the following diagram commutes.

$$\begin{array}{ccc} SA & \xrightarrow{\tau} & A \\ \tilde{d}\downarrow & & \downarrow d \\ SA & \xrightarrow{\tau} & A \end{array} \tag{4}$$

The coderivation construction, $d$ gives $\tilde{d}$, preserves commutators. This implies that if $d$ has degree $-1$ and squares to zero then $\tilde{d}$ also has degree$-1$ and squares to zero.

ii) Suppose $f$ is any linear map between graded commutative algebras $A$ to $B$. Then there exists a unique coalgebra morphism $\hat{f}$ from $SA$ to $SB$ such that the following diagram commutes.

$$\begin{array}{ccc} SA & \xrightarrow{\tau_A} & A \\ \hat{f}\downarrow & & \downarrow f \\ SB & \xrightarrow{\tau_B} & B \end{array} \tag{5}$$

Note that the uniqueness implies that the coalgebra construction , $f$ gives $\hat{f}$, respects composition.

iii) ([1] and [2]) Suppose $A$ and $B$ are graded commutative algebras and $d_A$ and $d_B$ are linear maps, not necessarily derivations, of degree $-1$ and square zero on $A$ and $B$ respectively. Suppose $f$ is a linear map, not necessarily an algebra map, from $A$ to $B$ such that $f \circ d_A = d_B \circ f$. Then $\hat{f}$ as in ii) is the unique homomorphism of differential graded coalgebras from $(SA, \tilde{d_A})$ to $(SB, \tilde{d_B})$ such that the following diagram commutes.

$$\begin{array}{ccc} (SA, \tilde{d_A}) & \xrightarrow{\tau_A} & (A, d_A) \\ \hat{f}\downarrow & & \downarrow f \\ (SB, \tilde{d_B}) & \xrightarrow{\tau_B} & (B, d_B) \end{array} \tag{6}$$

*Proof.* i) A linear map $d$ from $A$ to $A$ can be uniquely extended to a coderivation $D$ on $SA$ which commutes with the projection map to $A$ for the coalgebra structure on $SA$. That is there exists a unique $D$ such that the following diagram commutes.

$$\begin{array}{ccc} SA & \xrightarrow{\pi} & A \\ \tilde{d}\downarrow & & \downarrow d \\ SA & \xrightarrow{\pi} & A \end{array} \tag{7}$$

where $\pi$ is the projection map. $D$ is given by the following formula.

$$D(x_1 \wedge x_2 \wedge \dots x_n) = \sum_i \pm x_1 \wedge \dots \wedge d(x_i) \wedge \dots x_n$$

The signs in the sum depend on the degrees of $x_i$ and the degree of the map $d$. Consider the map $\tilde{d} := \tilde{\tau} \circ D \circ \tilde{\tau}^{-1}$. This map is the unique coderivation on $SA$ which makes (4) commute.

If $d_1$ and $d_2$ are two linear maps of some degrees from $A$ to $A$ and let $D_1$ and $D_2$ be the coderivations on $SA$ which commute with the projection maps. Since the bracket of two coderivations is a coderivation, $[D_1, D_2]$ is a coderivation. Besides this is the unique coderivation which extends $[d_1, d_2]$. So, $[\tilde{d_1}, \tilde{d_2}]$ is the unique coderivation with the above property for the map $[d_1, d_2]$. Thus if $d$ has degree $-1$ and squares to zero, then $\tilde{d}^2 = 1/2[\tilde{d}, \tilde{d}]$ is a coderivation which uniquely extends $d^2$. Since $d^2$ is zero, $\tilde{d}^2$ will also be zero. So maps of degree $-1$ which square to zero give coderivations which square to zero on $SA$.

ii) A linear map $f$ between $A$ and $B$ can be uniquely extended to a homomorphism of coalgebras $F$ from $SA$ to $SB$ which commutes with the projection maps. That is there exists a homomorphism of coalgebras such that the following diagram commutes.

$$\begin{array}{ccc} SA & \xrightarrow{\pi_A} & A \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ SB & \xrightarrow{\pi_B} & B \end{array} \tag{8}$$

$F$ is given by the following formula.

$$F(x_1 \wedge x_2 \wedge \dots x_n) = f(x_1) \wedge f(x_2) \wedge \dots f(x_n)$$

The map $\hat{f} = \tilde{\tau} \circ F \circ \tilde{\tau}^{-1}$ is then the unique coalgebra homomorphism which makes (5) commute.

iii) We only need to check that $\hat{f} \circ \tilde{d_A} = \tilde{d_B} \circ \hat{f}$. Let $D_A$ and $D_B$ be the coderivations on $SA$ and $SB$ respectively which commute with the projection maps and $F$ be the coalgebra homomorphism from $SA$ to $SB$ which commutes with the projections. Since $f \circ d_A = d_B \circ f$, by direct computation we have that $F \circ D_A = D_B \circ F$. It follows then that $\hat{f} \circ \tilde{d_A} = \tilde{d_B} \circ \hat{f}$. □

Conjugation by $\tilde{\tau}$ of the lift to a coalgebra map measures the deviation of the original map from being an algebra homeomorphism and that of the lift to a coderivation measures how far the original map was from being a derivation for the algebra structure.

