The Mayer-Vietoris Property in Differential Cohomology

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Abstract

In [1] it was shown that $K\hat{}$, a certain differential cohomology functor associated to complex K-theory, satisfies the Mayer-Vietoris property when the underlying manifold is compact. It turns out that this result is quite general. The work that follows shows the M-V property to hold on compact manifolds for any differential cohomology functor $J\hat{}$ associated to any Z-graded cohomology functor J(,Z) which, in each degree, assigns to a point a finitely generated group. The approach is to show that the result follows from Diagram 1, the commutative diagram we take as a definition of differential cohomology, and Diagram 2, which combines the three Mayer-Vietoris sequences for $J^*(,Z), J^*(,R)$ and $J^*(,R/Z)$.

Let $J = \sum \oplus J^k$ be a graded generalized cohomology functor. We assume each $J^k(\text{point})$ is finitely generated. By a differential cohomology functor associated to J we mean a functor $\hat{J}$ on the category of smooth manifolds with corners, together with four natural transformations, $i_1, i_2, \delta_1, \delta_2$, which satisfies the following commutative diagram of abelian group valued functors.

Diagram 1
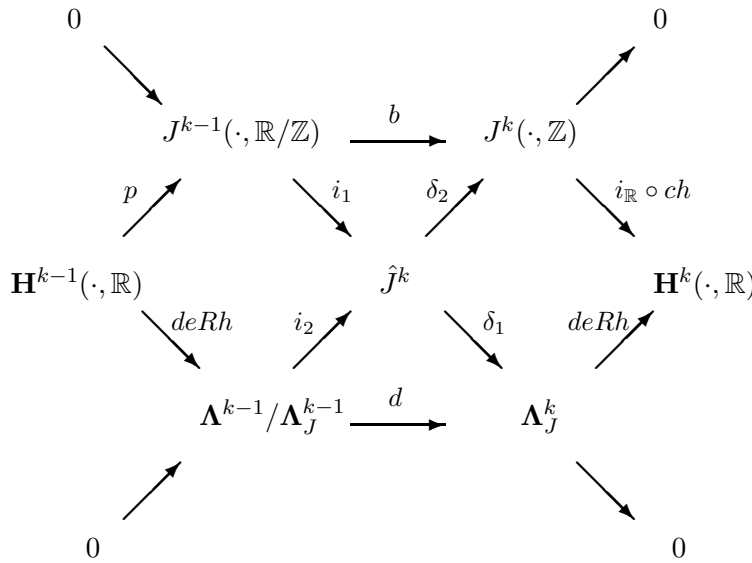
In the above the diagonals are short exact, and the upper and lower four-term sequences are also exact, and

$$\mathbf{H}^k(\cdot, \mathbb{R}) = \sum_{j=0} \oplus H^j(\cdot, J^{k-j}(\text{point}, \mathbb{R}))$$

$$\mathbf{\Lambda}^k = \sum_{j=0} \oplus \wedge^j(\cdot, J^{k-j}(\text{point}, \mathbb{R}))$$

$$\mathbf{\Lambda}_J^k = (\text{de Rham})^{-1}(\text{Im}(ch \circ i_{\mathbb{R}}))$$

$ch : J^k(\mathbb{Z}) \to \mathbf{H}^k(\mathbb{Q})$ is the canonical map, $i_{\mathbb{R}}$ is induced by $\mathbb{Q} \to \mathbb{R}$, p is induced by the coefficient sequence $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, and b denotes the associated Bockstein map. The maps $deRh$ and d are respectively the de Rham map and the exterior differential.

Theorem (Mayer-Vietoris Property): Let X be a compact smooth manifold. Assume $X = A \cup B$, and $A \cap B = D$, a co-dim 0 submanifold with collar neighborhoods in both A and B. Then, if $f_A \in \hat{J}^k(A)$ and $f_B \in \hat{J}^k(B)$ with $f_A|D = f_B|D$, then there exists $f \in \hat{J}^k(X)$ with $f|A = f_A$ and $f|B = f_B$.

