The homotopy invariance of the string topology loop product and string bracket

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Abstract

Let M^n be a closed, oriented, n-manifold, and LM its free loop space. In [4] a commutative algebra structure in homology, $H_*(LM)$, and a Lie algebra structure in equivariant homology $H_*^{S^1}(LM)$, were defined. In this paper we prove that these structures are homotopy invariants in the following sense. Let $f : M_1 \to M_2$ be a homotopy equivalence of closed, oriented n-manifolds. Then the induced equivalence, $Lf : LM_1 \to LM_2$ induces a ring isomorphism in homology, and an isomorphism of Lie algebras in equivariant homology. The analogous statement also holds true for any generalized homology theory h_* that supports an orientation of the M_i's.

Introduction

The term "string topology" refers to multiplicative structures on the (generalized) homology of spaces of paths and loops in a manifold. Let M^n be a closed, oriented, smooth n-manifold. The basic "loop homology algebra" is defined by a product

$$\mu : H_*(LM) \otimes H_*(LM) \longrightarrow H_*(LM)$$

of degree $-n$, and and the "string Lie algebra" structure is defined by a bracket

$$[\ ,\ ] : H_*^{S^1}(LM) \otimes H_*^{S^1}(LM) \longrightarrow H_*^{S^1}(LM)$$

of degree $2-n$. These were defined in [4]. Here $H_*^{S^1}(LM)$ refers to the equivariant homology, $H_*^{S^1}(LM) = H_*(ES^1 \times_{S^1} LM)$. More basic structures on the chain level were also studied in [4]. Furthermore, these structures were shown to exist for any multiplicative homology theory h_* that supports an orientation of M. (see [11]. [1]) Alternative descriptions of the basic structure were given

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[1]The string bracket for generalized homology theories was not explicitly discussed in [11], although in Theorem 2 of that paper there is a homotopy theoretic action of Voronov's cactus operad given, which, by a result of Getzler [14] yields a Batalin-Vilkovisky structure on the generalized homology, $h_*(LM)$, when M is h_*-oriented. According to [4], this is all that is needed to construct the string bracket. We will review this construction below.

in [11] and [6], but in the end they all relied on various perspectives of intersection theory of chains and homology classes.

The existence of various descriptions of these operations leads to the following:

Question. *To what extent are the the string topology operations sensitive to the smooth structure of the manifold, or even the homeomorphism structure?*

The main goal of this paper is to settle this question in the case of two of the basic operations: the string topology loop product and string bracket. We will in fact prove more: we will show that the loop homology algebra and string Lie algebra structures are oriented *homotopy invariants.*

We remark that it is still not known whether the full range of string topology operations [4], [5], [25], [10] are homotopy invariants. Indeed the third author has conjectured that they are not (see the postscript to [26]). More about this point will be made in the remark after theorem 2 below.

To state the main result, let h_* be a multiplicative homology theory that supports an orientation of M. Being a multiplicative theory means that the corresponding cohomology theory, h^*, admits a cup product, or more precisely, the representing spectrum of the theory is required to be a ring spectrum. An h_*-*orientation* of a closed n-manifold M can be viewed as a choice of fundamental class $[M] \in h_n(M)$ that induces a Poincaré duality isomorphism.

Theorem 1. *Let M_1 and M_2 be closed, h_*-oriented n-manifolds. Let $f : M_1 \to M_2$ be an h_*-orientation preserving homotopy equivalence. Then the induced homotopy equivalence of loop spaces, $Lf : LM_1 \to LM_2$ induces a ring isomorphism of loop homology algebras,*

$$(Lf)_* : h_*(LM_1) \xrightarrow{\cong} h_*(LM_2).$$

Indeed it is an isomorphism of Batalin-Vilkovisky (BV) algebras. Moreover the induced map on equivariant homology,

$$(Lf)_* : h^{S^1}_*(LM) \xrightarrow{\cong} h^{S^1}_*(LM).$$

is an isomorphism of graded Lie algebras,

Evidence for the above theorem came from the results of [11] and [9] which said that for simply connected manifolds M, there is an isomorphism of graded algebras,

$$H_*(LM) \cong H^*(C^*(M), C^*(M)),$$

where the right hand side is the Hochschild cohomology of $C^*(M)$, the differential graded algebra of singular cochains on M, with multiplication given by cup product. The Hochschild cohomology algebra is clearly a homotopy invariant.

However, the above isomorphism is defined in terms of the Pontrjagin-Thom construction arising from the diagonal embedding $M^S \subset M^T$ associated with each surjection of finite sets $T \to S$. Consequently, since the Pontrjagin-Thom construction uses the smooth structure, this isomorphism *a priori* seems to be sensitive to the smooth structure. Without an additional argument, one can only conclude from this isomorphism that the loop homology algebra of two homotopy equivalent simply connected closed manifolds are *abstractly* isomorphic. In summary, to prove homotopy invariance in the sense of Theorem 1, one needs a different argument.

The argument we present here does not need the simple connectivity hypothesis. This should prove of particular interest in the case of surfaces and 3-manifolds. Our argument uses the description of the loop product μ in terms of a Pontrjagin-Thom collapse map of an embedding

$$LM \times_M LM \hookrightarrow LM \times LM$$

given in [11]. Here $LM \times_M LM$ is the subspace of $LM \times LM$ consisting of those pairs of loops (α, β), with $\alpha(0) = \beta(0)$. In this description we are thinking of the loop space as the space of piecewise smooth maps $[0, 1] \to M$ whose values at 0 and 1 agree. This is a smooth, infinite dimensional manifold. The differential topology of such manifolds is discussed in [12] and [6].

This description quickly reduces the proof of the theorem to the question of whether the homotopy type of the complement of this embedding, $(LM \times LM) - (LM \times_M LM)$ is a stable homotopy invariant when considered as "a space over" $LM \times LM$. By using certain pullback properties, the latter question is then further reduced to the question of whether the complement of the diagonal embedding, $\Delta : M \to M \times M$, or somewhat weaker, the complement of the embedding

$$\Delta_k : M \to M \times M \times D^k$$
$$x \to (x, x, 0)$$

is a homotopy invariant when considered as a space over $M \times M$. For this we develop the notion of relative smooth and Poincare embeddings. This is related to the classical theory of Poincare embeddings initiated by Levitt [21] and Wall [27], and further developed by the second author in [16] and [17]. However, for our purposes, the results we need can be proved directly by elementary arguments. The results in Section 2 on relative embeddings are rather fundamental, but don't appear in the literature. These results may be of independent interest, and furthermore, by proving them here, we make the paper self contained.

Early on in our investigation of this topic, our methods led us to advertise the following question, which is of interest independent of its applications to string topology.

Let $F(M, q)$ be the configuration space of q-distinct, ordered points in a closed manifold M.

Question. *Assume that M_1 and M_2 be homotopy equivalent, simply connected closed n-manifolds. Are $F(M_1, q)$ and $F(M_2, q)$ homotopy equivalent?*

One knows that these configuration spaces have isomorphic cohomologies ([3]), stable homotopy types ([2], [7]) and have homotopy equivalent loop spaces ([8], [22]). But the homotopy invariance of the configuration spaces themselves is not yet fully understood. For example, when $q = 2$ and the manifolds are 2-connected, then one does have homotopy invariance ([22], [2]). On the other hand, the simple connectivity assumption in the above question is a necessity: a recent result of Longoni and Salvatore [23] shows that for the homotopy equivalent lens spaces $L(7,1)$ and $L(7,2)$, the configuration spaces $F(L(7,1),2)$ and $F(L(7,2),2)$ have distinct homotopy types.

