Structured Vector Bundles Define
Differential K-Theory

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Abstract

A equivalence relation, preserving the Chern-Weil form, is defined between connections on a complex vector bundle. Bundles equipped with such an equivalence class are called Structured Bundles, and their isomorphism classes form an abelian semi-ring. By applying the Grothedieck construction one obtains the ring $\hat{K}$, elements of which, modulo a complex torus of dimension the sum of the odd Betti numbers of the base, are uniquely determined by the corresponding element of ordinary K and the Chern-Weil form. This construction provides a simple model of differential K-theory, c.f. Hopkins-Singer (2005), as well as a useful codification of vector bundles with connection.
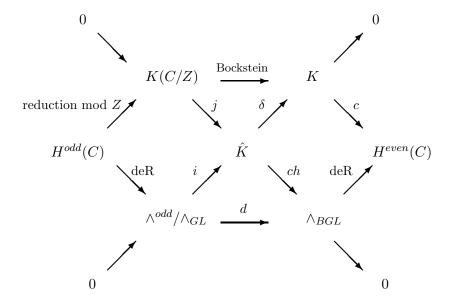
Introduction

This paper grew out of the effort to construct a simple geometric model for differential K-theory, the fibre product of usual K-theory with closed differential forms, [4],[5],[6]. The model which finally emerged also fulfilled our long standing wish for a simple and straightforward codification of complex vector bundles with connection.

Considering pairs of connections whose Chern-Simons difference form is exact defines an equivalence relation in the space of all connections on a given bundle. We call a pair, $\mathcal{V} = (V, \{\nabla\})$, consisting of a vector bundle together with a particular such equivalence class, a **structured bundle**. As is true for vector bundles, structured bundles have additive inverses up to trivial structured bundles: given $\mathcal{V}$ there is a $\mathcal{W}$ such that their direct sum is equivalent to a bundle with trivial holonomy (Theorem 1.15).

By defining Struct to be the commutative semi-ring of isomorphism classes of structured bundles, and using the standard Grothedieck device to turn Struct into a commutative ring, we obtain $\hat{K}$, a functor from smooth compact manifolds with corners into commutative rings. As in ordinary K, every element of $\hat{K}$ may be written as $\mathcal{V} - [n]$, where $\mathcal{V}$ is a structured bundle and $[n]$ is the trivial structured bundle of dim n. $\hat{K}$ achieves the above desired codification of connections and serves as the sought after geometric model of differential K-theory.

Defining four natural transformations into and out of $\hat{K}$ we develop in the first four sections the diagram with exact diagonals and boundaries,

$$
\begin{array}{ccccc}
0 & & & & 0 \\
& \searrow & & \nearrow & \\
& K(C/Z) & \xrightarrow{\text{Bockstein}} & K & \\
\nearrow_{\text{reduction mod } Z} & \searrow^{j} \quad \delta\nearrow & & \searrow^{c} & \\
H^{odd}(C) & & \hat{K} & & H^{even}(C) \\
\searrow_{\text{deR}} \quad i\nearrow & & \searrow^{ch} \quad \text{deR}\nearrow & & \\
& \wedge^{odd}/\wedge_{GL} & \xrightarrow{d} & \wedge_{BGL} & \\
& \nearrow & & \searrow & \\
0 & & & & 0
\end{array}
$$

where the sequence along the upper boundary may be identified (via $ch \otimes C$) with the Bockstein sequence for complex K-theory (the long exact sequence associated to the short exact sequence of coefficients $0 \to Z \to C \to C/Z \to 0$), and that along the lower boundary comes from de Rham theory. Here, $\wedge_{BGL}$ means all closed forms cohomologous to Chern characters of complex vector bundles, and $\wedge_{GL}$ means all closed forms cohomologous to pull-backs by maps into $GL =$ union of the $GL(n, C)$ of the transgression of the Chern character form. δ is the map which simply drops the connection, and ch is the Chern-Weil map applied to the Chern character polynomial. The fibre product statement above is related to the commutative square on the right half of the diagram.
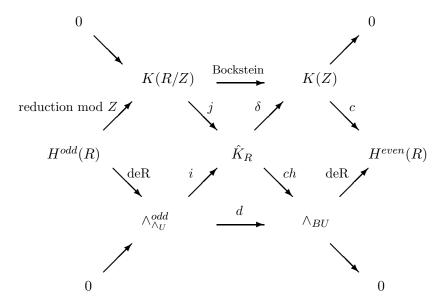
The work's main technical innovation is embodied in Proposition 2.6, where it is shown that all odd forms modulo $\wedge_{GL}$ arise as the Chern-Simons difference forms between the trivial connection and arbitrary connections on trivial bundles. A corollary, as implied by the diagram above, is that every element of $\wedge_{BGL}$ arises after stabilizing as the Chern character form of some connection in any bundle whose Chern character is the given cohomology class. In particular, if a bundle has zero characteristic classes over C, then there is a connection on that bundle, stabilized by adding in a trivial bundle, with vanishing Chern-Weil forms.

By considering the simultaneous kernel of ch and δ, the diagram also shows that the ambiguity in determining a structured bundle up to stabilizing solely by its characteristic forms and underlying element of K is measured by a complex torus, the dimension of which is the sum of the odd Betti numbers of the base manifold.

In showing that the kernel of ch is $K(C/Z)$ we were influenced by the work of Karoubi [2] and Lott [1], which gave a related description of $K(C/Z)$ involving a bundle with connection and an extra total odd form whose d is the Chern character form. Our proof is based on the characterization given in Appendix A of the homotopy fibre of a map in the homotopy category.

We also point out that the existence of a differential K-theory associated to K-theory, and indeed a differential theory associated to any exotic cohomology theory, was constructed in the paper of Hopkins and Singer [5]. Following their approach, Freed, as well as Hopkins and Singer and perhaps others like ourselves, were aware that a model for differential K-theory could be constructed based on pairs (E, O), where E is a bundle with connection and O is a total odd form with an equivalence relation generalizing that in [2]. One point of the present work is that this total odd form may be taken to be zero in the equivalent description of differential K-theory presented here.

There is a word for word variant of the above concerning complex vector bundles with Hermitian connection. Now there is a functor $\hat{K}_R$, four natural transformations and the diagram

$$
\begin{array}{ccccccc}
0 & & & & & & 0 \\
& \searrow & & & & \nearrow & \\
& & K(R/Z) & \xrightarrow{\text{Bockstein}} & K(Z) & & \\
\text{reduction mod } Z \nearrow & & j \searrow \quad \delta \nearrow & & & \searrow c & \\
H^{odd}(R) & & \hat{K}_R & & & & H^{even}(R) \\
& \text{deR} \searrow \quad i \nearrow & & ch \searrow \quad \text{deR} \nearrow & & \\
& & \wedge^{odd}_{\wedge_U} & \xrightarrow{d} & \wedge_{BU} & & \\
& \nearrow & & & & \searrow & \\
0 & & & & & & 0
\end{array}
$$

This is discussed briefly in Section 5. As a corollary, for any bundle over a closed Riemannian manifold after stabilizing, there is a unitary connection on the bundle whose Chern-Weil form is the harmonic representative of the Chern character of the bundle. Moreover, when the odd Betti numbers vanish, this structured bundle is unique up to adding factors with trivial holonomy.

Our model of $\hat{K}$ or $\hat{K}_R$ may relate to two questions:

1. Up to a natural transformation, are $\hat{K}$ or $\hat{K}_R$ uniquely determined by the diagram, as shown in [7] in the case of ordinary differential cohomology?

2. Can one enrich the families index theorem by passing from K to $\hat{K}$ or $\hat{K}_R$? c.f. [3], [4], [6].

Finally, this model of $\hat{K}$ or $\hat{K}_R$ might be helpful for certain quantum theories and M-theory, in which it has already been observed that actions can be written more appropriately in the language of differential K-theory than in that of differential forms [6], [8]. In this respect we note Theorem 3.9, showing that $\hat{K}$ and $\hat{K}_R$ satisfy the Mayer-Vietoris property, which relates to locality.

§1. Structured Bundles

Let $[V, \nabla]$ be a complex vector bundle with connection over a smooth manifold with corners, X, and let $R \in \wedge^2(X, \mathrm{End}(V))$ denote its curvature tensor.