Proof: Since $\delta_2(f_A)|D = \delta_2(f_B)|D$, the Mayer-Vietoris property for J shows there exists $v \in J^k(X)$ with $v|A = \delta_2(f_A)$ and $v|B = \delta_2(f_B)$. Choose $h \in \hat{J}^k(X)$ with $\delta_2(h) = v$. By naturality

$$\delta_2(h|A) = \delta_2(f_A) \quad \text{and} \quad \delta_2(h|B) = \delta_2(f_B)$$

thus by Diagram 1

1) $\quad h|A - f_A = i_2(\{\alpha_A\}) \qquad \{\alpha_A\} \in \mathbf{\Lambda}^{k-1}(A)/\mathbf{\Lambda}_J^{k-1}(A)$

$\quad h|B - f_B = i_2(\{\alpha_B\}) \qquad \{\alpha_B\} \in \mathbf{\Lambda}^{k-1}(B)/\mathbf{\Lambda}_J^{k-1}(B)$.

Under restriction to D the left hand sides are equal by hypothesis. Since i_2 is an injection, by naturality we see that $\{\alpha_A\}|D = \{\alpha_B\}|D$.

Suppose one can find $\bar{h} \in \hat{J}^k(X)$ with $\bar{h}|A = i_2(\{\alpha_A\})$ and $\bar{h}|B = i_2(\{\alpha_B\})$. Then by 1), $(h - \bar{h})|A = f_A$ and $(h - \bar{h})|B = f_B$, and the problem is solved.

The problem thus reduces to the case that $f_A = i_2(\{\alpha_A\})$ and $f_B = i_2(\{\alpha_B\})$. The remainder of the proof will be restricted to this case.

Since $\{\alpha_A\}|D - \{\alpha_B\}|D = 0$ we must have $\alpha_A|D - \alpha_B|D \in \mathbf{\Lambda}_J^{k-1}(D)$. If instead we had chosen α_A' and α_B' representing $\{\alpha_A\}$ and $\{\alpha_B\}$ then $\alpha_A - \alpha_A' \in \mathbf{\Lambda}_J^{k-1}(A)$ and $\alpha_B - \alpha_B' \in \mathbf{\Lambda}_J^{k-1}(B)$. Thus

$$(\alpha_A|D - \alpha_B|D) - (\alpha_A'|D - \alpha_B'|D) \in \mathbf{\Lambda}_J^{k-1}(A)|D + \mathbf{\Lambda}_J^{k-1}(B)|D$$

and therefore

$$w(\{\alpha_A\}, \{\alpha_B\}) = (\alpha_A|D - \alpha_B|D) \in \frac{\boldsymbol{\Lambda}_J^{k-1}(D)}{\boldsymbol{\Lambda}_J^{k-1}(A)|D + \boldsymbol{\Lambda}_J^{k-1}(B)|D}$$

is well defined. Suppose $w(\{\alpha_A\}, \{\alpha_B\}) = 0$. Then $\alpha_A|D - \alpha_B|D = \beta_A|D - \beta_B|D$, where $\beta_A \in \boldsymbol{\Lambda}_J^{k-1}(A)$ and $\beta_B \in \boldsymbol{\Lambda}_J^{k-1}(B)$. Thus $\{\alpha_A\} = \{\alpha_A - \beta_A\}$, $\{\alpha_B\} = \{\alpha_B - \beta_B\}$, and

$$(\alpha_A - \beta_A)|D = (\alpha_B - \beta_B)|D.$$

Since D has co-dim 0 and collar neighborhoods in both A and B, there exists a unique $\theta \in \boldsymbol{\Lambda}^{k-1}(X)$ with $\theta|A = \alpha_A - \beta_A$ and $\theta|B = \alpha_B - \beta_B$. Thus $\{\theta\}|A = \{\alpha_A\}$ and $\{\theta\}|B = \{\alpha_B\}$, which implies that $i_2(\{\theta\})|A = i_2(\{\alpha_A\})$ and $i_2(\{\theta\})|B = i_2(\{\alpha_B\})$.

We have therefore shown

2) $w(\{\alpha_A\}, \{\alpha_B\}) = 0 \implies$ problem is solved.