This paper is organized as follows. In Section 1 we will reduce the proof of the main theorem to a question about the homotopy invariance of the complement of the diagonal embedding, $\Delta_k : M \to M \times M \times D^k$. In Section 2 we develop the theory of relative smooth and Poincare embeddings, and then apply it to prove the homotopy invariance of these configuration spaces, and complete the proof of Theorem 1.

Remark. After the results of this paper were announced, two independent proofs of the homotopy invariance of the loop homology product were found by by Crabb [13], and by Gruher-Salvatore [15].

Conventions. A finitely dominated pair of spaces $(X, \partial X)$ is a *Poincare pair* of dimension d if there exists a pair $(\mathcal{L}, [X])$ consisting of a rank one abelian local coefficient system $\mathcal{L}$ on X and a "fundamental class" $[X] \in H_d(X, \partial X; \mathcal{L})$ such that the cap product homomorphisms

$$\cap[X] \colon H^*(X; \mathcal{M}) \to H_{d-*}(X, \partial X; \mathcal{L} \otimes \mathcal{M})$$

and

$$\cap[\partial X] \colon H^*(\partial X; \mathcal{M}) \to H_{d-1-*}(\partial X; \mathcal{L} \otimes \mathcal{M})$$

are isomorphisms for all local coefficient bundles $\mathcal{M}$ on X (respectively on ∂X). Here $[\partial X] \in H_{d-1}(\partial X; \mathcal{L})$ denotes the image of $[X]$ under the evident boundary homomorphism. If such a pair $(\mathcal{L}, [X])$ exists, then it is unique up to unique isomorphism.

1 A question about configuration spaces

In this section we state one of our main results about the homotopy invariance of certain configuration spaces, and then use it to prove Theorem 1. The theorem about configuration spaces will be proved in section 2.

Using an identification of the tangent bundle τ_M with the normal bundle of the diagonal, $\Delta \colon M \to M \times M$, we have an embedding of the disk bundle,

$$D(\tau_M) \subset M \times M \,,$$

which is identified with a compact tubular neighborhood of the diagonal. (To define the unit disk bundle, we use a fixed Euclidean structure on τ_M.) The closure of its complement will be denoted

$F(M,2)$. Notice that the inclusion $F(M,2) \subset M \times M - \Delta$ is a weak equivalence. We therefore have a decomposition,

$$M \times M = D(\tau_M) \cup_{S(\tau_M)} F(M,2),$$

where $S(\tau_M) = \partial D(\tau_M)$ is the unit sphere bundle. We now vary the configuration space in the following way.

Let D^k be a closed unit disk, and consider the generalized diagonal embedding,

$$\begin{aligned} \Delta_k : M &\to M \times M \times D^k \\ x &\to (x, x, 0). \end{aligned}$$

We may now identify the stabilized tangent bundle, $\tau_M \oplus \epsilon^k$ with the normal bundle of this embedding, where ϵ^k is the trivial k-dimensional bundle. This yields an embedding, $D(\tau_M \oplus \epsilon^k) \subset M \times M \times D^k$, which is identified with a closed tubular neighborhood of Δ_k. The closure of its complement is denoted by $F_{D^k}(M,2)$. The reader will notice that this is a model for the k-fold fiberwise suspension of the map $F(M,2) \to M \times M$. We now have a similar decomposition,

$$M \times M \times D^k = D(\tau_M \oplus \epsilon^k) \cup_{S(\tau_M \oplus \epsilon^k)} F_{D^k}(M,2).$$

Notice furthermore, that the boundary, $\partial(M \times M \times D^k) = M \times M \times S^{k-1}$ lies in the subspace, $F_{D^k}(M,2)$. In other words we have a commutative diagram,

$$\begin{array}{ccccc} S(\tau_M \oplus \epsilon^k) & \longrightarrow & F_{D^k}(M,2) & \longleftarrow & M \times M \times S^{k-1} \\ \downarrow & & \downarrow & & \\ D(\tau_M \oplus \epsilon^k) & \longrightarrow & M \times M \times D^k & & \end{array} \qquad (1)$$

where the commutative square is a pushout square. We refer to this diagram as $M(k)_\bullet$. We think of this more functorially as follows.

Consider the partially ordered set $\mathcal{F}$, with five objects, referred to as $\emptyset$, 0, 1, 01, and b, and the morphisms are generated by the following commutative diagram

$$\begin{array}{ccccc} \emptyset & \longrightarrow & 1 & \longleftarrow & b \\ \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 01\,. & & \end{array} \qquad (2)$$

Notice that 01 is a terminal object of this category.

Definition 1. We define an $\mathcal{F}$-space to be a functor $X : \mathcal{F} \to \text{Top}$, where Top is the category of topological spaces. The value of the functor at $S \subset \{0,1\}$ is denoted X_S. It will sometimes be convenient to specify X by maps of pairs

$$(X_b, \emptyset) \to (X_1, X_\emptyset) \to (X_{01}, X_0)\,,$$

where we are abusing notation slightly since the maps $X_\emptyset \to X_1$ and $X_0 \to X_{01}$ need not be inclusions.

A map (morphism) $\phi : X \to Y$ of $\mathcal{F}$-spaces is a natural transformation of functors. We say that ϕ is a weak equivalence, if it is an object-wise weak homotopy equivalence, i.e, it gives a a weak homotopy equivalence $\phi_i : X_i \xrightarrow{\simeq} Y_i$ for each object $i \in \mathcal{F}$. In general, we say that two $\mathcal{F}$-spaces are weakly equivalent if there is a finite zig-zag of morphisms connecting them,

$$X = X^1 \to X^2 \leftarrow X^3 \to \cdots \leftarrow X^i \to \cdots X^n = Y\,,$$

where each morphism is a weak equivalence.

Notice that diagram (1) defines a $\mathcal{F}$-space for each closed manifold M, and integer k. We call this $\mathcal{F}$- space $M(k)_\bullet$. In particular, $M(k)_{01} = M \times M \times D^k$.

The following is our main result about configuration spaces. It will be proved in section 2.

Theorem 2. *Assume M_1 and M_2 are closed manifolds and that $f : M_1 \to M_2$ is a homotopy equivalence. Then for k sufficiently large, the $\mathcal{F}$-spaces $M_1(k)$ and $M_2(k)$ are weakly equivalent in the following specific way.*

There is a $\mathcal{F}$-space $\mathcal{T}_\bullet$ that takes values in spaces of the homotopy type of CW-complexes, and morphisms of $\mathcal{F}$-spaces,

$$M_1(k)_\bullet \xrightarrow{\phi_1} \mathcal{T}_\bullet \xleftarrow{\phi_2} M_2(k)_\bullet$$

satisfying the following properties:

1. *The morphisms ϕ_1 and ϕ_2 are weak equivalences.*

2. *The terminal space $\mathcal{T}_{01}$ is defined as*

$$\mathcal{T}_{01} = T_{f\times f} \times D^k$$

where $T_{f\times f}$ is the mapping cylinder $(M_2\times M_2)\cup_{f\times f}(M_1\times M_1)\times I$. Furthermore on the terminal spaces, the morphisms, $\phi_1 : M_1 \times M_1 \times D^k \to T_{f\times f} \times D^k$ and $\phi_1 : M_2 \times M_2 \times D^k \to T_{f\times f} \times D^k$ are given by $\iota_1 \times 1$ and $\iota_2 \times 1$, where for $j = 1, 2$, $\iota_j : M_j \times M_j \to T_{f\times f}$ are the obvious inclusions as the two ends of the mapping cylinder.

