Axiomatic Characterization of Ordinary Differential Cohomology

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Introduction

The ring of differential characters, $\hat{H}(M, R/Z)$, a graded functor on the category of smooth manifolds together with smooth maps, was developed in [6], [7] and [8]. The motivation was to provide a home in the base for the bundle invariants constructed in [5].

$\hat{H}$ satisfies the Character Diagram shown in §1 and thus has natural transformations onto both closed differential forms with integral periods and integral cohomology. It was shown in [6] that an enriched version of the Weil homomorphism, carrying both an invariant polynomial and an associated integral homology class of the classifying space, naturally factors through $\hat{H}$ on its way to these targets. Based on this and other considerations, $\hat{H}$ has been shown to have useful applications to a number of areas, including conformal geometry and flat bundles. Most recently, the characters, perhaps generalized to relate to K-theory, have appeared in descriptions of quantizations of field theory related to super gravity and string theory, cf. [14].

In roughly the same time frame that differential characters were introduced, Deligne cohomology was developed as a tool in algebraic geometry, cf. [11]. When subsequently applied to the smooth category it was shown to yield a functor essentially equivalent to $\hat{H}$, cf. [11]. Thereafter other similar functors were developed, both in the smooth category and others, cf. [9], [10], [12], [13]. Those in the smooth category all satisfy the Character Diagram, and, while constructed in distinct manners, all have been shown to be naturally equivalent to $\hat{H}$.

Our purpose here is to show that, in the smooth category, the Diagram itself is sufficient to uniquely characterize all such functors. Theorem 1.1 shows that $\hat{H}$ is unique up to a unique natural equivalence, and Theorem 1.2 shows that this equivalence must preserve any graded product structure compatible with the natural pairings on the adjacent functors of the Diagram. The arguments are based on naturality and on largely classical facts about perturbing arbitrary singular cycles and homologies into embedded pseudomanifolds, both closed and with boundary, cf. [1], [2], [3], [4].

In [14] an analog of differential characters, called differential cohomology, is constructed for any generalized cohomology theory (meaning the Eilenberg Steenrod axioms are satisfied except that the groups of a point may not be concentrated in degree zero). Each flavor of differential cohomology satisfies an analog of the Character Diagram, which, in the ordinary case, agrees with ours. It would be interesting to know whether each of the extraordinary flavors is uniquely characterized by its diagram perhaps augmented by further axioms, i.e. whether or not there is a version of Theorem 1.1 for generalized differential cohomology.

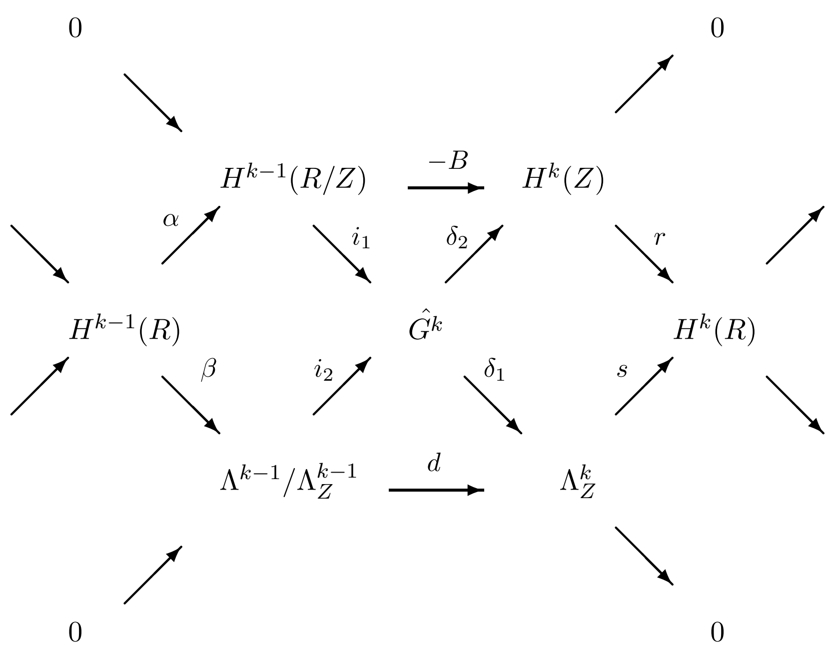
We are very happy to thank Jeff Cheeger for useful conversations in the early stages of the work, Blaine Lawson for acquainting us with the current status of the field, and Chris Bishop for his help with the proof of Fact 2.1.