Using the Chern-Weil homomorphism, the Chern character of V, $ch(V)$, may be represented by the total complex valued closed form on X, $ch(\nabla)$, defined by

$$1.1) \qquad ch(\nabla) = \sum_{j=0} \frac{1}{j!} \left(\frac{1}{2\pi i} \right)^j \mathrm{tr}(\overbrace{R \wedge \cdots \wedge R}^{j}) \quad \in \wedge^{even}(X, C).$$

For $t \in [0, 1]$ and $\gamma(t) = \nabla^t$ a smooth curve of connections, $(\nabla^t)' = A^t \in \wedge^1(X, \mathrm{End}(V))$, and we set

$$1.2) \qquad cs(\gamma) = \int_0^1 \sum_{j=1} \frac{1}{(j-1)!} \left(\frac{1}{2\pi i} \right)^j \mathrm{tr}(A^t \wedge \overbrace{R^t \wedge \cdots \wedge R^t}^{j-1}) \quad \in \wedge^{odd}(X, C).$$

It is a standard fact that

$$1.3) \qquad \mathrm{d} cs(\gamma) = ch(\nabla^1) - ch(\nabla^0).$$

There is a second formulation of 1.2) which will be useful in what follows.

Let $\Pi : X \times [0, 1] \to X$ be the standard projection, and set $W = \Pi^*(V)$. We may construct a connection, $\bar{\nabla}$, on W by defining $\bar{\nabla}_s = \nabla^t_{\Pi_*(s)}$ when s is tangent to the slice through t, and by making $\bar{\nabla}_{\partial/\partial t}(\Pi^*(f)) = 0$ for f any cross-section of V.

Let $\bar{R}$ be the curvature tensor of $\bar{\nabla}$. Then, if r, s are tangent to the slice through t,

$$1.4) \qquad \begin{aligned} \bar{R}_{r,s} &= R^t_{\Pi_*(r), \Pi_*(s)} \\[2mm] \bar{R}_{\partial/\partial t,\, s} &= A^t_{\Pi_*(s)}. \end{aligned}$$

The first is straightforward. To show the second, let $w \in W_{(x,t)}$, and extend it to be of the form $\Pi^*(f)$, where f is a cross-section of V. Also extend s to be the lift of a vector field on X. Clearly $[s, \partial/\partial t] = 0$. Thus

$$\bar{R}_{\partial/\partial t,\, s} w = \bar{\nabla}_{\partial/\partial t} \bar{\nabla}_s w - \bar{\nabla}_s \bar{\nabla}_{\partial/\partial t} w = \bar{\nabla}_{\partial/\partial t} \bar{\nabla}_s w = \frac{d}{dt} \nabla^t_{\Pi_*(s)} w = A^t_{\Pi_*(s)} w.$$

Now, let $\psi_t : X \to X \times [0, 1]$ be the slice map, $\psi_t(x) = (x, t)$. Then by 1.4)

$$\mathrm{tr}(A^t \wedge \overbrace{R^t \wedge \cdots \wedge R^t}^{j-1}) = \psi_t^*(\mathrm{tr}(i_{\partial/\partial t}\bar{R} \wedge \overbrace{\bar{R} \wedge \cdots \wedge \bar{R}}^{j-1}))$$

$$= \psi_t^*(i_{\partial/\partial t}(\tfrac{1}{j} \mathrm{tr}(\overbrace{\bar{R} \wedge \cdots \wedge \bar{R}}^{j}))).$$

From this we conclude

$$1.5) \qquad cs(\gamma) = \int_0^1 \psi_t^* \left(i_{\partial/\partial t} \, ch(\bar{\nabla}) \right).$$

The following proposition is almost certainly well known, but we did not find a reference.

Proposition 1.6: If α and γ are two paths connecting ∇^0 and ∇^1, then

$$cs(\alpha) = cs(\gamma) + \text{exact}.$$

Proof: It is sufficient to prove that if γ is a closed path of connections, then $cs(\gamma)$ is exact. By 1.3) $cs(\gamma)$ is obviously closed. To show it exact we show that $cs(\gamma)$ integrates to 0 on every cycle of X.

Let Z be such a cycle. Then by 1.5)

$$\int_Z cs(\gamma) = \int_{Z \times S^1} ch(\bar{\nabla}) = ch(W)(Z \times S^1) = \Pi^*(ch(V))(Z \times S^1) = ch(V)(\Pi_*(Z \times S^1)) = 0$$

Thus $cs(\gamma)$ is exact. ∎

Since ∇^0 and ∇^1 may always be joined by a smooth path, using Proposition 1.6, we may set

$$1.7) \qquad CS(\nabla^0, \nabla^1) = cs(\gamma) \bmod \text{exact}.$$

From Proposition 1.6) we also see

$$1.8) \qquad CS(\nabla^0, \nabla^1) + CS(\nabla^1, \nabla^2) = CS(\nabla^0, \nabla^2).$$

Definition: ∇^0 and ∇^1 will be called **equivalent**, and written $\nabla^0 \sim \nabla^1$, if $CS(\nabla^0, \nabla^1) = 0$. Equation 1.8) shows $\sim$ is an equivalence relation.

Definition: A pair $\mathcal{V} = [V, \{\nabla\}]$, where $\{\nabla\}$ is an equivalence class of connections on V will be called a **structured bundle**.

If ∇^W is a connection on W and $\sigma : V \to W$ is a bundle isomorphism covering the identity map of X, σ induces $\sigma^*(\nabla^W)$, a connection on V, and it is easily seen that $\{\sigma^*(\nabla^W)\} = \sigma^*(\{\nabla^W\})$. $\mathcal{V} = [V, \{\nabla^V\}]$ and $\mathcal{W} = [W, \{\nabla^W\}]$ are called **isomorphic** if $\sigma^*(\{\nabla^W\}) = \{\nabla^V\}$.

If $\psi : X \to Y$ is C^∞, and V is a bundle over Y with connections ∇^0 and ∇^1, then, in the usual manner, $\psi^*(\nabla^0)$ and $\psi^*(\nabla^1)$ are connections on $\psi^*(V)$. Clearly

$$CS(\psi^*(\nabla^0), \psi^*(\nabla^1)) = \psi^*(CS(\nabla^0, \nabla^1)).$$

Thus, if $\mathcal{V} = [V, \{\nabla\}]$ is a structured bundle over Y then $\psi^*(\mathcal{V}) = [\psi^*(V), \{\psi^*(\nabla)\}]$ is well defined as a structured bundle over X.

Suppose $\psi_t : X \to Y$ is a smooth 1-parameter family of maps. If $\mathcal{V} = [V, \{\nabla\}]$ is a structured bundle over Y, then $\mathcal{V}^t = [\psi_t^*(V), \psi_t^*(\{\nabla\})]$ is a 1-parameter family of structured bundles over X. Assume $t \in [0, 1]$ and let $\gamma_x : [0, 1] \to Y$ be the curve $\gamma_x(t) = \psi_t(x)$. Let $\sigma_t : \psi_0^*(V) \to \psi_t^*(V)$ be parallel transport along the curves γ_t. Then, letting $W = \psi_0^*(V)$ and $\nabla^t = \sigma_t^*(\psi_t^*(\nabla))$, $\mathcal{W}^t = [W, \{\nabla^t\}]$ is a 1-parameter family of structured bundles over X, isomorphic to the family $\mathcal{V}^t$, having the same underlying vector bundle.