3. *The induced weak equivalence,*

$$D(\tau_{M_1} \oplus \epsilon^k) = M_1(k)_0 \xrightarrow[\simeq]{\phi_2} \mathcal{T}_0 \xleftarrow[\simeq]{\phi_2} M_2(k)_0 = D(\tau_{M_2} \oplus \epsilon^k)$$

is homotopic to the composition

$$D(\tau_{M_1} \oplus \epsilon^k) \xrightarrow[\simeq]{\text{project}} M_1 \xrightarrow[\simeq]{f} M_2 \xrightarrow[\simeq]{\text{zero}} D(\tau_{M_2} \oplus \epsilon^k).$$

Notice that this theorem is a strengthening of the following homotopy invariance statement (see [2]).

Corollary 3. *Let $f : M_1 \to M_2$ be a homotopy equivalence of closed manifolds. Then for sufficiently large k, the configuration spaces, $F_{D^k}(M_1, 2)$ and $F_{D^k}(M_2, 2)$ are homotopy equivalent.*

As mentioned, we will delay the proof of Theorem 2 until the next section. Throughout the rest of this section we will assume its validity, and will use it to prove Theorem 1, as stated in the introduction.

Proof. Consider the equivalences of $\mathcal{F}$-spaces given in Theorem 2. Notice that we have the following commutative diagram of maps of pairs.

$$\begin{array}{ccccc}
(M_1(k)_{01},\, M_0(k)_b) & \longrightarrow & (M_1(k)_{01},\, M_1(k)_1) & \longleftarrow & (M_1(k)_0,\, M_1(k)_\emptyset) \\
\phi_1 \downarrow & & \downarrow \phi_1 & & \downarrow \phi_1 \\
(\mathcal{T}_{01},\, \mathcal{T}_b) & \longrightarrow & (\mathcal{T}_{01},\, \mathcal{T}_1) & \longleftarrow & (\mathcal{T}_0,\, \mathcal{T}_\emptyset) \\
\phi_2 \uparrow & & \uparrow \phi_2 & & \uparrow \phi_2 \\
(M_2(k)_{01},\, M_2(k)_b) & \longrightarrow & (M_2(k)_{01},\, M_2(k)_1) & \longleftarrow & (M_2(k)_0,\, M_2(k)_\emptyset)\,.
\end{array} \tag{3}$$

The vertical maps are weak homotopy equivalences of pairs, by Theorem 2. The horizontal maps are induced by the values of the $\mathcal{F}$-spaces on the morphisms in $\mathcal{F}$.

For ease of notation, for a pair (A, B) we write A/B for the homotopy cofiber (mapping cone) $A \cup cB$. By plugging in the values of these $\mathcal{F}$-spaces, and taking homotopy cofibers, we get a commutative diagram

$$\begin{array}{ccccc}
M_1 \times M_1 \times D^k / M_1 \times M_1 \times S^{k-1} & \longrightarrow & M_1 \times M_1 \times D^k / F_{D^k}(M_1, 2) & \xleftarrow{\simeq} & D(\tau_{M_1} \oplus \epsilon^k)/S(\tau_{M_1} \oplus \epsilon^k) \\
\phi_1 \downarrow & & \downarrow \phi_1 & & \downarrow \phi_1 \\
\mathcal{T}_{01}/\mathcal{T}_b & \longrightarrow & \mathcal{T}_{01}/\mathcal{T}_1 & \underset{\simeq}{\longleftarrow} & \mathcal{T}_0/\mathcal{T}_\emptyset \\
\phi_2 \uparrow & & \uparrow \phi_2 & & \uparrow \phi_2 \\
M_2 \times M_2 \times D^k / M_2 \times M_2 \times S^{k-1} & \longrightarrow & M_2 \times M_2 \times D^k / F_{D^k}(M_2, 2) & \xleftarrow{\simeq} & D(\tau_{M_2} \oplus \epsilon^k)/S(\tau_{M_2} \oplus \epsilon^k)
\end{array} \tag{4}$$

The right hand horizontal maps are equivalences, because the commutative squares defined by the $\mathcal{F}$-spaces $M_1(k)_\bullet$ and $M_2(k)_\bullet$ are pushouts, and therefore the commutative square defined by the $\mathcal{F}$-space $\mathcal{T}_\bullet$ is a homotopy pushout. By inverting these homotopy equivalences, as well as those induced by ϕ_2, we get a homotopy commutative square,

$$\begin{array}{ccc}
\Sigma^k((M_1 \times M_1)_+) & \xrightarrow{m_1} & \Sigma^k(M_1^{\tau_{M_1}}) \\
f_k \downarrow & & \downarrow f_k \\
\Sigma^k((M_2 \times M_2)_+) & \xrightarrow{m_2} & \Sigma^k(M_2^{\tau_{M_2}})
\end{array} \tag{5}$$

Here the maps f_k have the homotopy type of $\phi_2^{-1} \circ \phi_1$. The right hand spaces are the suspensions of the Thom spaces of the tangent bundles of M_1 and M_2 respectively.

Notice that property (2) in Theorem 2 regarding the morphisms ϕ_1 and ϕ_2 and the mapping cylinder $\mathcal{T}_{0,1}$ impies that the left hand map $f_k : \Sigma^k((M_1 \times M_1)_+) \to \Sigma^k((M_2 \times M_2)_+)$ is given by the k-fold suspension of the equivalence $f \times f : M_1 \times M_1 \xrightarrow{\simeq} M_2 \times M_2$. Consider the right hand vertical equivalence, $f_k : \Sigma^k(M_1)^{\tau_{M_1}} \to \Sigma^k(M_2^{\tau_{M_2}})$. By diagram (3) f_k is induced by a map of pairs, $(D(\tau_{M_1} \oplus \epsilon^k), S(\tau_{M_1} \oplus \epsilon^k)) \to (D(\tau_{M_2} \oplus \epsilon^k), S(\tau_{M_2} \oplus \epsilon^k))$ which on the ambient space, $D(\tau_{M_1} \oplus \epsilon^k) \to D(\tau_{M_2} \oplus \epsilon^k)$ is homotopic to the map determined by $f : M_1 \to M_2$ as in property 3 of Theorem 2. Therefore the induced map in cohomology $(f_k)^* : h^*(\Sigma^k(M_2)^{\tau_{M_2}}) \cong h^*(\Sigma^k(M_1)^{\tau_{M_1}})$ is an isomorphism as modules over $h^*(M_2)$, where the module structure on $h^*(\Sigma^k(M_1)^{\tau_{M_1}})$ is via the isomorphism $f^* : h^*(M_2) \cong h^*(M_1)$.

Moreover the isomorphism $(f_k)^* : h^*(\Sigma^k(M_2^{\tau_{M_2}})) \cong h^*(\Sigma^k(M_1^{\tau_{M_1}}))$ preserves the Thom class in cohomology. To see this, notice that the horizontal maps in diagram (5) yield the intersection product in homology, after applying the Thom isomorphism. This implies that the image of the fundamental classes $\Sigma^k([M_i] \times 1) \in h_{k+n}(\Sigma^k(M_i \times M_i))$ maps to the Thom classes in $h_{k+n}(\Sigma^k(M_i)^{\tau_{M_i}})$. Since the left hand vertical map is homotopic to $\Sigma^k(f \times f)$, and since the homotopy equivalence f preserves the h_*-orientations, it preserves the fundamental classes. Therefore by the commutativity of this diagram, $(f_k)_*$ preserves the Thom class. These facts imply that after applying the Thom isomorphism, the isomorphism $(f_k)^*$ is given by $f^* : h^*(M_2) \xrightarrow{\cong} h^*(M_1)$.