§1. Uniqueness Theorems

The proofs presented in this section depend on results outlined in §2.

Let $\mathcal{C}$ denote the category of smooth manifolds together with smooth maps. Let Λ denote the functor from $\mathcal{C}$ to differential graded algebras given by smooth real valued differential forms with the usual d operator and wedge product. Let $\Lambda_Z \subseteq \Lambda$ denote the functor which assigns to any manifold the graded algebra of closed differential forms with integral periods, and Λ/Λ_Z be the quotient functor, taking values in graded abelian groups. Finally, let $H(A)$ denote the smooth singular cohomology functor with coefficients in A, an abelian group or commutative ring.

<u>**Definition**</u>: A **Character Functor** is a 5-tuple $\{\hat{G}, i_1, i_2, \delta_1, \delta_2\}$, where $\hat{G}$ is a functor from $\mathcal{C}$ to graded abelian groups, and $i_1, i_2, \delta_1, \delta_2$ are natural transformations which make the following **Character Diagram** commutative and its diagonal sequences exact.



where, (α, B, r) is the Bockstein long exact sequence associated to the coefficient sequence $Z \to R \to R/Z$, and (β, d, s) is another long exact sequence in which β and s are obviously defined via the de Rham Theorem.

Remark: It will be shown in the proof of Theorem 1.1 and in Corollary 1.1 that δ_2 is actually uniquely determined by the other three natural transformations, and thus, strictly speaking, is not required in the above definition. Nonetheless, its inclusion completes the picture.

Differential Characters: As mentioned in the Introduction, differential characters $\hat{H}(M, R/Z)$ and their related mappings comprise a character functor. We recall their construction:

For $M \in \mathcal{C}$ let $C_k(M)$ and $Z_k(M)$ respectively denote the groups of smooth singular k-chains and k-cycles. For $\omega \in \Lambda^k(M)$ and $a \in C_k(M)$ we define $\tilde{\omega} \in C^k(M, R/Z)$ by

1.1) $\quad \tilde{\omega}(a) = \int_a \omega \mod Z.$

Following [8]* we define

1.2) $\quad \hat{H}^k(M, R/Z) = \{f \in \mathrm{Hom}(Z_{k-1}(M), R/Z) \quad | \quad f \circ \partial = \tilde{\omega}_f\}$

for some $\omega_f \in \Lambda^k(M)$. It is easily seen that ω_f is uniquely determined by f and that $\omega_f \in \Lambda^k_Z(M)$. In the notation of the Character Diagram

1.3) $\quad \delta_1(f) = \omega_f.$

Since $H^{k-1}(M, R/Z) \cong \mathrm{Hom}(H_k(M), R/Z)$ we may consider $H^{k-1}(M, R/Z) \subseteq \hat{H}^k(M, R/Z)$. From 1.1) and Stokes Theorem we see that $\theta \to \tilde{\theta}$ maps $\Lambda^{k-1}(M) \to \hat{H}^k(M, R/Z)$, and since the kernel of this map is $\Lambda^{k-1}_Z(M)$ we may consider $\Lambda^{k-1}(M)/\Lambda^{k-1}_Z(M) \subseteq \hat{H}^k(M, R/Z)$.

1.4) $\quad$ The above inclusions define i_1 and i_2.

Since R is divisible we can find $T \in C^{k-1}(M, R)$ with $\tilde{T}|Z_{k-1}(M) = f$. It is easily shown that $\omega_f - \delta T \in C^k(M, Z)$ and is closed. Moreover its cohomology class, $\{\omega_f - \delta T\} \in H^k(M, Z)$, is independent of the choice of T. By definition, $\delta_2(f) = \{\omega_f - \delta T\}$.