Letting $\gamma_x'(t)$ denote the tangent vector to γ_x at t, and using 1.5), we conclude

$$1.9) \qquad CS(\nabla^0, \nabla^1) = \int_0^1 \psi_t^*(i_{\gamma_x'(t)} \, ch(\nabla)) \, dt.$$

If ∇^V and ∇^W are connections on V and W they determine connections on $V \oplus W$ and $V \otimes W$, denoted by $\nabla^V \oplus \nabla^W$ and $\nabla^W \otimes \nabla^W$. For f, g cross-sections in V and W, and r, a tangent vector to X,

$$
\begin{aligned}
(\nabla^V \oplus \nabla^W)_r(f, g) &= (\nabla_r^V f, \nabla_r^W g) \\
(\nabla^V \otimes \nabla^W)_r(f \otimes g) &= \nabla_r^V(f) \otimes g + f \otimes \nabla_r^W(g).
\end{aligned}
$$

It is well known that

$$
\begin{aligned}
1.10) \quad ch(\nabla^V \oplus \nabla^W) &= ch(\nabla^V) + ch(\nabla^W) \\
1.11) \quad ch(\nabla^V \otimes \nabla^W) &= ch(\nabla^V) \wedge ch(\nabla^W).
\end{aligned}
$$

<u>Lemma 1.12</u>: Let $\nabla^V, \bar{\nabla}^V, \nabla^W, \bar{\nabla}^W$ be connections on the indicated bundles. Then

$$
\begin{aligned}
a) \quad & CS(\nabla^V \oplus \nabla^W, \bar{\nabla}^V \oplus \bar{\nabla}^W) = CS(\nabla^V, \bar{\nabla}^V) + CS(\nabla^W, \bar{\nabla}^W) \\
b) \quad & CS(\nabla^V \otimes \nabla^W, \bar{\nabla}^V \otimes \bar{\nabla}^W) = ch(\nabla^V) \wedge CS(\nabla^W, \bar{\nabla}^W) + ch(\bar{\nabla}^W) \wedge CS(\nabla^V, \bar{\nabla}^V)
\end{aligned}
$$

<u>Proof</u>: Using 1.8)

$$CS(\nabla^V \oplus \nabla^W, \bar{\nabla}^V \oplus \bar{\nabla}^W) = CS(\nabla^V \oplus \nabla^W, \nabla^V \oplus \bar{\nabla}^W) + CS(\nabla^V \oplus \bar{\nabla}^W, \bar{\nabla}^V \oplus \bar{\nabla}^W).$$

Direct calculation of each term using 1.2) shows a).

Again using 1.8)

$$CS(\nabla^V \otimes \nabla^W, \bar{\nabla}^V \otimes \bar{\nabla}^W) = CS(\nabla^V \otimes \nabla^W, \nabla^V \otimes \bar{\nabla}^W) + CS(\nabla^V \otimes \bar{\nabla}^W, \bar{\nabla}^V \otimes \bar{\nabla}^W)$$

and again from 1.2), direct calculation shows b). ∎

From Lemma 1.12 one immediately sees

Proposition 1.13: If $\mathcal{V} = [V, \{\nabla^V\}]$ and $\mathcal{W} = [W, \{\nabla^W\}]$ are structured bundles, then the equivalence classes $\{\nabla^V \oplus \nabla^W\}$ and $\{\nabla^V \otimes \nabla^W\}$ are independent of the choices of $\nabla^V \in \{\nabla^V\}$ and $\nabla^W \in \{\nabla^W\}$, and so

$$
\begin{aligned}
\mathcal{V} \oplus \mathcal{W} &= [V \oplus W, \{\nabla^V \oplus \nabla^W\}] \qquad \text{and} \\
\mathcal{V} \otimes \mathcal{W} &= [V \otimes W, \{\nabla^V \otimes \nabla^W\}]
\end{aligned}
$$

are well defined structured bundles.

Definition: We define **Struct(X)** to be the set of isomorphism classes of structured bundles over X. By Proposition 1.13, the operations $\oplus$ and $\otimes$ make Struct(X) an abelian semi-group with commutative, distributive multiplication. A smooth map ψ from X to Y induces $\psi^* : \text{Struct}(Y) \to \text{Struct}(X)$ preserving these operations. Thus, Struct is a **functor** on the category of smooth compact manifolds with corners into that of commutative semi-rings.

We conclude from 1.3) that $ch : \text{Struct}(X) \to \wedge^{even}(X)$ is a well defined natural transformation, and from 1.10) and 1.11)

$$
\begin{aligned}
ch(\mathcal{V} \oplus \mathcal{W}) &= ch(\mathcal{V}) + ch(\mathcal{W}) \\
1.14) \\
ch(\mathcal{V} \otimes \mathcal{W}) &= ch(\mathcal{V}) \wedge ch(\mathcal{W}).
\end{aligned}
$$

Definition: A connection ∇ on V will be called **flat** if its holonomy around every closed path is the identity. This implies the curvature $R \equiv 0$ and that V is naturally isomorphic to the product bundle with the trivial product connection. $\mathcal{V} = [V, \{\nabla\}]$ will be called flat if some $\nabla \in \{\nabla\}$ is flat. Since any two such of dim n are isomorphic, we shall denote this isomorphism class by $[n] \in \text{Struct}(X)$.

The following theorem is based on a related result in [11], stated without giving the proof. We employ that proof here in Lemma 1.16 below.

Theorem 1.15: Given any $\mathcal{V} \in \text{Struct}(X)$ there is a $\mathcal{W} \in \text{Struct}(X)$ such that $\mathcal{V} \oplus \mathcal{W} = [n]$ for some n. Any such $\mathcal{W}$ will be called an **inverse** of $\mathcal{V}$.

To prove the Theorem we need

Lemma 1.16: Let ∇ be a connection on $V \oplus W$ with curvature R. Let ∇^V and ∇^W be the connections on V and W induced by ∇. E.g. if $\Pi^V : V \oplus W \to V$ is the projection, and f is a cross-section in V then $\nabla_r^V f = \Pi^V(\nabla_r f)$. Suppose $R_{r,s}(V) \subseteq V$ and $R_{r,s}(W) \subseteq W$ for all tangent vectors r, s at any point of X. Then,

$$\nabla^V \oplus \nabla^W \sim \nabla.$$

Proof: We may write

$$\nabla = \nabla^V \oplus \nabla^W + A$$

where $A \in \wedge^1(X, \text{End}(V \oplus W))$. For f a cross-section in V we see

$$A_r f = \nabla_r f - \Pi^V(\nabla_r f) = \Pi^W(\nabla_r f) \in W.$$

As the same holds for W, we see

$$1.17) \qquad A_r(V) \subseteq W \qquad \text{and} \qquad A_r(W) \subseteq V.$$

Setting $\bar\nabla = \nabla^V \oplus \nabla^W$, let $\bar R$ denote its curvature and $\bar d$ denote its exterior differentiation operator. Since $\bar\nabla$ preserves V and W, 1.17) implies

1.18) $\qquad \bar d A_{r,s}(V) \subseteq W \qquad and \qquad \bar d A_{r,s}(W) \subseteq V.$

The usual formula computing the curvature of one connection from that of another shows

$$R = \bar R + A \wedge A + \bar d A.$$

By hypothesis, R preserves V and W. So does $\bar R$, being the curvature of a direct sum connection, and so does $A \wedge A$ by 1.17). This implies that $\bar d A$ preserves them as well, but 1.18) shows the opposite. Thus $\bar d A = 0$ and

1.19) $\qquad R = \bar R + A \wedge A.$

Let $\nabla^t = \bar\nabla + tA$, a curve of connections joining $\nabla^V \oplus \nabla^W$ to ∇. Letting R^t denote the associated curvature, we see from 1.19)

1.20) $\qquad R^t = \bar R + t^2 A \wedge A.$

In the notation of 1.2), $A^t = (\nabla^t)' = A$, and so the CS integrand consists of terms of the form

$$\mathrm{tr}(A \wedge \overbrace{R^t \wedge \cdots \wedge R^t}^{j-1}).$$

But, by 1.20) R^t preserves both V and W, and, since A reverses them, all such trace terms must vanish. Thus $CS(\nabla, \bar\nabla) = 0$. $\qquad\blacksquare$

Proof of Theorem 1.15:

The classifying spaces $B_k GL(n,C) = GL(n+k,C)/GL(n,C) \times GL(k,C)$ come with natural bundles, V^n and V^k, of dimension n and k, and connections ∇^n and ∇^k induced by the standard flat connection on $V^n \oplus V^k$. Lemma 1.16 shows that $\mathcal{V}^n = [V^n, \{\nabla^n\}]$ and $\mathcal{V}^k = [V^k, \{\nabla^k\}]$ are inverses of each other.