This observation will be useful, as we will eventually lift the map of $\mathcal{F}$-spaces given in Theorem 2 up to the level of loop spaces, and we'll consider the analogue of the diagram (5).

To understand why this is relevant, recall from [4], [11] that the loop homology product $\mu : h_*(LM) \times h_*(LM) \to h_*(LM)$ can be defined in the following way. Consider the pullback square

$$\begin{array}{ccc} LM \times_M LM & \xrightarrow{\iota} & LM \times LM \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e \times e} \\ M & \xrightarrow[\Delta]{} & M \times M \end{array}$$

where $e : LM \to M$ is the fibration given by evaluation at the basepoint: $e(\gamma) = \gamma(0)$. Let $\eta(\Delta)$ be a tubular neighborhood of the diagonal embedding of M, and let $\eta(\iota)$ be the inverse image of this neighborhood in $LM \times LM$. The normal bundle of Δ is the tangent bundle, τ_M. Recall that the evaluation map $e : LM \to M$ is a locally trivial fiber bundle [20]. Therefore the tubular neighborhood $\eta(\iota)$ of $\iota : LM \times_M LM \hookrightarrow LM \times LM$ is homeomorphic to total space of the pullback of the tangent bundle, $e^*(TM)$. We therefore have a Pontrjagin-Thom collapse map,

$$\tau : LM \times LM \longrightarrow LM \times LM/((LM \times LM) - \eta(\iota)) \cong (LM \times_M LM)^{\tau_M} \qquad (6)$$

where $(LM \times_M LM)^{\tau_M}$ is the Thom space of the pullback $e^*(\tau_M) \to LM \times_M LM$.

Now as pointed out in [4], there is a natural map

$$j \colon LM \times_M LM \to LM$$

given by sending a pair of loops (α, β) with the same starting point, to the concantenation of the loops, $\alpha * \beta$. The loop homology product is then defined to be the composition

$$\mu_*\colon h_*(LM \times LM) \xrightarrow{\tau_*} h_*((LM \times_M LM)^{TM}) \xrightarrow[\cong]{\cap u} h_{*-n}(LM \times_M LM) \xrightarrow{j_*} h_{*-n}(LM) \tag{7}$$

where $\cap u$ is the Thom isomorphism given by capping with the Thom class.

Now consider the fiber bundles, $LM_i \times LM_i \times D^k \to M_i \times M_i \times D^k = M_i(k)_{01}$ for $i = 1, 2$. By restricting this bundle to the spaces $M_i(k)_j$, $j \in \mathrm{Ob}(\mathcal{F})$, we obtain $\mathcal{F}$-spaces which we call $\mathcal{L}M_i(k)_\bullet$, for $i = 1, 2$. So we have morphisms of $\mathcal{F}$-space, $e : \mathcal{L}M_i(k)_\bullet \to M_i(k)_\bullet$ which on every object is a fiber bundle, and every morphism induces a pull-back square.

Similarly, let $\mathcal{L}\mathcal{T}_\bullet$ be the $\mathcal{F}$ space obtained by restricting the fibration $L(T_{f\times f}) \times D^k \xrightarrow{e\times 1} T_{f\times f} \times D^k = \mathcal{T}_{01}$ to the spaces $\mathcal{T}_j$, for $j \in \mathrm{Ob}(\mathcal{F})$.

The morphisms ϕ_i of Theorem 2 then lift to give weak equivalences of $\mathcal{F}$-spaces, $\mathcal{L}\phi_i : \mathcal{L}M_i(k)_\bullet \to \mathcal{L}\mathcal{T}_\bullet$ that make the following diagram of $\mathcal{F}$-spaces commute:

$$\begin{array}{ccccc} \mathcal{L}M_1(k)_\bullet & \xrightarrow{\mathcal{L}\phi_1} & \mathcal{L}\mathcal{T}_\bullet & \xleftarrow{\mathcal{L}\phi_2} & \mathcal{L}M_2(k)_\bullet \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ M_1(k)_\bullet & \xrightarrow{\phi_1} & \mathcal{T}_\bullet & \xleftarrow{\phi_2} & M_2(k)_\bullet \end{array} \tag{8}$$

The commutative diagram of maps of pairs (3) lifts to give a corresponding diagram with spaces $\mathcal{L}M_i(k)_\bullet$ replacing $M_i(k)_\bullet$, and $\mathcal{L}\mathcal{T}_\bullet$ replacing $\mathcal{T}_\bullet$. There is also a corresponding commutative diagram of quotients, that lifts the diagram (4). The result is a homotopy commutative square, which lifts square (5).

$$\begin{array}{ccc} \Sigma^k((LM_1 \times LM_1)_+) & \xrightarrow{\tau_1} & \Sigma^k(LM_1 \times_{M_1} LM_1)^{\tau_{M_1}} \\ {\scriptstyle \tilde{f}_k}\downarrow & & \downarrow{\scriptstyle \tilde{f}_k} \\ \Sigma^k((LM_2 \times LM_2)_+) & \xrightarrow{\tau_1} & \Sigma^k(LM_2 \times_{M_2} LM_2)^{\tau_{M_2}} \end{array} \tag{9}$$

Here the maps $\tilde{f}_k$ have the homotopy type of $L\phi_2^{-1} \circ L\phi_1$.

Now as argued above, the description of the maps $L\phi_i : \mathcal{L}M_i(k)_{(0,1)} \to \mathcal{L}\mathcal{T}_{(0,1)}$, that is, $LM_i \times LM_i \times D^k \to LT_{f\times f} \times D^k$ as the loop functor applied to the inclusion as the ends of the cylinder, implies that the map $\tilde{f}_k : \Sigma^k((LM_1 \times LM_1)_+) \to \Sigma^k((LM_2 \times LM_2)_+)$ is homotopic to the k-fold suspension of $Lf \times Lf : LM_1 \times LM_1 \xrightarrow{\simeq} LM_2 \times LM_2$. Moreover, in cohomology, the map $\tilde{f}_k^* : h^*(\Sigma^k(LM_2 \times_{M_2} LM_2)^{\tau_{M_2}}) \to h^*(\Sigma^k(LM_1 \times_{M_1} LM_1)^{\tau_{M_1}})$ preserves Thom classes because the bundles are pulled back from bundles over M_1 and M_2 respectively, and as seen above, $(f_k)^* : h^*(\Sigma^k(M_2^{\tau_{M_2}})) \cong h^*(\Sigma^k(M_1^{\tau_{M_1}}))$ preserves Thom classes. Also, since this map is, up to homotopy, induced by a map of pairs

$$L\phi_2^{-1} \circ L\phi_1 : (D(e^*(\tau_{M_1} \oplus \epsilon^k)),\, S(e^*(\tau_{M_1} \oplus \epsilon^k))) \to (D(e^*(\tau_{M_2} \oplus \epsilon^k)),\, S(e^*(\tau_{M_2} \oplus \epsilon^k)))$$

it induces an isomorphism of $h^*(D(e^*(\tau_{M_2} \oplus \epsilon^k)))$ modules, where this ring acts on $h^*(\Sigma^k(LM_1 \times_{M_1} LM_1)^{\tau_{M_1}})$ via the homomorphism, $(L\phi_2^{-1} \circ L\phi_1)^* : h^*(D(e^*(\tau_{M_2} \oplus \epsilon^k))) \xrightarrow{\cong} h^*(D(e^*(\tau_{M_1} \oplus \epsilon^k)))$.