**Proposition 1.** Let $d$ be a linear map from $A$ to $A$ and $\tilde{d}$ be the coderivation constructed in the previous theorem. Let $f$ be a linear map from $A$ to $B$ and $\hat{f}$ be homomorphisms constructed from $f$ in the last theorem.

i) $d$ is a derivation if and only if

$$\tilde{d}(x_1 \wedge x_2 \wedge \dots x_n) = \sum_i \pm x_1 x_2 \dots d(x_i) \dots x_n$$

ii) $f$ is a homomorphism of algebras if and only if

$$\hat{f}(x_1 \wedge x_2 \wedge \dots x_n) = \tau(f(x_1) \wedge f(x_2) \wedge \dots f(x_n)) = f(x_1)f(x_2)\dots f(x_n)$$

*Proof.* i) In the proof of the last theorem we defined the coderivation $D$ such that it satisfies the formula

$$\tau \circ D(x_1 \wedge x_2 \wedge \dots x_n) = \sum_i \pm x_1 x_2 \dots d(x_i) \dots x_n$$

This expression is equal to $d(x_1 x_2 \dots x_n) = d \circ \tau(x_1 \wedge x_2 \wedge \dots x_n)$ if and only if $d$ is a derivation for the algebra $A$. Thus we have that $d \circ \tau = \tau \circ D$ if and only if $d$ is a derivation for the algebra $A$. Since $\tilde{d}$ is the unique coderivation that commutes with $\tau$, $\tilde{d} = D$ if and only if $d$ is a derivation.

ii) Similar to the proof of the first part

$$\tau \circ F(x_1 \wedge x_2 \wedge \ldots x_n) = \tau(f(x_1) \wedge f(x_2) \wedge \ldots f(x_n)) = f(x_1)f(x_2)\ldots f(x_n)$$

which is equal to $f(x_1 x_2 \ldots x_n)$ if and only if $f$ is a homomorphism. By uniqueness of $\tilde{f}$ we have that it is equal to $F$ if and only if $f$ is a homomorphism.

□

## 3 More detailed Proof of Theorem 1

*Proof.* The proof is similar to the proof of Lemma 2. Define $\tilde{\tau}_C$ to be $\tilde{I} \circ \tilde{\tau} \circ \iota$. This map is a composition of differential coalgebra maps and hence is itself a differential coalgebra map. Let $F_n = \bigoplus_{i=1}^{n} \wedge^i C$, where $\wedge^i C$ is the graded symmetric product of $i$ copies of $C$. The $F_n$ is the filtration on $SC$. We will use induction on the degree of the filtration to prove that $\tilde{\tau}_C$ is an isomorphism. As $\iota$ preserves the filtration $\tilde{\tau}_C$ also preserves the filtration. Also as $\iota$ agrees with $i$ on $C = F_1$ and since $\tilde{\tau}$ is identity on $A$, $\tilde{\tau}_C$ is identity on $F_1$. Now suppose $\tilde{\tau}_C$ is an isomorphism when restricted to $F_{n-1}$. Since $\tilde{\tau}_C$ is a coalgebra mapping, for $v = x_1 \wedge x_2 \wedge \ldots x_n$ in $\wedge^n C$ we have that $\tilde{\tau}_C^{\wedge n} \circ \Delta^{n-1}(v) = \Delta^{n-1} \circ \tilde{\tau}_C(v) = v$. This implies that $\tilde{\tau}_C(v) = v+$ lower order terms. Then consider the short exact sequence

$$0 \to F_{n-1} \hookrightarrow F_n \to \wedge^n C \to 0$$

By induction hypothesis $\tilde{\tau}_C$ is an isomorphism on $F_{n-1}$ and by the above argument it induces identity on $\wedge^n C$. Hence it is an isomorphism on $F_n$ for every $n$ which implies it is a coalgebra isomorphism of $SC$.

□

## 4 Related Work

Our focus on the cumulant bijection was directly inspired by a lecture of Jae Suk Park at CUNY November 2011. The ideas in the lecture have been developed in the two papers [1] and [2].

The "induced cumulant bijection" is used in [3] to set up potential algorithms for computing 3D fluid motion based on differential forms and the integration deformation retract to cochains.

## References


[1] Gabriel C. Drummond-Cole, Jae-Suk Park, John Terilla. "Homotopy Probability Theory I". preprint arXiv February 2013

[2] Gabriel C. Drummond-Cole, Jae-Suk Park, John Terilla. "Homotopy Probability Theory II". preprint arXiv February 2013

[3] D. Sullivan. “3D Incompressible Fluids: Combinatorial Models, Eigenspace Models, and a Conjecture about Well-posedness of the 3D Zero Viscosity Limit”. to appear in JDG Hirzebruch Volume 2014