The theorem of Narasimhan-Ramanan [9] shows that for sufficiently large k, an n-dim $\mathcal{V} \in \mathrm{Struct}(X)$ may be obtained as the pull-back of $\mathcal{V}^n$ via a C^∞ map of $X \to B_k GL(n,C)$. The pull-back of $\mathcal{W}^k$ will then be an inverse of $\mathcal{V}$ in the sense of Theorem 1.15. $\qquad\blacksquare$

§2. The Stably Trivial Case

Let $GL = \lim_n GL(n,C)$, the stabilized complex general linear group and $\mathcal{G}$ its Lie algebra. $\mathcal{G}$ consists of complex valued matrices, all but a finite number of whose entries are 0. Let $\theta \in \wedge^1(GL,\mathcal{G})$ denote the canonical left invariant $\mathcal{G}$-valued form on GL. Set

2.1) $\qquad \Theta = \sum_{j=1} b_j \, \mathrm{tr}(\overbrace{\theta \wedge \theta \wedge \cdots \wedge \theta}^{2j-1}) \in \wedge^{odd}(GL)$

where

$$b_j = \frac{1}{(j-1)!} \left(\frac{1}{2\pi i}\right)^j \int_0^1 (t^2 - t)^{j-1}\, dt.$$

It is well known that Θ is a bi-invariant closed odd form, and the free abelian group generated by all distinct products of its components represent the entire complex cohomology ring of GL. We define $\wedge_{GL} \subseteq \wedge^{odd}$ by

2.2) $\wedge_{GL}(X) = \{g^*(\Theta)\} + \wedge^{odd}_{exact}$

where $g : X \to GL$ runs through all C^∞ maps.

Note that if g, h map X into GL, then $g \oplus h : X \to GL$ may be defined, and $(g \oplus h)^*(\Theta) = g^*(\Theta) + h^*(\Theta)$. Moreover, $(g^{-1})^*(\Theta) = -g^*(\Theta)$. Thus $\wedge_{GL}(X)$ is an abelian group.

Lemma 2.3: Let V be a trivial bundle with the two flat connections ∇ and $\bar{\nabla}$. Then

$$CS(\nabla, \bar{\nabla}) \in \wedge_{GL} / \wedge^{odd}_{exact}.$$

Proof: Since ∇ and $\bar{\nabla}$ each have trivial holonomy, we can find a cross-section $g \in \mathrm{Aut}(V)$ such that

$$\bar{\nabla}_t(f) = g^{-1}(\nabla_t(g(f))).$$

Expressing g as a matrix with respect to a ∇-parallel framing of V, we see

$$\bar{\nabla} = \nabla + g^{-1}\, dg.$$

Now, regarding $g : X \to GL$, one easily sees that $g^{-1}dg = g^*(\theta)$. Thus

$$\bar{\nabla} = \nabla + g^*(\theta).$$

Setting $\bar{\nabla}^t = \nabla + tg^*(\theta)$, we see

$$\bar{R}^t = R + t\, dg^*(\theta) + t^2\, g^*(\theta) \wedge g^*(\theta).$$

But, either calculating on GL, or directly with $g^{-1}dg$, we see that $dg^*(\theta) = -g^*(\theta) \wedge g^*(\theta)$. Moreover, since ∇ has trivial holonomy, $R \equiv 0$. Thus

$$\bar{R}^t = (t^2 - t)g^*(\theta) \wedge g^*(\theta).$$

It then follows from 1.2) that $CS(\nabla, \bar{\nabla}) = g^*(\Theta)$.   ∎

Definition:

$$\mathrm{Struct}_{\mathrm{ST}}(X) = \{\, [V, \{\nabla\}] \in \mathrm{Struct}(X) \mid V \text{ is stably trivial} \,\}.$$

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For $\mathcal{V} \in \mathrm{Struct}_{\mathrm{ST}}(X)$, let F and H be trivial bundles such that $V \oplus F = H$ and let ∇^F, ∇^H be flat connections on F and H. We define

$$\widehat{CS} : \mathrm{Struct}_{\mathrm{ST}}(X) \to \wedge^{odd}/\wedge_{GL}$$

by

$$\widehat{CS}(\mathcal{V}) = CS(\nabla^H, \nabla \oplus \nabla^F) \bmod \wedge_{GL}/\wedge_{exact}^{odd}.$$

Proposition 2.4: $\widehat{CS}$ is a well defined homomorphism.

Proof: Suppose $\bar{F}, \bar{H}, \nabla^{\bar{F}}, \nabla^{\bar{H}}$ are another pair of trivial bundles with flat connections with $V \oplus \bar{F} = \bar{H}$. Using 1.7), Lemma 1.12 and Lemma 2.3, and working mod $\wedge_{GL}$, we see

$$
\begin{aligned}
CS(\nabla^{\bar{H}}, \nabla \oplus \nabla^{\bar{F}}) &= CS(\nabla^{\bar{H}} \oplus \nabla^F, \nabla \oplus \nabla^{\bar{F}} \oplus \nabla^F) \\
&= CS(\nabla^H \oplus \nabla^{\bar{F}}, \nabla \oplus \nabla^{\bar{F}} \oplus \nabla^F) \\
&= CS(\nabla^H, \nabla \oplus \nabla^F).
\end{aligned}
$$

Thus $\widehat{CS}$ is well defined. That $\widehat{CS}$ is a homomorphism follows immediately from Lemma 1.10. $\blacksquare$

Definition: $\mathcal{V} \in \mathrm{Struct}(X)$ is called **stably flat** if there exists flat $\mathcal{F}$ and $\mathcal{H}$ such that $\mathcal{V} \oplus \mathcal{F} = \mathcal{H}$. The set of these objects will be denoted by $\mathrm{Struct}_{\mathrm{SF}}(X)$. Clearly $\mathrm{Struct}_{\mathrm{SF}}(X) \subseteq \mathrm{Struct}_{\mathrm{ST}}(X)$ and is a sub semi-group.

Proposition 2.5: $\ker(\widehat{CS}) = \mathrm{Struct}_{\mathrm{SF}}(X)$.

Proof: Obviously $\mathrm{Struct}_{\mathrm{SF}} \subseteq \ker(\widehat{CS})$. Now suppose $\widehat{CS}(\mathcal{V}) = 0$. Let F, H, ∇^F and ∇^H be as in the definition of $\widehat{CS}$. Now, $\widehat{CS}(\mathcal{V}) = 0$ implies

$$CS(\nabla^H, \nabla \oplus \nabla^F) = g^*(\Theta) \bmod \wedge_{exact}^{odd}$$

for some $g : X \to GL$. Again as in the proof of Lemma 2.3, choosing a ∇^H-parallel framing of H, we may regard $g \in \mathrm{Aut}(H)$ and set

$$\bar{\nabla}^H = g^{-1}(\nabla^H \circ g).$$

As in the Lemma we see $CS(\nabla^H, \bar{\nabla}^H) = g^*(\Theta)$ and thus $CS(\bar{\nabla}^H, \nabla^H) = -g^*(\Theta)$. Therefore

$$
\begin{aligned}
CS(\bar{\nabla}^H, \nabla \oplus \nabla^F) &= CS(\bar{\nabla}^H, \nabla^H) + CS(\nabla^H, \nabla \oplus \nabla^F) \\
&= -g^*(\Theta) + g^*(\Theta) = 0 \bmod \text{exact}.
\end{aligned}
$$

Setting $\bar{\mathcal{H}} = [H, \{\bar{\nabla}^H\}]$ and $\mathcal{F} = [F, \{\nabla^F\}]$ we see $\mathcal{V} \oplus \mathcal{F} = \bar{\mathcal{H}}$ and thus $\mathcal{V} \in \mathrm{Struct}_{\mathrm{SF}}(X)$. $\blacksquare$

Proposition 2.6: $\mathrm{Im}(\widehat{CS}) = \wedge^{odd}(X)/\wedge_{GL}(X)$.

Proof: If L is a trivialized line bundle over X then any connection on L is simply a complex valued 1-form, w. Since $w \wedge w = 0$, the associated curvature, R^w, is dw, and $\{w\} = \{w + df \mid f \in C^\infty(X, C)\}$.