If $\phi : M \to N$ is C^∞, $a \in Z_{k-1}(M)$ and $f \in \hat{H}^k(M, R/Z)$ then $\phi^* : \hat{H}^k(N, R/Z) \to \hat{H}^k(M, R/Z)$ is simply defined by

1.5) $\quad \phi^*(f)(a) = f(\phi_*(a)).$

*The indexing has been shifted by 1 from that in [8].

Theorem 1.1: Any character functor $\{\hat{G}, i_1, i_2, \delta_1, \delta_2\}$ is equivalent to $\{\hat{H}, i_1, i_2, \delta_1, \delta_2\}$ via a natural transformation $\Phi : \hat{G} \to \hat{H}$ which commutes with the identity map on all other functors in the Diagram; such Φ is unique.

Proof:

For any singular chain, x, we will denote by $|x|$ the union of the images of all the simplexes of x.

Let $a \in Z_{k-1}(M)$. Using Fact 2.1 we may choose a $(k-1)$-good open neighborhood U of $|a|$. Let $\lambda : U \to M$ denote the inclusion map. Since $H^k(U, Z) = 0$, the Character Diagram shows

1.6) $\qquad \lambda^*(g) = i_2(\{\theta\})$

where $\theta \in \Lambda^{k-1}(U)$ and is determined up to an element of $\Lambda_Z^{k-1}(U)$. We set

1.7) $\qquad \Phi(g)(a) = \tilde{\theta}(a) = \int_a \theta \mod Z.$

We proceed by showing

1.8) Φ is a well defined homomorphism from $\hat{G}^k(M)$ into $\mathrm{Hom}(Z_k(M), R/Z)$.

1.9) $\mathrm{Im}(\Phi) \subseteq \hat{H}^k(M, R/Z) \subseteq \mathrm{Hom}(Z_k(M), R/Z)$.

1.10) If $\varphi : M \to N$ is C^∞, then $\Phi \circ \varphi^* = \varphi^* \circ \Phi$, where φ^* denotes the maps induced by φ of the functors $\hat{G}$ and $\hat{H}$.

1.11) $\Phi \circ i_1 = i_1$, $\Phi \circ i_2 = i_2$, $\delta_1 \circ \Phi = \delta_1$.

1.12) Φ is an isomorphism.

1.13) $\delta_2 \circ \Phi = \delta_2$.

1.14) Φ is the unique natural transformation from $\hat{G}$ to $\hat{H}$ satisfying 1.11), 1,12) and 1.13).

To show 1.8), note that clearly 1.7) is independent of the choice of $\theta \in \{\theta\}$. To show it is independent of the choice of U, let U' be another $(k-1)$-good neighborhood of $|a|$. Again by Fact 2.1 we can choose a third $(k-1)$-good neighborhood of $|a|$, $U'' \subseteq U \cap U'$. Let $U'' \xrightarrow{\eta} U \xrightarrow{\lambda} M$ be the inclusion maps. By 1.5) and the naturality of $\hat{G}$

$$(\lambda \circ \eta)^*(g) = \eta^*(\lambda^*(g)) = \eta^*(i_2(\{\theta\})) = i_2(\eta^*(\{\theta\})) = i_2(\{\eta^*(\theta)\}).$$

Since $\widetilde{\eta^*(\theta)}(a) = \tilde{\theta}(a)$ we see that the definitions of Φ using U and U'' agree. Since the same is true for U' and U'', we have shown that the definition of Φ is independent of the choice of U.