Set $J_o^k(X) = \{v \in J^k(X) \mid v|A = 0 = v|B\}$. Let $v \in J_o^k(X)$ and choose $h \in \hat{J}^k(X)$ with $\delta_2(h) = v$. By naturality, $\delta_2(h|A) = 0 = \delta_2(h|B)$. Thus

$$\begin{aligned} h|A &= i_2(\{\gamma_A\}), \\ h|B &= i_2(\{\gamma_B\}), \text{ and} \\ \{\gamma_A\}|D &= \{\gamma_B\}|D. \end{aligned}$$

Set

$$\Omega(v) = w(\{\gamma_A\}, \{\gamma_B\}).$$

To see that Ω is well defined, let $\bar{h} \in \hat{J}^k(X)$ with $\delta_2(\bar{h}) = v$. Then $\bar{h} = h + i_2(\{\rho\})$ for some $\rho \in \boldsymbol{\Lambda}^{k-1}(X)$. So $\bar{h}|A = i_2\{\gamma_A + \rho|A\}$ and $\bar{h}|B = i_2\{\gamma_B + \rho|B\}$. Since $(\rho|A)|D = \rho|D = (\rho|B)|D$, the definition of w shows $w(\{\gamma_A + \rho|A\}, \{\gamma_B + \rho|B\} = w(\{\gamma_A\}, \{\gamma_B\})$. Thus,

$$\Omega : J_o^k(X) \to \frac{\boldsymbol{\Lambda}_J^{k-1}(D)}{\boldsymbol{\Lambda}_J^{k-1}(A)|D + \boldsymbol{\Lambda}_J^{k-1}(B)|D}$$

is well defined, and is clearly a homomorphism.

Now, given $\{\alpha_A\}, \{\alpha_B\}$ with $\{\alpha_A\}|D = \{\alpha_B\}|D$, suppose we can find $v \in J_o^k(X)$ with $\Omega(v) = w(\{\alpha_A\}, \{\alpha_B\})$. Then, choosing h with $\delta_2(h) = B$ and letting $h|A = \{\gamma_A\}$ and $h|B = \{\gamma_B\}$, we see

$$w(\{\alpha_A - \gamma_A\}, \{\alpha_B - \gamma_B\}) = 0.$$

By 2), this implies there exists $\theta \in \boldsymbol{\Lambda}^{k-1}(X)$ with

$$\begin{aligned} \{\theta\}|A &= \{\alpha_A - \gamma_A\} = \{\alpha_A\} - \{\gamma_A\} \\ \{\theta\}|B &= \{\alpha_B - \gamma_B\} = \{\alpha_B\} - \{\gamma_B\} \end{aligned}$$

Thus

$$(i_2(\{\theta\}) + h)|A = i_2(\{\alpha_A\})$$
$$(i_2(\{\theta\}) + h)|B = i_2(\{\alpha_B\})$$

and so $i_2(\{\theta\}) + h$ solves the problem for the coherent pair $i_2(\{\alpha_A\}), i_2(\{\alpha_B\})$.

The proof of the theorem will clearly be completed if we can show

$*)$ Ω is surjective.

The remainder of the work will be devoted to proving $*)$.

We consider the following diagram in which the rows are Mayer-Vietoris exact sequences of the various cohomology functors.

Diagram 2

$$
\begin{array}{ccccccc}
J^{k-2}(A,\mathbb{R}/\mathbb{Z})\oplus J^{k-2}(B,\mathbb{R}/\mathbb{Z}) & \xrightarrow{\Delta_1} & J^{k-2}(D,\mathbb{R}/\mathbb{Z}) & \xrightarrow{d_1^*} & J^{k-1}(X,\mathbb{R}/\mathbb{Z}) & \xrightarrow{\Sigma_1} & J^{k-1}(A,\mathbb{R}/\mathbb{Z})\oplus J^{k-1}(B,\mathbb{R}/\mathbb{Z}) \\
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
J^{k-1}(A,\mathbb{Z})\oplus J^{k-1}(B,\mathbb{Z}) & \xrightarrow{\Delta_2} & J^{k-1}(D,\mathbb{Z}) & \xrightarrow{d_2^*} & J^{k}(X,\mathbb{Z}) & \xrightarrow{\Sigma_2} & J^{k}(A,\mathbb{Z})\oplus J^{k}(B,\mathbb{Z}) \\
\downarrow ch & & \downarrow ch & & \downarrow ch & & \downarrow ch \\
\mathbf{H}^{k-1}(A,\mathbb{Q})\oplus\mathbf{H}^{k-1}(B,\mathbb{Q}) & \xrightarrow{\Delta_3} & \mathbf{H}^{k-1}(D,\mathbb{Q}) & \xrightarrow{d_3^*} & \mathbf{H}^{k}(X,\mathbb{Q}) & \xrightarrow{\Sigma_3} & \mathbf{H}^{k}(A,\mathbb{Q})\oplus\mathbf{H}^{k}(B,\mathbb{Q})
\end{array}
$$