Let $\mathcal{L}_w = [L, \{w\}]$. Using tw as a curve of connections joining w to the trivial connection, and noting that $R^{tw} = tR^w$, 1.2) shows

$$2.7) \qquad \widehat{CS}(\mathcal{L}_w) = \sum_{j=1}^{j} \frac{1}{j!}\left(\frac{1}{2\pi i}\right)^j w \wedge (dw)^{j-1}.$$

We first suppose $X = R^n$. If $w = f\,dx$ then $w \wedge dw = 0$ and thus $\widehat{CS}(\mathcal{L}_{fdx}) = f\,dx$. Moreover, since $\widehat{CS}$ is a homomorphism

$$\widehat{CS}\left(\sum_i \oplus \mathcal{L}_{f_i dx_i}\right) = \sum f_i dx_i.$$

Thus $\wedge^1(R^n)/\wedge_G(R^n) \subseteq \mathrm{Im}(\widehat{CS})$.

Proceeding by induction on k, suppose

$$2.8) \qquad \left(\sum_{j=1}^{k} \wedge^{2j-1}(R^n)\right) / \wedge_{GL}(R^n) \subseteq \mathrm{Im}(\widehat{CS}).$$

Let $w = x_1 dx_2 + x_3 dx_4 + \cdots + x_{2k-1}dx_{2k} + f\,dx_{2k+1}$.

Claim: $w \wedge (dw)^k = (k+1)! f\,dx_1 \wedge \cdots \wedge dx_{2k+1} + \text{exact}$.

To show this, let $\gamma = dx_1 \wedge dx_2 + \cdots + dx_{2k-1} \wedge dx_{2k}$, and note

$$dw = \gamma + df \wedge dx_{2k+1} \qquad \Rightarrow \qquad (dw)^k = (\gamma + df \wedge dx_{2k+1})^k.$$

Since all powers of $df \wedge dx_{2k}$ vanish,

$$(dw)^k = \gamma^k + k\gamma^{k-1} \wedge df \wedge dx_{2k+1} = k!dx_1 \wedge \cdots \wedge dx_{2k} +$$
$$k!\left[\sum_{j=1}^{k} dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_{2j-1} \wedge dx_{2j}} \wedge \cdots \wedge dx_{2k-1} \wedge dx_{2k}\right] \wedge df \wedge dx_{2k+1}$$

Thus,

$$w \wedge (dw)^k = k!f\,dx_1 \wedge \cdots \wedge dx_{2k+1} +$$
$$k!\left[\sum_{j=1}^{k} dx_1 \wedge \cdots \wedge dx_{2j-2} \wedge x_{2j-1} \wedge dx_{2j} \wedge \cdots \wedge dx_{2k}\right] \wedge df \wedge dx_{2k+1}$$
$$= k!\,f\,dx_1 \wedge \cdots \wedge dx_{2k+1} - k!\sum_{j=1}^{k} dx_1 \wedge \cdots \wedge dx_{2j-1} \wedge x_{2j-1}df \wedge dx_{2j} \wedge \cdots \wedge dx_{2k+1}$$
$$= (k+1)!\,f\,dx_1 \wedge \cdots \wedge dx_{2k+1} + \text{exact}.$$

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Thus, working mod $\wedge_{GL}(R_n)$,

$$\widehat{CS}(\mathcal{L}_{(2\pi i)^{k+1}w}) = f\,dx_1 \wedge \cdots \wedge dx_{2k+1} + \theta,$$

where

$$\theta \in \sum_{j=1}^{k} \wedge^{2j-1}(R^n).$$

By induction, $\theta = \widehat{CS}(\mathcal{V})$ for some $\mathcal{V} \in \text{Struct}_{\text{ST}}(R^n)$. Theorem 1.15 shows $\mathcal{V}$ has an inverse $\mathcal{V}^{-1}$. Clearly $\mathcal{V}^{-1} \in \text{Struct}_{\text{ST}}(R^n)$ and by Proposition 2.4, $\widehat{CS}(\mathcal{V}^{-1}) = -\theta$. Thus

$$\widehat{CS}(\mathcal{L}_{(2\pi i)^{k+1}w} \oplus \mathcal{V}^{-1}) = f\,dx_1 \wedge \cdots \wedge dx_{2k+1}.$$

The general element of $\wedge^{2k+1}(R^n)$ is the sum of such terms, and thus is the image under $\widehat{CS}$ of the direct sum of the inverse images of each of these terms.

For the general case let $\psi : X \to R^n$ be an imbedding. Since $\psi^* : \wedge^{odd}(R^n) \to \wedge^{odd}(X)$ is onto, and $\psi^*(\wedge_{GL}(R^n)) \subseteq \wedge_{GL}(X)$, $\psi^* : \wedge^{odd}(R^n)/\wedge_{GL}(R_n) \to \wedge^{odd}(X)/\wedge_{GL}(X)$ is onto. Moreover, $\psi^*(\text{Struct}_{\text{ST}}(R^n)) \subseteq \text{Struct}_{\text{ST}}(X)$, and finally $\widehat{CS} \circ \psi^* = \psi^* \circ \widehat{CS}$. Thus if $\rho \in \wedge^{odd}(X)/\wedge_{GL}(X)$, we can find $\bar{\rho} \in \wedge^{odd}(R_n)/\wedge_{GL}(R^n)$ with $\psi^*(\bar{\rho}) = \rho$. By the special case, $\bar{\rho} = \widehat{CS}(\mathcal{V})$ for some $\mathcal{V} \in \text{Struct}_{\text{ST}}(R^n)$. Then

$$\rho = \psi^*(\widehat{CS}(\mathcal{V})) = \widehat{CS}(\psi^*(\mathcal{V})). \quad \blacksquare$$

From Propositions 2.4, 2.5, 2.6 we see

Theorem 2.7:

$$\widehat{CS} : \text{Struct}_{\text{ST}}(X)/\text{Struct}_{\text{SF}}(X) \quad \overset{\cong}{\underset{\longrightarrow}{}} \quad \wedge^{odd}(X)/\wedge_{GL}(X).$$

§3. $\hat{K}(X)$

Using the standard construction of K, which transforms an abelian semi-group into a group, we define

$$\hat{K} = K(\text{Struct}(X)).$$

$\hat{K}(X)$ is the free abelian group generated by isomorphism classes of structured bundles, modulo the relation $\mathcal{V} + \mathcal{W} - (\mathcal{V} \oplus \mathcal{W})$. Equivalently defined, $\hat{K}(X)$ is the quotient of the semi-group under $\oplus$ consisting of all pairs $(\mathcal{V}, \mathcal{W})$ modulo the sub semi-group consisting of pairs $(\mathcal{V}, \mathcal{V})$. Since $(0, \mathcal{V})$ is obviously the additive inverse of $(\mathcal{V}, 0)$, we write $(\mathcal{V}, \mathcal{W})$ as $\mathcal{V} - \mathcal{W}$.

Using Theorem 1.15 it is straightforward using the pairs definition to show

3.1)   Every element of $\hat{K}(X)$ is of the form $\mathcal{V} - [n]$.

3.2) $\mathcal{V} - [n] = 0 \Leftrightarrow \mathcal{V}$ is stably flat and $n = \dim(\mathcal{V})$.

Again using the pairs definition, one sees that $\otimes$ is well defined in $\hat{K}(X)$, and thus $\hat{K}(X)$ becomes a commutative ring. (Defining $(\mathcal{V}, \mathcal{W}) \otimes (\mathcal{V}', \mathcal{W}')$ to be $(\mathcal{V} \otimes \mathcal{V}' \oplus \mathcal{W} \otimes \mathcal{W}', \mathcal{W} \otimes \mathcal{V}' \oplus \mathcal{V} \otimes \mathcal{W}')$ one sees $\{(\mathcal{W}, \mathcal{W})\}$ is an ideal.)

We define $\wedge_{BGL} \subseteq \wedge^{even}$ by

$$\wedge_{BGL}(X) = \{ch(\mathcal{V})\} + \wedge_{exact}^{even}$$

where $\mathcal{V}$ ranges over all elements of $\mathrm{Struct}(X)$. From 1.10) and 1.11) and Theorem 1.15 we see that $\wedge_{BGL}(X)$ is a commutative ring.