But by the lifting of property 3 in Theorem 2, this map is homotopic to the compositon,

$$(D(e^*(\tau_{M_1} \oplus \epsilon^k)) \xrightarrow{\text{project}} LM_1 \times_{M_1} LM_1 \xrightarrow{Lf \times Lf} LM_2 \times_{M_2} LM_2 \xrightarrow{\text{zero}} (D(e^*(\tau_{M_2} \oplus \epsilon^k)).$$

Hence when one applies the Thom isomorphism to both sides, the isomorphism

$$\tilde{f}_k^* : h^*(\Sigma^k(LM_2 \times_{M_2} LM_2)^{\tau_{M_2}}) \to h^*(\Sigma^k(LM_1 \times_{M_1} LM_1)^{\tau_{M_1}})$$

is given by $(Lf \times Lf)^* : h^*(LM_2 \times_{M_2} LM_2) \xrightarrow{\cong} h^*(LM_1 \times_{M_1} LM_1)$.

By the definition of the loop product (7), to prove that $(Lf)_* : h_*(LM_1) \to h_*(LM_2)$ is a ring isomorphism, we need to show that the diagram

$$\begin{array}{ccccccc}
h_*(LM_1 \times LM_1) & \xrightarrow{(\tau_1)_*} & h_*((LM_1 \times LM_1)^{\tau_{M_1}}) & \xrightarrow[\cong]{\cap u} & h_{*-n}((LM_1 \times_{M_1} LM_1)) & \xrightarrow{j_*} & h_{*-n}(LM_1) \\
{\scriptstyle (Lf \times Lf)_*}\downarrow & & {\scriptstyle (\tilde{f}_k)_*}\downarrow & & \downarrow{\scriptstyle (Lf \times Lf)_*} & & \downarrow{\scriptstyle (Lf)_*} \\
h_*(LM_2 \times LM_2) & \xrightarrow{(\tau_2)_*} & h_*((LM_2 \times LM_2)^{\tau_{M_2}}) & \xrightarrow[\cong]{\cap u} & h_{*-n}((LM_2 \times_{M_2} LM_2)) & \xrightarrow{j_*} & h_{*-n}(LM_2)
\end{array}$$

commutes. We have now verified that the left and middle squares commute. But the right hand square obviously commutes. Thus $(Lf)_* : h_*(LM_1) \to h_*(LM_2)$ is a ring isomorphism as claimed.

To prove that Lf is a map of BV- algebras, recall that the BV-operator Δ is defined in terms of the S^1-action. Clearly Lf preserves this action, and hence induces an isomorphism of BV-algebras. This will imply that Lf induces an isomorphism of the string Lie algebras for the following reason. Recall the definition of the Lie bracket from [4]. Given $\alpha \in h_p^{S^1}(LM)$ and $\beta \in h_q^{S^1}(LM)$, then the bracket $[\alpha, \beta]$ is the image of $\alpha \times \beta$ under the composition,

$$\begin{aligned} h_p^{S^1}(LM) \times h_q^{S^1}(LM) &\xrightarrow{\mathrm{tr}_{S^1} \times \mathrm{tr}_{S^1}} h_{p+1}(LM) \times h_{q+1}(LM) \\ &\xrightarrow{\text{loop product}} h_{p+q+2-n}(LM) \xrightarrow{j} h_{p+q+2-n}^{S^1}(LM). \end{aligned} \tag{10}$$

Here $\mathrm{tr}_{S^1} \colon h_*^{S^1}(LM) \to h_{*+1}(LM)$ is the S^1 transfer map (called "M" in [4]), and $j \colon h_*(LM) \to h_*^{S^1}(LM)$ is the usual map that descends nonequivariant homology to equivariant homology (called "E" in [4]). We refer the reader to [1] for a concise definition of the S^1-transfer.

We now know that Lf preserves the loop product, and since it is an S^1-equivariant map, it preserves the transfer map tr_{S^1} and the map j. Therefore it preserves the string bracket. □

2 Relative embeddings and the proof of Theorem 1

Theorem 2 reduces the proof of the homotopy invariance of the loop product and the string bracket (Theorem 1) to the homotopy invariance of the $\mathcal{F}$-spaces associated with the embeddings diagonal of M_1 and M_2. The goal of the present section is to prove Theorem 2.

2.1 Relative smooth embeddings

Let N be a compact smooth manifold of dimension n whose boundary ∂N comes equipped with a smooth manifold decomposition

$$\partial N = \partial_0 N \cup \partial_1 N$$

in which $\partial_0 N$ and $\partial_1 N$ are glued together along their common boundary

$$\partial_{01} N \ := \ \partial_0 N \cap \partial_1 N \, .$$

Assume that K is a space obtained from $\partial_0 N$ by attaching a finite number of cells. Hence we have a relative cellular complex

$$(K, \partial_0 N) \, .$$

It then makes sense to speak of the *relative dimension*

$$\dim(K, \partial_0 N) \leq k$$

as being the maximum dimension of the attached cells.

Let

$$f \colon K \to N$$

be a map of spaces which extends the identity map of $\partial_0 N$.

Definition 2. We call these data, $(K, \partial_0 N, f \colon K \to N)$, a *relative smooth embedding problem*

Definition 3. A *solution* to the relative smooth embedding problem consists of

- a codimension zero compact submanifold

  $$W \subset N$$

  such that $\partial W \cap \partial N = \partial_0 N$ and this intersection is transversal, and

- a homotopy of f, fixed on $\partial_0 N$, to a map of the form

  $$K \xrightarrow{\ \sim\ } W \xrightarrow{\ \subset\ } N$$

  in which the first map is a homotopy equivalence.

Lemma 4. *If $2k < n$, then there is a solution to the relative smooth embedding problem.*

We remark that Lemma 4 is essentially a simplified version of a result of Hodgson [Ho] who strengthens it by r dimensions when the map f is r-connected.

Proof of the Lemma 4. First assume that $K = \partial_0 N \cup D^k$ is the effect of attaching a single k-cell to $\partial_0 N$. Then the restriction of f to the disk gives a map

$$(D^k, S^{k-1}) \to (N, \partial_0 N)$$

and, by transversality, we can assume that its restriction $S^{k-1} \to \partial_0 N$ is a smooth embedding. Applying transversality again, the map on D^k can be generically deformed relative to S^{k-1} to a smooth embedding. Call the resulting embedding g. Let W be defined by taking a regular neighborhood of $\partial_0 N \cup g(D^k) \subset N$. Then g and W give the desired solution in this particular case.

The general case is by induction on the set of cells attached to $\partial_0 N$. The point is that if a solution $W \subset N$ has already been achieved on a subcomplex L of K given by deleting one of the top cells, then removing the interior of W from N gives a new manifold N', such that $\partial N'$ has a boundary decomposition. The attaching map $S^{k-1} \to L$ can be deformed (again using transversality) to a map into $\partial_0 N'$. Then we have reduced to a situation of solving the problem for a map of the form $D^k \cup \partial_0 N' \to N'$, which we know can be solved by the previous paragraph. ☐

We now thicken the complex K by crossing $\partial_0 N$ with a disk. Namely, for an integer $j \geq 0$, define the space

$$K_j \simeq K \cup_{\partial_0 N} (\partial_0 N) \times D^j \, ,$$

where we use the inclusion $\partial_0 N \times 0 \subset (\partial_0 N) \times D^j$ to form the amalgamated union. Then $(K, \partial_0 N) \subset (K_j, (\partial_0 N) \times D^j)$ is a deformation retract, and the map $f \colon K \to N$ extends in the evident way to a map

$$f_j \colon K_j \to N \times D^j$$

that is fixed on $(\partial_0 N) \times D^j$.