The Δ's are the differences of the restrictions to D of the individual components. d^* is the Mayer-Vietoris promotion map. $\sum$ restricts an element to each of A and B and takes their direct sum. b is the Bockstein map, and ch is defined in Diagram 1. It is well known that all 2×2 boxes commute up to appropriate sign in the graded sense. Note that $\text{Im}(ch)$ is a spanning lattice in $\mathbf{H}^*(\cdot, \mathbb{Q})$.

The proof of $*)$ will now follow from a series of lemmas.

Lemma 1:

$$
\frac{\mathbf{\Lambda}_J^{k-1}(D)}{\mathbf{\Lambda}_J^{k-1}(A)|D + \mathbf{\Lambda}_J^{k-1}(B)|D} \xrightarrow[\cong]{\text{de Rham}} \frac{i_\mathbb{R} \circ ch(J^{k-1}(D,\mathbb{Z}))}{i_\mathbb{R} \circ ch(\text{Im}(\Delta_2))} \xleftarrow[\cong]{i_\mathbb{R}} \frac{ch(J^{k-1}(D,\mathbb{Z}))}{ch(\text{Im}(\Delta_2))}
$$

Proof: Since de Rham maps the denominator of the first expression into that of the second, the first map is well defined and is onto since the map of the numerator is onto. If $\theta \in \mathbf{\Lambda}^{k-1}(D)$ maps to an element of $i_\mathbb{R} \circ ch(\text{Im}(\Delta_2))$ there must be an $\eta \in \mathbf{\Lambda}_J^{k-1}(A)|D + \mathbf{\Lambda}_J^{k-1}(B)|D$ and $\mu \in \mathbf{\Lambda}_{exact}^{k-1}$ with $\theta = \eta + \mu$. But any exact form on D is the restriction of an exact form on A. Moreover $\mathbf{\Lambda}_{exact}^* \subseteq \mathbf{\Lambda}_J^*$, and thus $\mu \in \mathbf{\Lambda}_J^{k-1}(A)|D$, and so $\theta \in \mathbf{\Lambda}_J^{k-1}(A)|D + \mathbf{\Lambda}_J^{k-1}(B)|D$. Therefore de Rham is $1:1$, and so an isomorphism. That $i_\mathbb{R}$ induces an isomorphism is straightforward. ∎

By the above, we may consider

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3) $\quad \Omega: J_o^k(X) \to \dfrac{ch(J^{k-1}(D,\mathbb{Z}))}{ch(\mathrm{Im}(\Delta_2))}.$

Note that by Diagram 2,

$$ch(\mathrm{Im}(\Delta_2)) = \Delta_3(\mathrm{Im}(ch)) \subseteq \mathrm{Im}(\Delta_3).$$

<u>Lemma 2</u>: Let

$$\phi: \frac{ch(J^{k-1}(D,\mathbb{Z}))}{ch(\mathrm{Im}(\Delta_2))} \to \frac{\mathbf{H}^{k-1}(D,\mathbb{Q})}{\mathrm{Im}(\Delta_3)}$$

be the map induced by inclusion. Then

$$\ker(\phi) = \mathrm{tor}\left(\frac{ch(J^{k-1}(D,\mathbb{Z}))}{ch(\mathrm{Im}(\Delta_2))}\right).$$