To show $\Phi(g) \in \mathrm{Hom}(Z_{k-1}, R/Z)$ we must show that for $a, b \in Z_{k-1}$ that $\Phi(g)(a+b) = \Phi(g)(a) + \Phi(g)(b)$. Using all of the distinct individual simplices of a and b, create a chain c such that $|a| \cup |b| \subseteq |c|$. Pick U, a $(k-1)$-good neighborhood of $|c|$. Since $|a+b| \subseteq |a| \cup |b|$ we have each of $|a|$, $|b|$, $|a+b| \subseteq U$. Choosing θ as in 1.6)

$$\Phi(g)(a+b) = \tilde{\theta}(a+b) = \tilde{\theta}(a) + \tilde{\theta}(b) = \Phi(g)(a) + \Phi(g)(b).$$

Since it is clear that $\Phi(g_1 + g_2) = \Phi(g_1) + \Phi(g_2)$ we have now shown 1.8).

To prove 1.9) we will show that if $a \equiv \partial e$ then

$$*) \qquad \Phi(g)(a) = \widetilde{\delta_1(g)}(e).$$

Since both sides vanish if $k-1 = \dim M$, we may assume $k-1 < \dim M$. Let U be a $(k-1)$-good neighborhood of $|a|$. Using Fact 2.2 we can find a $(k-1)$ dim imbedded pseudomanifold $P \subseteq U$ and a k-chain b with $|b| \subseteq U$ such that

$$a = \partial b + P$$

where we identify P with its fundamental cycle. Letting θ be defined as in 1.6) we see

$$\begin{aligned} 1.15) \quad \Phi(g)(a) &= \tilde{\theta}(\partial b) + \Phi(g)(P) = \widetilde{d\theta}(b) + \Phi(g)(P) = \delta_1(\widetilde{\lambda^*(g)})(b) + \Phi(g)(P) \\ &= \widetilde{\delta_1(g)}(b) + \Phi(g)(P) \end{aligned}$$

where we used Stokes Theorem, the fact that $d = \delta_1 \circ i_2$, and naturality of $\hat{G}$.

Since a is a boundary in M and P is homologous to a, P is a boundary in M. We then use Fact 2.3 to find U', a $(k-1)$-good neighborhood of P with $\lambda : U' \to N$ the inclusion, and a k-chain y with $|y| \subseteq U'$ and $P = \partial y$.

As in 1.6), $\lambda^*(g) = i_2\{\theta'\}$ for some $\theta' \in \Lambda^{k-1}(U')$, and by the same argument as above

$$\Phi(g)(P) = \tilde{\theta}'(\partial y) = \widetilde{\delta_1(g)}(y).$$

Combining this with 1.15) we see

$$\Phi(g)(a) = \widetilde{\delta_1(g)}(b+y) = \widetilde{\delta_1(g)}(b+y-e+e) = \widetilde{\delta_1(g)}(b+y-e) + \widetilde{\delta_1(g)}(e).$$

But, since $\partial(b+y) = a = \partial e$, $b+y-e$ is a cycle, and $\widetilde{\delta_1(g)} \in \Lambda^k_Z(M)$, it follows the first term vanishes. This proves $*$) and thus 1.9).

To prove 1.10), let $g \in \hat{G}^k(N)$, $a \in Z_{k-1}(M)$, $U \subseteq N$, a $(k-1)$-good neighborhood of $|\phi_*(a)|$ and $W \subseteq \phi^{-1}(U)$, a $(k-1)$-good neighborhood of $|a|$. The various inclusions are labelled in the commutative diagram

$$\begin{array}{ccccc} W & \overset{\gamma}{\subset} & \phi^{-1}(U) & \overset{\eta}{\subset} & M \\ & & \downarrow \phi & & \downarrow \phi \\ & & U & \overset{\lambda}{\subset} & N \end{array}$$

Now, $\lambda^*(g) = i_2(\{\theta\})$ for $\theta \in \Lambda^{k-1}(U)$. By naturality of g

$$(\eta \circ \gamma)^*(\phi^*(g)) = \gamma^* \circ \eta^* \circ \phi^*(g) = \gamma^* \circ \phi^* \circ \lambda^*(g) = (\phi \circ \gamma)^*(i_2(\{\theta\})) = i_2(\{(\phi \circ \gamma)^*(\theta)\}).$$