By analogy with the definition of $\wedge_{GL}$, and using the theorem of Narasimhan-Ramanan [9], we could alternatively have defined

$$\wedge_{BGL}(X) = \{\phi^*(\Omega)\} + \wedge_{exact}^{even}$$

where $\phi : X \to BGL$ ranges over all C^∞ maps, and Ω is the Chern character form of the standard connection on the classifying bundle over BGL.

Clearly, ch extends to $\hat{K}(X)$, and maps it to $\wedge_{BGL}(C)$. We also define

$$\delta : \hat{K}(X) \to K(X)$$

by

$$\delta([V, \{\nabla\}] - [W, \{\bar{\nabla}\}]) = V - W.$$

Letting $c : K(X) \to H^{even}(X, C)$ be the natural transformation defined by the Chern character, and $\mathrm{deR} : \wedge_{BGL}(X) \to H^{even}(X, C)$ be that defined by the de Rham Theorem, we see

3.3)

$$
\begin{array}{ccc}
 & K(X) & \\
\delta \nearrow & & \searrow c \\
\hat{K}(X) & \otimes & H^{even}(X, C) \\
ch \searrow & & \nearrow \mathrm{deR} \\
 & \wedge_{BGL}(X) & \\
\end{array}
$$

is a commutative diagram.

<u>Proposition 3.4</u>: $\ker(\delta) \cong \wedge^{odd}(X)/\wedge_{GL}(X)$.

<u>Proof</u>: Define $\Gamma : \text{Struct}_{ST}/\text{Struct}_{SF} \to \hat{K}$ by

$$\Gamma(\{\mathcal{V}\}) = \mathcal{V} - \dim(\mathcal{V}).$$

By 3.2), Γ is well defined and is an injection. Moreover, it is clear that $\text{Im}(\Gamma) = \ker(\delta)$. Thus from Theorem 2.7,

$$\Gamma \circ \widehat{CS}^{-1} : \wedge^{odd}(X)/\wedge_{GL}(X) \xrightarrow{\cong} \ker(\delta). \qquad \blacksquare$$

Let $i = \Gamma \circ \widehat{CS}^{-1}$. Since δ is clearly onto,

$$3.5) \qquad 0 \longrightarrow \wedge^{odd}(X)/\wedge_{GL}(X) \xrightarrow{i} \hat{K}(X) \xrightarrow{\delta} K(X) \longrightarrow 0$$

is an exact sequence.

<u>Proposition 3.6</u>: $ch \circ i = d$, and ch is onto.

<u>Proof</u>:   To show the first, note that from the definition of $\widehat{CS}$,

$$d\widehat{CS}(\mathcal{V}) = ch(\mathcal{V}) - \dim(\mathcal{V}) = ch(\Gamma(\mathcal{V}))$$
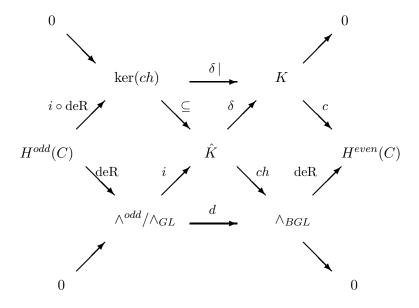
for any $\mathcal{V} \in \text{Struct}_{ST}(X)$. Thus, for $\theta \in \wedge^{odd}(X)$ and $\{\theta\}$ its equivalence class mod $\wedge_{GL}(X)$,

$$ch(i(\{\theta\})) = ch(\Gamma(\widehat{CS}^{-1}(\{\theta\}))) = d(\{\theta\}) = d\theta.$$

To show the second, let $\mu \in \wedge_{BGL}(X)$. By definition, $\exists\, \mathcal{V} \in \text{Struct}(X)$ and $\theta \in \wedge^{odd}$ so that $\mu = ch(\mathcal{V}) + d\theta$. By the above, $\mu = ch(\mathcal{V} + i(\{\theta\}))$. $\qquad \blacksquare$

Let deR $: H^{odd}(X, C) \to \wedge^{odd}(X)/\wedge_{GL}(X)$ be the obvious map induced by the de Rham Theorem. Since the image of deR consists of closed forms, $d \circ \text{deR} = 0$, which by Proposition 3.6, implies $ch \circ i \circ \text{deR} = 0$. Thus, $i \circ \text{deR}(H^{odd}(X, C)) \subseteq \ker(ch)$. We have now established

<u>Proposition 3.7</u>: The following diagram of functors and natural transformations is commutative, and its diagonals are exact.

$$
\begin{array}{ccccc}
0 & & & & 0 \\
& \searrow & & \nearrow & \\
& \mathrm{ker}(ch) & \xrightarrow{\ \delta\,|\ } & K & \\
{\scriptstyle i\,\circ\,\mathrm{deR}}\nearrow & \ \searrow{\scriptstyle \subseteq}\quad {\scriptstyle \delta}\nearrow & & \searrow{\scriptstyle c} & \\
H^{odd}(C) & & \hat{K} & & H^{even}(C) \\
& \searrow{\scriptstyle \mathrm{deR}}\quad {\scriptstyle i}\nearrow & & \searrow{\scriptstyle ch}\quad {\scriptstyle \mathrm{deR}}\nearrow & \\
& \wedge^{odd}/\wedge_{GL} & \xrightarrow{\ d\ } & \wedge_{BGL} & \\
& \nearrow & & \searrow & \\
0 & & & & 0
\end{array}
$$

Corollary 3.8: The outside sequences

$$
H^{odd}(C) \ \xrightarrow{\ i\circ\mathrm{deR}\ } \ \mathrm{ker}(ch) \ \xrightarrow{\ \delta\,|\ } \ K \ \xrightarrow{\ c\ } \ H^{even}(C)
$$

$$
H^{odd}(C) \ \xrightarrow{\ \mathrm{deR}\ } \ \wedge^{odd}/\wedge_{GL} \ \xrightarrow{\ d\ } \ \wedge_{BGL} \ \xrightarrow{\ \mathrm{deR}\ } \ H^{even}(C)
$$

are exact.

Proof: Exactness of the first follows from diagram chasing, and that of the second from the de Rham Theorem. ∎

In the decomposition below we assume that D is a codimension zero or one submanifold with collar neighborhoods in each of A and B. Thus a smooth form on D can be extended to a smooth form on either A or B.

Theorem 3.9 (Mayer-Vietoris):

Let $A, B \subseteq X$ with $A \cap B = D$ and $A \cup B = X$. If $\mu_A \in \hat{K}(A)$ and $\mu_B \in \hat{K}(B)$ with $\mu_A|D = \mu_B|D$, then there exists $\mu \in \hat{K}(X)$ with $\mu|A = \mu_A$ and $\mu|B = \mu_B$.

Proof: Following the diagram in Proposition 3.7, since $\delta(\mu_A)\,|\,D = \delta(\mu_B)\,|\,D$, the Mayer-Vietoris property for K produces $V - [n] \in K(X)$ with $(V - [n])\,|\,A = \delta(\mu_A)$ and $(V - [n])\,|\,B = \delta(\mu_B)$. Choose $\bar{\mu} \in \hat{K}(X)$ with $\delta(\bar{\mu}) = V - [n]$.

Now, $\delta(\bar{\mu}\,|\,A) = \delta(\bar{\mu})\,|\,A = \delta(\mu_A)$, and similarly for B. Thus, by the diagram

$$
\begin{aligned}
\bar{\mu}\,|\,A &= \mu_A + i(\{\alpha_A\}) \\
*) \qquad & \\
\bar{\mu}\,|\,B &= \mu_B + i(\{\alpha_B\})
\end{aligned}
$$

where α_A, $\alpha_B \in \wedge^{odd}(A)$, $\wedge^{odd}(B)$ and $\{\alpha_A\}$, $\{\alpha_B\}$ represent their equivalence classes mod $\wedge_{GL}(A)$, $\wedge_{GL}(B)$.

By the above,

$$i(\{\alpha_A \,|\, D\}) - i(\{\alpha_B \,|\, D\}) = i(\{\alpha_A\}) \,|\, D - i(\{\alpha_B\}) \,|\, D = (\bar{\mu} \,|\, A) \,|\, D - (\bar{\mu} \,|\, B) \,|\, D - \mu_A \,|\, D + \mu_B \,|\, D.$$

The first pair vanishes since each term is $\bar{\mu} \,|\, D$, and the second pair vanishes by hypothesis. Since i is an injection,

$$\alpha_A \,|\, D = \alpha_B \,|\, D + w$$

where $w \in \wedge_{GL}(D)$.