<u>Proof</u>: Clearly torsion $\subseteq \ker(\phi)$ since the image of ϕ lies in a rational vector space. Let $x_1, \cdots, x_n$ be a set of generators of $J^{k-1}(A,\mathbb{Z}) \oplus J^{k-1}(B,\mathbb{Z})$. Then $\{ch(x_i)\}$ span $\mathbf{H}^{k-1}(A,\mathbb{Q}) \oplus \mathbf{H}^{k-1}(B,\mathbb{Q})$, and thus $\{\Delta_3(ch(x_i))\} = \{ch(\Delta_2(x_i))\}$ generate $\mathrm{Im}(\Delta_3)$. Therefore if $y \in ch(J^{k-1}(D,\mathbb{Z}))$, and $y \in \mathrm{Im}(\Delta_3)$, $y = \sum q_i ch(\Delta_2(x_i))$ for some choice of rational $\{q_i\}$. Clearing denominators leads to integers $m, m_1, \cdots, m_n$ with $my = \sum m_i ch(\Delta_2(x_i))$. Thus y represents a torsion element in $ch(J^{k-1}(D,\mathbb{Z}))/ch(\mathrm{Im}(\Delta_2))$. ∎

Let

$$\Pi: \frac{ch(J^{k-1}(D,\mathbb{Z}))}{ch(\mathrm{Im}(\Delta_2))} \to \frac{ch(J^{k-1}(D,\mathbb{Z}))}{ch(\mathrm{Im}(\Delta_2))}\,/\,\mathrm{Torsion}$$

<u>Lemma 3</u>: $\Pi \circ \Omega$ is surjective.

<u>Proof</u>: From Diagram 2 we derive

By $\frac{ch}{ch}$ we mean the application of ch to both numerator and denominator. Clearly $\Pi \circ \frac{ch}{ch}$ is onto. Since $J_o^k(X, \mathbb{Z}) = \ker(\sum_2)$, Diagram 2 shows that, as used above, d_2^* is an isomorphism and d_3^* is $1:1$. By Lemma 2, ϕ is $1:1$.

Recalling the definition of w and Ω, an element $v \in J_o^k(X, \mathbb{Z})$ and a choice of $h \in \hat{J}^k(X)$ with $\delta_1(h) = v$ gives rise to elements $\gamma_A, \gamma_B \in \mathbf{\Lambda}^{k-1}(A), \mathbf{\Lambda}^{k-1}(B)$, with $\gamma_A|D - \gamma_B|D \in \mathbf{\Lambda}_J^{k-1}(D)$, the de Rham image of which lies in $H^{k-1}(D, \mathbb{Q})$. Letting $[\gamma_A|D - \gamma_B|D]$ represent its rational cohomology class, and using 3), we may write

$$\Omega(v) = [\gamma_A|D - \gamma_B|D] \mod ch(\text{Im}(\Delta_2)).$$

From Diagram 1, we see $\delta_1(h|A) = d\gamma_A$, and $\delta_1(h|B) = d\gamma_B$. Since

$$d\gamma_A|D - d\gamma_B|D = d(\gamma_A|D - \gamma_B|D) = 0$$

we may define the closed form η on X by $\eta|A = d\gamma_A$ and $\eta|B = d\gamma_B$. Clearly $\eta = \delta_1(h)$ and thus $[\eta] = i_\mathbb{R}(ch(v))$. Let d^* denote the Mayer-Vietoris promotion map in $\mathbf{H}^*(\cdot, \mathbb{R})$. From the definition of $[\eta]$, we see that

$$d^*(i_\mathbb{R}([\gamma_A|D - \gamma_B|D])) = i_\mathbb{R}(ch(v)).$$

Since $d^* \circ i_\mathbb{R} = i_\mathbb{R} \circ d_3^*$ we see that $d_3^*([\gamma_A|D - \gamma_B|D]) = ch(v)$, and this implies that in the above diagram

4) $d_3^* \circ \phi \circ \Pi \circ \Omega(v) = ch(v).$

To show that $\Pi \circ \Omega$ is surjective, let $x \in ch(J^{k-1}(D, \mathbb{Z}))/ch(\text{Im}(\Delta_2)) \mod$ torsion, and choose y with $\Pi \circ \frac{ch}{ch}(y) = x$. Then by 4)

$$d_3^* \circ \phi \circ \Pi \circ \Omega(d_2^*(y)) = ch(d_2^*(y)) = d_3^* \circ ch(y) = d_3^* \circ \phi \circ \Pi \circ \frac{ch}{ch}(y).$$

Since $d_3^* \circ \phi$ is $1:1$ we must have $\Pi \circ \Omega(d_2^*(y)) = \Pi \circ \frac{ch}{ch}(y) = x$.   ∎

By the commutativity of Diagram 2 we note that $b(\text{Im}(d_1^*)) = d_2^*(\text{Im}(b)) \subseteq \ker(\sum_2) = J_o^k(X, \mathbb{Z})$.