Thus, by the definition of Φ and 1.5)

$$(\Phi(\phi^*(g)))(a) = \widetilde{(\phi \circ \gamma)^*(\theta)}(a) = \tilde{\theta}((\phi \circ \gamma)_*(a)) = \tilde{\theta}(\phi_*(a)) = (\Phi(g))(\phi_*(a)) = \phi^*(\Phi(g))(a).$$

To prove 1.11) let $\mu \in H^{k-1}(M, R/Z)$, $a \in Z_{k-1}$ and U a $(k-1)$-good neighborhood of $|a|$. Since $H^k(U) = 0$, by the exactness of the Bockstein sequence, $\mu = \alpha(x)$, where $x \in H^{k-1}(R)$. By commutativity of the Character Diagram for $\hat{G}$

$$i_i(\mu) = i_i(\alpha(x)) = i_2(\beta(x)).$$

Thus, by 1.6)

$$\Phi(i_1(\mu))(a) = \tilde{\theta}(a) \qquad \text{for any } \theta \in \{\beta(x)\}.$$

But, since β is defined by the de Rham Theorem, and using i_1 of $\hat{H}$ as defined in 1.4)

$$\tilde{\theta}(a) = x(a) \bmod Z = \alpha(x)(a) = \mu(a) = i_1(\mu)(a)$$

Thus $\Phi \circ i_1 = i_1$. That $\Phi \circ i_2 = i_2$ follows immediately from 1.5) and 1.6), and that $\delta_1 \circ \Phi = \delta_1$ follows from $*$) in the proof of 1.8). This shows 1.11).

To prove 1.12) we note that the following diagram of exact sequences is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{k-1}(M,R/Z) & \xrightarrow{i_1} & \hat{G}^k(M) & \xrightarrow{\delta_1} & \Lambda^k_Z(M) & \longrightarrow & 0 \\ & & \| & & \downarrow \Phi & & \| & & \\ 0 & \longrightarrow & H^{k-1}(M,R/Z) & \xrightarrow{i_1} & H^k(M,R/Z) & \xrightarrow{\delta_1} & \Lambda^k_Z(M) & \longrightarrow & 0 \end{array}$$

Thus by the Five Lemma Φ is an isomorphism.

To prove 1.13) we need

Lemma 1.1: For any character functor, $\hat{G}$, δ_2 is uniquely determined by i_1, i_2 and δ_1.

Proof: Let $\delta_2' : \hat{G}^k(Z) \to H^k(Z)$ be another such δ_2, which, together with i_1, i_2 and δ_1 make $\hat{G}$ a character functor. Since each of δ_2 and δ_2' induce isomorphisms from $\hat{G}^k/i_2(\Lambda^{k-1}/\Lambda^{k-1}_Z)$ onto $H^k(Z)$,

$$\delta_2' \circ \delta_2^{-1} : H^k(Z) \to H^k(Z)$$

is an automorphism.

Let $T^k(Z) \subseteq H^k(Z)$ denote the torsion subgroup. For $\tau \in T^k(Z)$, the Bockstein exact sequence shows $\tau = B(\mu)$ for some $\mu \in H^{k-1}(R/Z)$. Thus, by the commutativity of the Character Diagram

$$\delta_2' \circ \delta_2^{-1}(\tau) = \delta_2' \circ \delta_2^{-1}(B(\mu)) = \delta_2' \circ \delta_2^{-1}(\delta_2 \circ i_1(\mu)) = \delta_2' \circ i(\mu) = B(\mu) = \tau.$$

Therefore $(\delta_2' \circ \delta_2^{-1})|T^k(Z)$ is the identity.

By naturality of $\hat{G}$, equipped with either δ_2 or δ_2', we see that $\delta_2' \circ \delta_2^{-1}$ is a natural automorphism of $H^k(Z)$ which holds fixed $T^k(Z)$. By Fact 1.1 below this can only hold if $\delta_2' \circ \delta_2^{-1}$ is the identity. Thus $\delta_2' = \delta_2$. $\diamondsuit$

Fact 1.1: Any natural (homotopy invariant) automorphism of the integral cohomology functor on $\mathcal{C}$ which is either identity on torsion, or the identity after tensoring with R must be the identity.