Theorem 5. *Let $f \colon K \to N$ be as above, but without the dimension restrictions. Then for sufficiently large $j \geq 0$, the embedding problem for the map $f_j \colon K_j \to N \times D^j$ admits a solution.*

Proof. The relative dimension of $(K_j, (\partial_0 N) \times D^j)$ is k, but for sufficiently large j we have $2k \leq n+j$. The result follows from the previous lemma. ☐

2.2 Relative Poincaré embeddings

Now suppose more generally that $(N, \partial N)$ is a (finite) Poincaré pair of dimension n equipped with a *boundary decomposition* such that $\partial_0 N$ is a smooth manifold. By this, we mean we have an expression of the form

$$\partial N \;=\; \partial_0 N \cup_{\partial_{01} N} \partial_1 N$$

in which $\partial_0 N$ is a manifold with boundary $\partial_{01} N$ and also $(\partial_1 N, \partial_{01} N)$ is a Poincaré pair. Furthermore, we assume that the fundamental classes for each of theses pairs glue to a fundamental class for ∂N. These fundamental classes lie in ordinary homology.

As above, let

$$f\colon K \to N$$

be a map which is fixed on $\partial_0 N$. We will assume that the relative dimension of $(K, \partial_0 N)$ is at most $n-3$. Call these data a *relative Poincaré embedding problem.*

Definition 4. A *solution* of a relative Poincaré embedding problem as above consists of

- a Poincare pair $(W, \partial W)$, and a Poincaré decomposition

  $$\partial W = \partial_0 N \cup_{\partial_{01} N} \partial_1 W$$

  such that each of the maps $\partial_{01} N \to \partial_1 W$ and $\partial_0 N \to W$ is 2-connected;

- a Poincare pair $(C, \partial C)$ with Poincaré decomposition

  $$\partial C = \partial_1 W \cup_{\partial_{01} N} \partial_1 N\,;$$

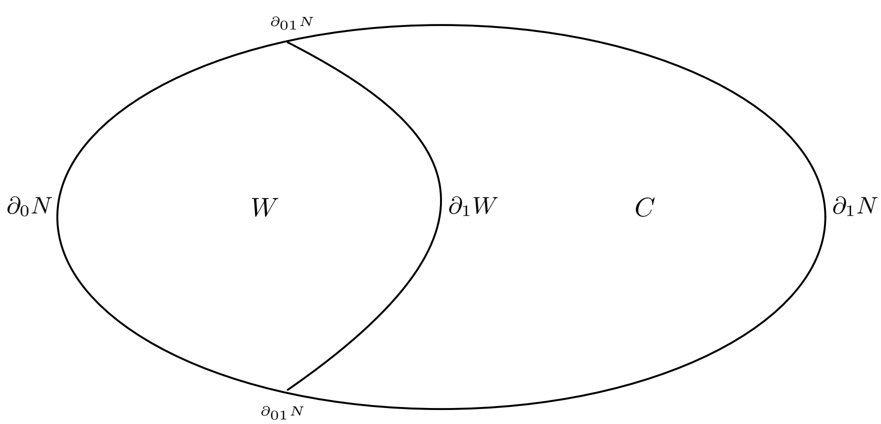
- a weak equivalence $h\colon K \to W$ which is fixed on $\partial_0 N$;

- a weak equivalence

  $$e\colon W \cup_{\partial_1 W} C \to N$$

  which is fixed on ∂N, such that $e \circ h$ is homotopic to f by a homotopy fixing $\partial_0 N$.

The above is depicted in the following schematic homotopy decomposition of N:



The space C is called the *complement*, which is a Poincaré space with boundary $\partial_1 W \cup \partial_1 N$. The above spaces assemble to give a strictly commutative square which is homotopy cocartesian:

$$\begin{array}{ccc} (\partial_1 W, \partial_{01} N) & \longrightarrow & (C, \partial_1 N) \\ \downarrow & & \downarrow \\ (W, \partial_0 N) & \longrightarrow & (N, \partial N) \end{array} \tag{11}$$

(compare [17]). From here through the rest of the paper we refer to such a commutative square as a "homotopy pushout".

As above, we can construct maps $f_j\colon K_j \to N \times D^j$, which define a family of relative Poincare embedding problems. Our goal in this section is to prove the analogue of Theorem 5 that shows that for sufficiently large j one can find solutions to these problems.

We begin with the following result, comparing the smooth to the Poincare relative embedding problems.

Lemma 6. *Assume that is M is a compact smooth manifold equipped with a boundary decomposition. Let*

$$\phi\colon (N; \partial_0 N, \partial_1 N) \to (M, \partial_0 M, \partial_1 M)$$

be a homotopy equivalence whose restriction $\partial_0 N \to \partial_0 M$ is a diffeomorphism.

Then the relative Poincaré embedding problem for f admits a solution if the relative smooth embedding problem for $\phi \circ f$ admits a solution.

Proof. A solution of the smooth problem together with a choice of homotopy inverse for h extending the inverse diffeomorphism on $\partial_0 M$ gives solution to the relative Poincaré embedding problem. □

Now suppose that $D(\nu) \to \partial_0 N$ is the unit disk bundle of the normal bundle of an embedding of ∂N into codimension ℓ Euclidean space, $\mathbb{R}^{n+\ell}$. The zero section then gives an inclusion $\partial_0 N \subset D(\nu)$. Set

$$K_\nu := K \cup_{\partial_0 N} D(\nu).$$

Clearly, K_ν is canonically homotopy equivalent to K.

Assuming ℓ is sufficiently large, there exists Spivak normal fibration [24]

$$S(\xi) \to N$$

whose fibers have the homotopy type of an $\ell - 1$ dimensional sphere. Then, by the uniqueness of the Spivak fibration [24], we have a fiber homotopy equivalence over ∂N

$$S(\nu) \xrightarrow{\simeq} S(\xi_{|\partial_0 N}).$$

Let $D(\xi)$ denote the fiberwise cone fibration of $S(\xi) \to N$. Then we have a canonical map

$$f_\nu\colon K_\nu \to D(\xi)$$

which is fixed on the $D(\nu)$. Note that

$$\partial D(\xi) = D(\nu) \cup S(\xi)$$

is a decomposition of Poincaré spaces such that $D(\nu)$ has the structure of a smooth manifold. Let us set $\partial_0 D(\xi) := D(\nu)$ and $\partial_1 D(\xi) := S(\xi)$.

Then the classical construction of the Spivak fibration (using regular neigbhorhood theory in Euclidean space) shows that there is a homotopy equivalence

$$(D(\xi);\partial_0 D(\xi),\partial_1 D(\xi)) \xrightarrow{\sim} (M;\partial_0 M,\partial_1 M)$$

in which M is a compact codimension zero submanifold of some Euclidean space. Furthermore, the restriction $\partial_0 D(\xi) \to \partial_0 M$ is a diffeomorphism.

Consequently, by Lemma 4 and lemma 6 we obtain

Proposition 7. *If the rank of ν is sufficiently large, then the relative Poincaré embedding problem for f_ν has a solution.*

Let η denote a choice of inverse for ξ in the Grothendieck group of reduced spherical fibrations over N. For simplicity, we may assume that the fiber of η is a sphere of dimension $\dim N - 1$. Then ξ restricted to $\partial_0 N$ is fiber homotopy equivalent to $\tau_{\partial_0 N} \oplus \epsilon$, where $\tau_{\partial_0 N}$ is the tangent sphere bundle of $\partial_0 N$ and ϵ is the trivial bundle with fiber S^0. For simplicity, we will assume that ξ restricted to $\partial_0 N$ has been identified with $\tau_{\partial_0 N} \oplus \epsilon$. Similarly, we will choose an identification of $\xi_{\partial_1 N}$ with $\tau_{\partial_1 N} \oplus \epsilon$, where $\tau_{\partial_1 N}$ is any spherical fibration over $\partial_1 N$ that represents the Spivak tangent fibration.