Case I: $w = d\rho$

Extend ρ to all of A, and set $\tilde{\alpha}_A = \alpha_A + d\rho$. Thus $\{\tilde{\alpha}_A\} = \{\alpha_A\}$, and $\tilde{\alpha}_A \,|\, D = \alpha_B \,|\, D$. The latter equation implies there is a unique $\alpha \in \wedge^{odd}(X)$ with $\alpha \,|\, A = \tilde{\alpha}_A$ and $\alpha \,|\, B = \tilde{\alpha}_B$. Thus by $*$)

$$\begin{aligned} \bar{\mu} \,|\, A &= \mu_A + i(\{\alpha\}) \,|\, A \\ \bar{\mu} \,|\, B &= \mu_B + i(\{\alpha\}) \,|\, B \end{aligned}$$

which implies that $\mu = \bar{\mu} - i(\{\alpha\})$ satisfies the conditions of the theorem.

Case II: $w = g^*(\Theta) + d\rho$, where $g : D \to GL$, and $g^*(\Theta)$ is not exact.

Using the clutching construction, we may construct a vector bundle V over X with the properties that $V \,|\, A$ and $V \,|\, B$ are each trivialized by cross-sections $\{E_i^A\}$ and $\{E_i^B\}$, and

$$**) \qquad E_j^B \,|\, D = \sum_i g_{ij} E_i^A \,|\, D.$$

Choose a connection, ∇', on V, set $\mathcal{V} = [V, \{\nabla'\}]$ and $\mu' = \mathcal{V} - [\dim(\mathcal{V})] \in \hat{K}(X)$.

By construction, $\delta(\mu' \,|\, A) = 0 = \delta(\mu' \,|\, B)$, and thus

$$\begin{aligned} \mu' \,|\, A &= i(\{\alpha_A'\}) \\ \mu' \,|\, B &= i(\{\alpha_B'\}) \end{aligned}$$

where α_A', $\alpha_B' \in \wedge^{odd}(A)$, $\wedge^{odd}(B)$ and $\{\alpha_A'\}$, $\{\alpha_B'\}$ represent their equivalence classes modulo $\wedge_{GL}(A)$, $\wedge_{GL}(B)$.

Let ∇^{AF} and ∇^{BF} be the flat connections on $V \,|\, A$ and $V \,|\, B$ defined by making $\{E_i^A\}$, $\{E_i^B\}$ parallel. By the definition of i, and working mod exact, we may take

$$\begin{aligned} \alpha_A' &= CS(\nabla^{AF}, \nabla' \,|\, A) \\ \alpha_B' &= CS(\nabla^{BF}, \nabla' \,|\, B). \end{aligned}$$

Now, continuing to work mod exact,

$$\alpha'_A \,|\, D - \alpha'_B \,|\, D = CS(\nabla^{AF} \,|\, D, \nabla' \,|\, D) - CS(\nabla^{BF} \,|\, D, \nabla' \,|\, D) = CS(\nabla^{AF} \,|\, D, \nabla^{BF} \,|\, D) = g^*(\Theta)$$

by $**$) and the argument of Lemma 2.3.

Thus, by taking $\bar{\bar{\mu}} = \bar{\mu} - \mu'$ and referring to $*$) we see

$$\begin{aligned}
\bar{\bar{\mu}} \,|\, A &= \mu_A + i(\{\alpha_A - \alpha'_A\}) \\
\bar{\bar{\mu}} \,|\, B &= \mu_B + i(\{\alpha_B - \alpha'_B\})
\end{aligned}$$

and

$$(\alpha_A - \alpha'_A) \,|\, D = (\alpha_B - \alpha'_B) \,|\, D + \text{exact}.$$

The problem is now reduced to Case I. ∎

Corollary 3.10: $\ker(ch)$ also satisfies the Mayer-Vietoris property.

Proof: In the theorem above, if $ch(\mu_A) = 0 = ch(\mu_B)$, then $ch(\mu) \,|\, A = 0 = ch(\mu) \,|\, B$. Since $ch(\mu)$ is a differential form, this implies $ch(\mu) = 0$. ∎

Proposition 3.11: $\ker(ch)$ is a homotopy functor.

Proof: Any element of $\ker(ch)$ is of the form $\mathcal{V} - [\dim(\mathcal{V})]$, where $ch(\mathcal{V}) = \dim(\mathcal{V})$. By 1.9) the pull backs of $\mathcal{V}$ under two smoothly homotopic C^∞ maps would be isomorphic, and so of course would pull backs of $[\dim(\mathcal{V})]$. ∎

§4. $\ker(ch)$ has classifying space the homotopy fibre of
$$BGL \xrightarrow{\ ch\ } \Pi_{n=1}^{\infty} K(C, 2n)$$

We begin by introducing a relative group denoted $\pi_n(BGL, \ker(ch))$ related to the characterization of homotopy fibers discussed in Appendix A. This group will consist of equivalence classes of stable complex vector bundles with C-linear connections over the n-disk D^n so that the Chern-Weil form vanishes on ∂D^n. Two of these (E, ξ) and (E', ξ') are equivalent if for some stable bundle isomorphism over D^n, $E \xrightarrow{\ p\ } E'$ the CS form $CS(\xi, p^*\xi')$ which is closed on ∂D^n is already exact on ∂D^n. We can add equivalence classes by direct sum, and these form a group using Theorem 1.15 applied to pairs, denoted $\pi_n(BGL, \ker(ch))$.

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Proposition 4.1:

$$\pi_n(BGL, \ker(ch)) = \begin{cases} 0 & n \text{ odd} \\ C & n \text{ even} \end{cases}$$

with the isomorphism given by the class in the cohomology of the boundary of the Chern-Simons difference form with the flat connection.

Proof: Since D^n is contractible, bundles over D^n are trivial, the bundle isomorphism $E \xrightarrow{p} E'$ exists and is unique up to homotopy. The CS form over ∂D^n $CS(\xi, p^*\xi')$ is odd and therefore exact for n odd. Thus $\pi_n(BGL, \ker(ch)) = 0$ for n odd.

For n even, the CS form $CS(\xi, p^*\xi')$ is closed and defines an element in $H^{n-1}(\partial D^n, C) \simeq H^n(D, \partial D^n; C)$. (E, ξ) and (E') are equivalent iff this class is zero. All classes occur in this by Proposition 2.6 applied to the boundary of the n-disk. This proves Proposition 4.1. ∎

Proposition 4.2: The functor $\ker(ch)$ is naturally equivalent on pointed compact manifolds with corners to the based homotopy classes of maps into some classifying space $GL(C/Z)$.

Proof: $\ker(ch)$ of a point is zero. $\ker(ch)$ is a homotopy functor satisfying Mayer-Vietoris by Corollary 3.10. $\ker(ch)$ also sends finite disjoint unions to finite products. It follows from Brown's theorem [10] that $\ker(ch)$ on compact manifolds with corners has a classifying space which we denote $GL(C/Z)$.

∎

Proposition 4.3: $GL(C/Z)$ is homotopy equivalent to the homotopy fibre of the Chern character map

$$BGL \xrightarrow{ch} \Pi_{n=1}^\infty K(C, 2n).$$

Proof: The map $\ker(ch) \xrightarrow{\delta} \tilde{K}(Z)$ implies a map $GL(C/Z) \xrightarrow{\delta} BGL$ where $\tilde{K}(Z)$ is the kernel of the restriction to the base point which is classified by maps into BGL.

The composition $GL(C/Z) \xrightarrow{\delta} BGL \xrightarrow{ch} \Pi_{n=1}^\infty K(C, 2n)$ is null homotopic from the definition of $\ker(ch)$ as representing structured bundles with Chern character form identically zero. In fact we may consider that we are provided with a preferred homotopy class of null homotopies for this composition $ch \circ \delta$.