Proof:

One knows any integral k-cohomology class of a finite dimensional manifold M is induced by pulling back a universal class $u(k)$ on a fixed universal space $K(Z,k)$, by a map f of M into a finite skeleton of $K(Z,k)$. The space $K(Z,k)$ is characterized by having one non zero homotopy group Z in dimension k. This finite skeleton of $K(Z,k)$ has the same homotopy type as a manifold V, as can be seen by embedding the skeleton into a euclidean space and forming a regular neighborhood. The map into this smooth neighborhood can be deformed to a smooth map which we also denote by f. Now we calculate the automorphism. The k-th cohomology of this manifold neighborhood V is Z in dimension k, by the Hurewicz and universal coefficient theorems applied to $K(Z,k)$. It follows any natural automorphism must multiply the universal class $u(k)$ and thus the general class $f^*(u(k))$ by plus or minus one. The minus one possibility is ruled out by either one of our further hypotheses by considering a manifold with a nonzero odd order class or an infinite order class in degree k. $\diamondsuit$

To prove 1.13) we set $\delta_2' = \delta_2 \circ \Phi$ and note that using 1.10), 1.11) and 1.12) one easily shows $\{\hat{G}, i_1, i_2, \delta_1, \delta_2'\}$ is a character functor. By Lemma 1.1 we see $\delta_2 \circ \Phi = \delta_2$.

To complete the proof of the theorem we must show 1.14). If Φ' were another such then $\Phi' \circ \Phi^{-1}$: $\hat{H}^k(M, R/Z) \to \hat{H}^{k-1}(M, R/Z)$ would be a natural automorphism holding fixed the other terms in the Character Diagram. Thus, for $f \in \hat{H}^{k-1}(M, R/Z)$, $a \in Z_{k-1}(M)$, and U a $(k-1)$-good open neighborhood of $|a|$ with $\lambda : U \to M$,

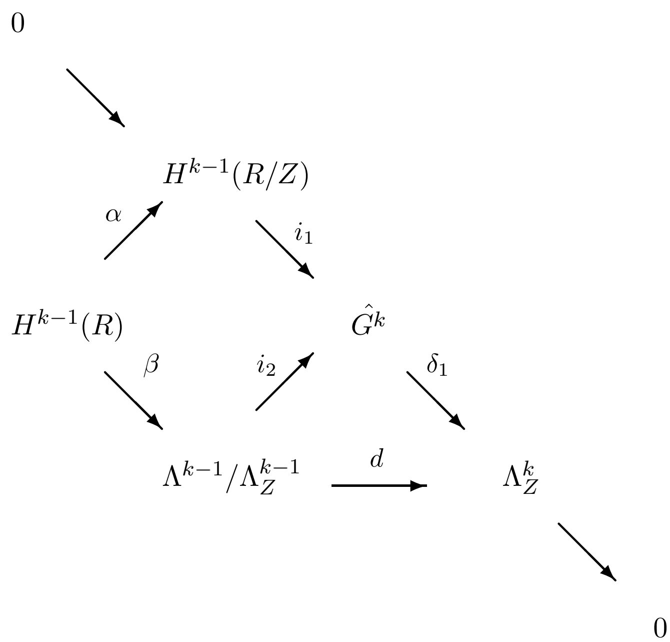
$$\begin{aligned}
(\Phi' \circ \Phi^{-1})(f)(a) &= (\Phi' \circ \Phi^{-1})(\lambda^*(f))(a) \\
&= (\Phi' \circ \Phi^{-1})(i_2(\{\theta\})(a)) \\
&= i_2(\Phi' \circ \Phi^{-1}(\{\theta\})(a) \\
&= i_2(\{\theta\})(a) = (\lambda^*(f))(a) = f(a).
\end{aligned}$$

Thus $\Phi = \Phi'$. $\diamondsuit$

As was indicated in the Remark following the definition of the character functor, the result below shows that the definition actually requires less data.