Since $\xi \oplus \eta$ is trivializable, for some integer j we get a homotopy equivalence

$$(D(\xi\oplus\eta); D(\nu\oplus\tau_{\partial_0 N}); D(\nu\oplus\tau_{\partial_1 N}) \cup S(\xi\oplus\eta)) \to (N\times D^j; (\partial_0 N)\times D^j; (\partial_1 N)\times D^j \cup N \times S^{j-1})$$

which restricts to a diffeomorphism $D(\nu\oplus\tau_{\partial_0 N}) \to (\partial_0 N)\times D^j$.

Now a choice of solution of the relative Poincaré embedding problem for f_ν, as given by Proposition 7, guarantees that the relative problem for $f_{\nu\oplus\tau}$ has a solution. But clearly, the latter is identified with the map $f_j\colon K_j \to N\times D^j$. Consequently, we have proven the following.

Theorem 8. *If $j \gg 0$ is sufficiently large, then the relative Poincaré embedding problem for $f_j\colon K_j \to N\times D^j$ has a solution.*

2.3 Application to diagonal maps and a proof of Theorem 2

We now give a proof of Theorem 2. By the results of section 1, this will complete the proof of Theorem 1.

Let $f\colon M_1 \to M_2$ be a homotopy equivalence of closed smooth manifolds. Using an identification of the tangent bundle τ_{M_1} with the normal bundle of the diagonal, $\Delta\colon M_1 \to M_1\times M_1$, we have an embedding

$$D(\tau_{M_1}) \subset M_1 \times M_1\,,$$

which is identified with a compact tubular neighborhood of the diagonal. The closure of its complement will be denoted $F(M_1,2)$. Notice that the inclusion $F(M_1,2) \subset M_1^{\times 2} - \Delta$ is a weak equivalence of spaces over $M_1^{\times 2}$ (i.e., it is a morphism of spaces over $M_1^{\times 2}$ whose underlying map of spaces is a weak homotopy equivalence). Notice also that we have a decomposition

$$M_1^{\times 2} = D(\tau_{M_1}) \cup_{S(\tau_{M_1})} F(M_1, 2).$$

Making the same construction with M_2, we also have a decomposition

$$M_2^{\times 2} = D(\tau_{M_2}) \cup_{S(\tau_{M_2})} F(M_2, 2)\,.$$

Notice that since $f\colon M_1 \to M_2$ is a homotopy equivalence, the composite

$$D(\tau_{M_1}) \xrightarrow{\text{projection}} M_1 \xrightarrow{\ f\ } M_2 \xrightarrow[\hookrightarrow]{\text{zero section}} D(\tau_{M_2})$$

is also a homotopy equivalence. Let T be the *mapping cylinder* of this composite map. Then we have a pair

$$(T, D(\tau_{M_1}) \amalg D(\tau_{M_2}))\,.$$

Furthermore, up to homotopy, we have a preferred identification of T with the mapping cylinder of f.

The map $f^{\times 2}\colon M_1^{\times 2} \to M_2^{\times 2}$ also has a mapping cylinder $T^{(2)}$ which contains the manifold

$$\partial T^{(2)} := M_1^{\times 2} \amalg M_2^{\times 2}.$$

Then $(T^{(2)}, \partial T^{(2)})$ is a Poincaré pair. Furthermore,

$$\partial T^{(2)} = (D(\tau_{M_1}) \amalg D(\tau_{M_2})) \cup (F(M_1, 2) \amalg F(M_2, 2))$$

is a manifold decomposition. Let us set $\partial_0 T^{(2)} = D(\tau_{M_1}) \amalg D(\tau_{M_2})$ and $\partial_1 T^{(2)} = (F(M_1, 2) \amalg F(M_2, 2))$.

Since the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\ f\ } & M_2 \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ M_1 \times M_1 & \xrightarrow[f \times f]{} & M_2 \times M_2 \end{array}$$

commutes, we get an induced map of mapping cylinders. This map, together with our preferred identification of the cylinder of f with T, allows the construction of a map

$$g\colon T \to T^{(2)}$$

which extends the identity map of $\partial_0 T^{(2)}$. In other words, g is a relative Poincaré embedding problem.

By Proposition 7, there exists an integer $j \gg 0$ such that the associated relative Poincaré embedding problem

$$g_j\colon T_j \to T^{(2)} \times D^j$$

has a solution. Here,

$$T_j \ := \ T \cup (\partial_0 T^{(2)}) \times D^j \ ,$$

and

$$\partial_0(T^{(2)} \times D^j) := \big(D(\tau_{M_1}) \times D^j\big) \amalg \big(D(\tau_{M_2}) \times D^j\big) \qquad \partial_1(T^{(2)} \times D^j) := F_{D^j}(M_1,\, 2) \amalg F_{D^j}(M_2,\, 2).$$

where, for convenience, we are redefining $F_{D^j}(M,\, 2)$ as $M \times M \times S^{j-1} \cup F(M,2) \times D^j$ (cf. S1).

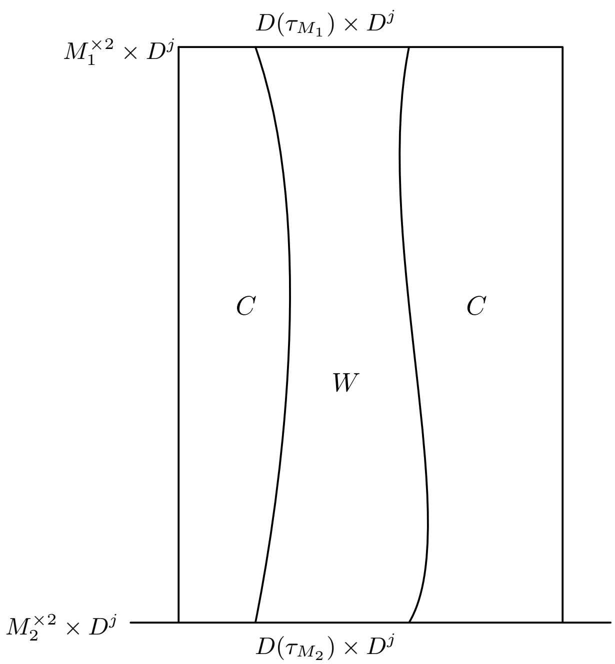
This makes $T^{(2)} \times D^j$ a Poincaré space with boundary decomposition

$$\partial(T^{(2)} \times D^j) \ = \partial_0(T^{(2)} \times D^j) \cup \partial_1(T^{(2)} \times D^j)\,.$$

By definition 8, a solution to this Poincaré embedding problem yields Poincaré pairs $(W, \partial W)$ and $(C, \partial C)$, with the following properties.