Using this null homotopy gives a map $(BGL) \cup \text{cone}\, GL(C/Z) \longrightarrow \Pi_{n=1}^\infty K(C, 2n)$ which then gives a map

$$\pi_n(BGL, GL(C/Z)) \xrightarrow{CH} \pi_n(\Pi_{k=1}^\infty K(C, 2k)).$$

In Proposition 4.1 and the paragraph before we have interpreted $\pi_n(BGL, GL(C/Z))$ geometrically as the group $\pi_n(BGL, \ker(ch))$ and shown CH is an isomorphism.

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By Appendix A, this means $GL(C/Z) \xrightarrow{\delta} BGL$ is homotopy equivalent to the homotopy fibre of $BGL \xrightarrow{ch} \Pi_{n=1}^{\infty} K(C, 2n)$.   ∎

Thus we have

Theorem 4.4: $\ker(ch)$ is naturally equivalent to $K^{odd}(C/Z) =$ complex K-theory with coefficients in C/Z.

Proof: This will follow from the definition of $K^{odd}(C/Z)$ and the above. It is correct to define $K^{odd}(C/Z)$ as classified by the homotopy fibre of the map of classifying spaces corresponding to the map of reduced theories (the kernel of restrictions to the base points)

$$\tilde{K}(Z) \xrightarrow{\otimes C} \tilde{K}(Z) \otimes_Z C \equiv \tilde{K}(C)$$

i.e.

$$BGL \xrightarrow{ch} \Pi_{n=1}^{\infty} K(C, 2n)$$

where ch is the composition of $\otimes C$ with the Chern equivalence of $\tilde{K}(C)$ and $\Pi_{n=1}^{\infty} H^{2n}(, C)$.

For then the long exact sequence of the fibration

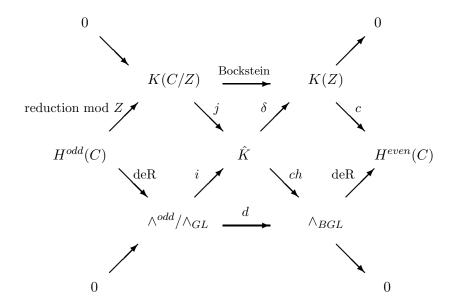$$\text{homotopy fibre} \to BGL \to \Pi_{n=1}^{\infty} K(C, 2n)$$

becomes

$$K^{odd}(C) \to K^{odd}C/Z \to K^{even}(Z) \to K^{even}(C)$$

or after applying the Chern character equivalence over C

$$H^{odd}(C) \to K^{odd}C/Z \to K^{even}(Z) \to H^{even}(C).   ∎$$

We gather all of this together to arrive at the following result.

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Corollary: We have the diagram with exact diagonals and exact upper and lower boundaries:

$$
\begin{array}{ccccc}
0 & & & & 0 \\
& \searrow & & \nearrow & \\
& K(C/Z) & \xrightarrow{\text{Bockstein}} & K(Z) & \\
\text{reduction mod } Z \nearrow & \quad j \searrow \quad \delta \nearrow & & \searrow c & \\
H^{odd}(C) & & \hat{K} & & H^{even}(C) \\
\text{deR} \searrow \quad i \nearrow & & ch \searrow \quad \text{deR} \nearrow & & \\
& \wedge^{odd}/\wedge_{GL} & \xrightarrow{d} & \wedge_{BGL} & \\
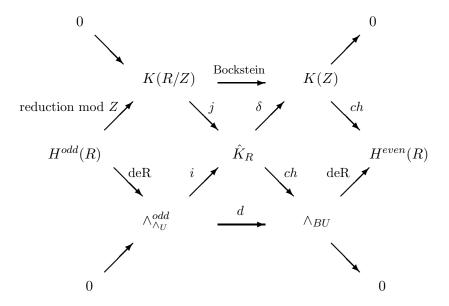& \nearrow & & \searrow & \\
0 & & & & 0
\end{array}
$$

Proof: The natural equivalence between $\ker(ch)$ and $K(C/Z)$ respects the Bockstein sequence because the construction in Appendix A relates the long exact homotopy sequence of the pair (total space, fibre) and the long exact sequence of homotopy groups of a fibration. ∎

§5. Hermitian Vector Bundles

In all that preceded, the the basic objects were complex vector bundles with connection. The entire approach immediately applies to Hermitian bundles with inner product preserving connection. The same definition of equivalence goes through and gives rise to a Hermitian version of Struct. Analogs of all results remain true, with proofs following identical lines.

Letting $\hat{K}_R = K$ (Hermitian Struct), we obtain the following commutative diagram,

where $\wedge_U$ and $\wedge_{BU}$ are real valued forms, defined analogously to $\wedge_{GL}$ and $\wedge_{BGL}$.

Corollary 5.1:

For any bundle over a closed Riemannian manifold after stabilizing, there is a unitary connection on the bundle whose Chern-Weil form is the harmonic representative of the Chern character of the bundle. Moreover, when the odd Betti numbers vanish, this structured bundle is unique up to adding factors with trivial holonomy.

Appendix A

Recall in the homotopy theory of spaces homotopy equivalent to CW complexes a map $X \to Y$ is homotopy equivalent to the projection map of a Serre fibration. To see this let us assume X and Y are connected. First replace $X \xrightarrow{p} Y$ by $X \xrightarrow{\tilde{p}} \tilde{Y}$ where $\tilde{p}$ is an inclusion by replacing Y by (the mapping cylinder of p) $= X \times I \cup_\sim Y$ where $X \times 1$ is collapsed by p onto its image in Y.

Then replace X by $\tilde{X}$ where $\tilde{X}$ is all the paths in $\tilde{Y}$ that start in X. Then $\tilde{X}$ maps into $\tilde{Y}$ (continue to call it $\tilde{p}$) with the Serre path lifting property by evaluating a path at its endpoint in $\tilde{Y}$. Clearly, $\tilde{Y} \sim Y$, $\tilde{X} \sim X$, and $\tilde{p} \sim p$.

The fibre $F \to X$ of $X \xrightarrow{p} Y$ is defined up to homotopy to be the inclusion into $\tilde{X}$ of the paths in $\tilde{Y}$ starting in X and ending at a specific point $y \in Y$ (or $\tilde{Y}$).

Question: What properties characterize the homotopy fibre $F \xrightarrow{i} X$ of a map $X \xrightarrow{p} Y$?

Proposition: Suppose we have a map $F' \xrightarrow{i'} X$ and further suppose the composition $F' \xrightarrow{i'} X \xrightarrow{p} Y$ is provided with a null homotopy so that the induced map of homotopy sets

$$\pi_i(X, F') \to \pi_i(Y, \text{base point})$$

are bijections $i = 1, 2, \dots$. Then $F' \xrightarrow{i'} X$ is homotopy equivalent to the homotopy fibre $F \to X$ of $X \xrightarrow{p} Y$.

Proof: In this proof we assume X and Y are connected and p is onto π_1. Thus F is connected and we assume F' is also connected. By the path lifting property of Serre fibrations, the null homotopy of the composition $F' \xrightarrow{i} X \xrightarrow{p} Y$ defines a canonical homotopy class of maps $F' \to F$ so that

$$
\begin{array}{ccccc}
F' & \xrightarrow{i'} & X & \xrightarrow{p} & Y \\
\downarrow & & \| & & \| \\
F & \xrightarrow{i} & X & \xrightarrow{p} & Y
\end{array}
$$

is homotopy commutative.

Now we look at the exact sequence of homotopy groups and sets

$$
\begin{array}{ccccccccccccc}
\cdots & \to & \pi_2 X & \to & \pi_2(X, F') & \to & \pi_1 F' & \to & \pi_1 X & \to & \pi_1(X, F') & \to & \pi_0 F' & \cong & \text{pt} \\
& & \| & & \downarrow & & \downarrow & & \| & & \downarrow & & \cong\downarrow & & \\
\cdots & \to & \pi_2 X & \to & \pi_2(X, F) & \to & \pi_1 F & \to & \pi_1 X & \to & \pi_1(X, F) & \to & \pi_0 F & \cong & \text{pt}
\end{array}
$$

In a fibration the Serre path lifting implies the homotopy sets $\pi_i(X, F)$ are isomorphic to $\pi_i(Y, \text{base point})$ and thus become groups. By the above commutative diagram the maps $\pi_i(X, F') \to \pi_i(X, F)$ are bijections. Thus the proposition follows from the 5-lemma.   ∎

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