Lemma 4: $b(\text{Im}(d_1^*)) \subseteq \ker(\Omega)$.

Proof: Let $x \in J^{k-2}(D, \mathbb{R}/\mathbb{Z})$, and $b(d_1^*(x)) = v \in J_o^k(X, \mathbb{Z})$. To compute $\Omega(v)$ we need $h \in \hat{J}^k(X)$ with $\delta_2(h) = v$ and consider $h|A$ and $h|B$. By Diagram 1 we may take $h = i_1(d_1^*(x))$. But $i_1(d_1^*(x))|A = i_1((d_1^*(x)|A) = 0$ since $d_1^*(x) \in \ker(\sum_1)$. Similarly for B. Thus $\Omega(v) = 0$.   ∎

Thus, we may regard

5) $\Omega : \dfrac{J_o^k(X, \mathbb{Z})}{b(\text{Im}(d_1^*))} \to \dfrac{ch(J^{k-1}(D, \mathbb{Z}))}{ch(\text{Im}(\Delta_2))}.$

Lemma 5:

$$\frac{J^{k-1}(D,\mathbb{Z})}{\mathrm{Im}(\Delta_2)+\mathrm{Tor}(J^{k-1}(D,\mathbb{Z}))} \quad \begin{array}{c} \xrightarrow[\cong]{\;d_2^*\;} \\[4pt] \searrow{\scriptstyle ch} \\ {\scriptstyle \cong} \end{array} \quad \begin{array}{c} \dfrac{J^k(X,\mathbb{Z})}{b(\mathrm{Im}(d_1^*))} \\[10pt] \big\downarrow{\scriptstyle \cong} \\[10pt] \dfrac{ch(J^{k-1}(D,\mathbb{Z}))}{ch(\mathrm{Im}(\Delta_2))} \end{array}$$

Proof: In the upper case we note that $d_2^* : J^{k-1}(D,\mathbb{Z})/\mathrm{Im}(\Delta_2) \xrightarrow{\cong} J_o^k(X,\mathbb{Z})$, and $d_2^*(\mathrm{Tor}(J^{k-1}(D,\mathbb{Z}))) = d_2^*(\mathrm{Im}(b)) = b(\mathrm{Im}(d_1^*))$. In the lower case we note that $ch : J^{k-1}(D,\mathbb{Z})/\mathrm{Tor}(J^{k-1}(D,\mathbb{Z})) \xrightarrow{\cong} ch(J^{k-1}(D,\mathbb{Z}))$. The vertical isomorphism then follows. $\blacksquare$

Lemma 6: $\quad \Omega : \mathrm{Tor}\left(\dfrac{J^{k-1}(D,\mathbb{Z})}{b(\mathrm{Im}(d_1^*))}\right) \xrightarrow{\cong} \mathrm{Tor}\left(\dfrac{ch(J^{k-1}(D,\mathbb{Z}))}{ch(\mathrm{Im}(d_1^*))}\right).$

Proof: Since $b(\mathrm{Im}(d_1^*)) \subseteq \mathrm{Tor}(J_o^k(X,\mathbb{Z}))$,

$$\mathrm{Tor}\left(\frac{J_o^k(X,\mathbb{Z})}{b(\mathrm{Im}(d_1^*))}\right) = \frac{\mathrm{Tor}(J_o^k(X,\mathbb{Z}))}{b(\mathrm{Im}(d_1^*))} = \frac{\mathrm{Im}(b) \cap \ker(\sum_2)}{b(\mathrm{Im}(d_1^*))}$$