- $\partial W \ = \ \partial_0(T^{(2)} \times D^j) \cup \partial_1 W$, where $\partial_0(T^{(2)} \times D^j) \ = \ \big(D(\tau_{M_1}) \times D^j\big) \amalg \big(D(\tau_{M_2}) \times D^j\big)$ and $\partial_1 W \hookrightarrow W$ is 2-connected,
- $\partial C = \partial_1 W \cup \partial_1(T^{(2)} \times D^j)$, where $\partial_1(T^{(2)} \times D^j) = F_{D^j}(M_1,\, 2) \amalg F_{D^j}(M_2,\, 2)$. Notice that $\partial_{01}(T^{(2)} \times D^j) = \partial(D(\tau_{M_1} \times D^j)) \amalg \partial(D(\tau_{M_2} \times D^j))$.
- There is a weak equivalence, $h\colon T_j \xrightarrow{\simeq} W$, fixed on $\big(D(\tau_{M_1}) \times D^j\big) \amalg \big(D(\tau_{M_2}) \times D^j\big)$.
- There is a weak equivalence
$$e\colon W \cup_{\partial_1 W} C \to T^{(2)} \times D^j$$
which is fixed on $\partial(T^{(2)} \times D^j)$, such that $e \circ h$ is homotopic to $g_j\colon T_j \to T^{(2)} \times D^j$ by a homotopy fixing $\big(D(\tau_{M_1}) \times D^j\big) \amalg \big(D(\tau_{M_2}) \times D^j\big)$.

The above homotopy decomposition of $T^{(2)} \times D^j$ is indicated in the following schematic diagram:



Furthermore, the complement C and the normal data $\partial_1 W$ of the solution sits in a commutative diagram of pairs

$$\begin{array}{ccccc}
(M_1^{\times 2} \times S^{j-1}, \emptyset) & \xrightarrow[\sim]{\subset} & (T^{(2)} \times S^{j-1}, \emptyset) & \xleftarrow[\sim]{\supset} & (M_2^{\times 2} \times S^{j-1}, \emptyset) \\
\downarrow \cap & & \downarrow \cap & & \downarrow \cap \\
(F_{D^j}(M_1, 2), S(\tau_{M_1} + \epsilon^j)) & \xrightarrow{\subset} & (C, \partial_1 W) & \xleftarrow{\supset} & (F_{D^j}(M_2, 2), S(\tau_{M_2} + \epsilon^j)) \\
\downarrow \cap & & \downarrow e & & \downarrow \cap \\
(M_1^{\times 2} \times D^j, D(\tau_{M_1}) \times D^j) & \xrightarrow[\sim]{\subset} & (T^{(2)} \times D^j, T_j) & \xleftarrow[\sim]{\supset} & (M_2^{\times 2} \times D^j, D(\tau_{M_2}) \times D^j)\,.
\end{array}$$

Here each (horizontal) arrow marked with $\sim$ is a weak homotopy equivalence. Each column describes an $\mathcal{F}$-space (cf. Definition 1). In fact, the outer columns are the $\mathcal{F}$-spaces $M_i(j)$ described in §1. Furthermore, the horizontal maps describe morphisms of $\mathcal{F}$-spaces.

Consequently, to complete the proof of Theorem 2, it suffices to show these morphisms of $\mathcal{F}$-spaces are weak equivalences. We are therefore reduced to showing that the horizontal arrows in the second row are weak homotopy equivalences.

By symmetry, it will suffice to prove that the left map in the second row,

$$(F_{D^j}(M_1, 2), S(\tau_{M_1} + \epsilon^j)) \to (C, \partial_1 W)$$

is a weak equivalence.

We will prove that the map $F_{D^j}(M_1, 2) \to C$ is a weak equivalence; the proof that $S(\tau_{M_1} + \epsilon^j) \to \partial_1 W$ is a weak equivalence is similar and will be left to the reader. To do this, consider the following commutative diagram.

$$\begin{array}{ccccc} F_{D^j}(M_1,2) & \xrightarrow{=} & F_{D^j}(M_1,2) & \xrightarrow{\hookrightarrow} & C \\ \cap\downarrow & & \downarrow\cap & & \downarrow e \\ M_1^{\times 2}\times D^j & \underset{\hookrightarrow}{\longrightarrow} & (M_1^{\times 2}\times D^j)\cup_{(D(\tau_{M_1})\times D^j)} W & \underset{\hookrightarrow}{\longrightarrow} & T^{(2)}\times D^j\,. \end{array} \tag{12}$$

Lemma 9. *Each of the commutative squares in diagram* (12) *is a homotopy pushout.*

Before we prove this lemma, we show how we will use it to complete the proof of Theorem 2.

Proof of Theorem 2. By the lemma, since each of the squares of this diagram is a homotopy pushout, then so is the outer diagram,

$$\begin{array}{ccc} F_{D^j}(M_1,2) & \xrightarrow{\hookrightarrow} & C \\ \cap\downarrow & & \downarrow e \\ M_1^{\times 2}\times D^j & \underset{\hookrightarrow}{\longrightarrow} & T^{(2)}\times D^j. \end{array}$$

Now recall that $T^{(2)}$ is the mapping cylinder of the homotopy equivalence, $f^{\times 2}\colon M_1^{\times 2} \to M_2^{\times 2}$. Therefore the inclusion, $M_1^{\times 2} \to T^{(2)}$ is an equivalence, and hence so is the bottom horizontal map in this pushout diagram, $M_1^{\times 2}\times D^j \hookrightarrow T^{(2)}\times D^j$. Furthermore, the inclusion $F_{D^j}(M_1,2) \to M_1^{\times 2}\times D^j$ is 2-connected, assuming the dimension of M is 2 or larger. Therefore by the pushout property of this square and the Blakers-Massey theorem, we conclude that the top horizontal map in this diagram, $F_{D^j}(M_1,2) \hookrightarrow C$ is a homotopy equivalence.

As described before, this is what was needed to complete the proof of Theorem 2. □

Proof of Lemma 9. We first consider the right hand commutative square. By the properties of the solution to the relative embedding problem given above in (2.3), we know that $e\colon C \to T^{(2)}\times D^j$ extends to an equivalence, $e\colon C\cup_{\partial_1 W} W \xrightarrow{\simeq} T^{(2)}\times D^j$. Now notice that the intersection of $\partial_1 W$ with $F_{D_j}(M_1,2)$ is the boundary, $\partial(D(\tau_{M_1})\times D^j)$. But

$$(F_{D_j}(M_1,2))\cup_{\partial(D(\tau_{M_1})\times D^j)} W = (M_1^{\times 2}\times D^j)\cup_{(D(\tau_{M_1})\times D^j)} W.$$

This proves that the right hand square is a homotopy pushout.

We now consider the left hand diagram. Again, by using the properties of the solution of the relative embedding problem given above in (2.3), we know that the homotopy equivalence $h\colon T_j \xrightarrow{\simeq} W$ extends to a homotopy equivalence,

$$h\colon (M_1^{\times 2}\times D^j)\cup_{(D(\tau_{M_1})\times D^j)} T_j \xrightarrow{\simeq} (M_1^{\times 2}\times D^j)\cup_{(D(\tau_{M_1})\times D^j)} W.$$

But by construction, T_j is homotopy equivalent to the mapping cylinder of the composite homotopy equivalence, $D(\tau_{M_1}) \xrightarrow{\text{project}} M_1 \xrightarrow{f} M_2 \xrightarrow{\text{zero section}} D(\tau_{M_2})$. This implies that the inclusion

$D(\tau_{M_1}) \times D^j \hookrightarrow T_j$ is a homotopy equivalence, and so $(M_1^{\times 2} \times D^j) \cup_{(D(\tau_{M_1}) \times D^j)} T_j$ is homotopy equivalent to $M_1^{\times 2} \times D^j$. Thus the inclusion given by the bottom horizontal map in the square in question, $M_1^{\times 2} \times D^j \hookrightarrow (M_1^{\times 2} \times D^j) \cup_{(D(\tau_{M_1}) \times D^j)} W$ is also a homotopy equivalence. Since the top horizontal map is the identity, this square is also a homotopy pushout. This completes the proof of Lemma 9, which was the last step in the proof of Theorem 2. □

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