Let $x \in \mathrm{Im}(b) \cap \ker(\sum_2)$, i.e. $x = b(u)$, where $u \in J^{k-1}(X,\mathbb{R}/\mathbb{Z})$ and $b(u)|A = 0 = b(u)|B$. From Diagram 1 and naturality we see $\delta_2(i_1(u)|A) = 0 = \delta_2(i_1(u)|B)$, and so $i_1(u)|A = i_2(\{\theta_A\})$ and $i_1(u)|B = i_2(\{\theta_B\})$. Since $\delta_1 \circ i_1 = 0$ and $\delta_1 \circ i_2 = d$, we see

$$\theta_A \in \mathbf{\Lambda}_{\mathrm{closed}}^{k-1}(A),\ \theta_B \in \mathbf{\Lambda}_{\mathrm{closed}}^{k-1}(B) \qquad \text{and} \qquad \theta_A|D - \theta_B|D \in \mathbf{\Lambda}_J^{k-1}(D).$$

Using the original formulation of Ω

$$\Omega(x) = \theta_A|D - \theta_B|D \mod \mathbf{\Lambda}_J^{k-1}(A)|D + \mathbf{\Lambda}_J^{k-1}(B)|D.$$

Now suppose $\Omega(x) = 0$. This implies one can find $\gamma_A \in \mathbf{\Lambda}_J^{k-1}(A)$ and $\gamma_B \in \mathbf{\Lambda}_J^{k-1}(B)$ with

$$\theta_A|D - \theta_B|D = \gamma_A|D - \gamma_B|D.$$

Since $\{\theta_A\} = \{\theta_A - \gamma_A\}$ and $\{\theta_B\} = \{\theta_B - \gamma_B\}$ we see

$$i_1(u)|A = i_2(\{\theta_A - \gamma_A\})$$
$$i_1(u)|B = i_2(\{\theta_B - \gamma_B\})$$

where $(\theta_A - \gamma_A)|D = (\theta_B - \gamma_B)|D)$. Thus we may define $\sigma \in \mathbf{\Lambda}_{\mathrm{closed}}^{k-1}(X)$ by $\sigma|A = \theta_A - \gamma_A$ and $\sigma|B = \theta_B - \gamma_B$.

Let $[\sigma] \in \mathbf{H}^{k-1}(X,\mathbb{R})$ be the de Rham class represented by σ. Referring to Diagram 1 we have $p([\sigma]) \in J^{k-1}(X,\mathbb{R}/\mathbb{Z})$ and

$$i_1(u|A) = i_2(\mathrm{deRh}([\sigma]|A)) = i_1(p([\sigma]|A))$$
$$i_1(u|B) = i_2(\mathrm{deRh}([\sigma]|B)) = i_1(p([\sigma]|B)).$$

By injectivity of i_1 we see

$$(u - p([\sigma]))|A = 0 = (u - p([\sigma]))|B.$$

Thus $u - p([\sigma]) \in \text{Im}(d_1^*)$. Since $\text{Im}(p) = \ker(b)$ we see

$$x = b(u) = b(u - p([\sigma])) \in b(\text{Im}(d_1^*)).$$

Thus $\Omega|\text{Tor}(\frac{J_o^k(X,\mathbb{Z})}{b(\text{Im}(d_1^*))})$ is $1 : 1$. By the assumption that $J^k(\text{point})$ is finitely generated and X is compact it follows that $\text{Tor}\left(\frac{J_o^k(X,\mathbb{Z})}{b(\text{Im}(d_1^*))}\right)$ is finite.

By Lemma 5

$$\text{card}\left(\text{Tor}\left(\frac{J_o^k(X,\mathbb{Z})}{b(\text{Im}(d_1^*))}\right)\right) = \text{card}\left(\text{Tor}\left(\frac{ch(J^{k-1}(D,\mathbb{Z}))}{ch(\text{Im}(\Delta_2))}\right)\right).$$

Surjectivity thus follows from injectivity, proving the Lemma. ∎

The proof of $*$), and thus of the Theorem, follows immediately from Lemma 3 and Lemma 6.

<div align="center">Q.E.D.</div>

Reference

1. J. Simons and D. Sullivan. "Structured Vector Bundles Define Differential K-Theory". Quanta of Maths. AMS and Clay Mathematics Institute. 2010. pp. 577-597.