Corollary 1.1

If $\hat{G}$ is a functor from $\mathcal{C}$ to graded abelian groups and i_1, i_2 and δ_1 are natural transformations making the diagram below commute and its diagonal exact



then i_2 is $1:1$, and there exists one and only one $\delta_2 : \hat{G} \to H(Z)$ so that $\{\hat{G}, i_1, i_2, \delta_1, \delta_2\}$ is a character functor.

<u>Proof:</u>

$$i_2(\{\theta\}) = 0 \Rightarrow \delta_1 \circ i_2(\{\theta\}) = 0 \Rightarrow d\theta = 0 \Rightarrow \{\theta\} = \beta(x) \Rightarrow i_1 \circ \alpha(x) = 0$$
$\Rightarrow \alpha(x) = 0 \Rightarrow x = r(u)$ for $u \in H^{k-1}(Z)$, and thus $0 = \beta(x) = \{\theta\}$. Thus i_2 is $1:1$.

We next show that if $U \subseteq M$ is k-good and $\lambda : U \to M$ is the inclusion then, for $g \in \hat{G}^k(M)$

$*$) $\qquad \lambda^*(g) = i_2(\{\theta\}) \qquad$ for unique $\{\theta\}$.

To show $*$), note that since U is k-good $\delta_1(\lambda^*(g)) = d(\{\rho\})$ and thus $\delta_1(i_2(\{\rho\})) = \delta_1(\lambda^*(g)) \Rightarrow \lambda^*(g) = i_2\{\rho\} + i_1(u)$, where $u \in H^k(U, R/Z)$. Since $H^k(U, Z) = 0$ the Bockstein shows $u = \alpha(x)$. Thus $i_1(u) = i_1(\alpha(x)) = i_2(\beta(x))$. Therefore $\lambda^*(g) = i_2(\{\rho\} + \alpha(x)) = i_2(\{\theta\})$ for some $\{\theta\}$. Since i_2 is $1:1$, $\{\theta\}$ is unique.

We may then define $\Phi : \hat{G} \to \hat{H}$ as in 1.7), and follow the proof of Theorem 1.1 through 1.12). By then setting $\delta_2 = \delta_2 \circ \Phi$, $\hat{G}$ becomes a character functor, and the uniqueness of δ_2 is assured by Lemma 1.1. $\diamondsuit$

Ring Structure

In [7] $\hat{H}$ was shown to possess a natural associative graded commutative ring structure, $*$, cf. [9]-[13]. For $f \in \hat{H}^k$ and $g \in \hat{H}^l$ this ring structure also satisfies, cf. [8],

1.16) $f * g \in \hat{H}^{k+l}$

1.17) $f * g = (-1)^{kl}\, g * f$

1.18) $\delta_1(f * g) = \delta_1(f) \wedge \delta_1(g)$

1.19) $\delta_2(f * g) = \delta_2(f) \cup \delta_2(g)$

1.20) $f * i_1(u) = (-1)^k\, i_1\, (\delta_2(f) \cup u)$

1.21) $f * i_2(\{\theta\}) = (-1)^k\, i_2\, (\{\delta_1(f) \wedge \theta\})$

Theorem 1.2: A character functor, $\hat{G}$, possesses at most one natural ring structure satisfying 1.16) - 1.21).

Proof: Suppose $*$ and $\dagger$ are two such. Set $B(f,g) = f * g - f \dagger g$. From 1.16), 1.18) and the Character Diagram we see that

$$B(f,g) \in \mathrm{Ker}(\delta_1) = i_1(H^{k+l-1}(R/Z)).$$

From 1.17) and 1.21) we see that if $f \in i_2(\Lambda^{k-1}/\Lambda_Z^{k-1})$ or $g \in i_2(\Lambda^{l-1}/\Lambda_Z^{l-1})$

$$B(f,g) = 0.$$

Since the diagram shows $\hat{G}^k/i_2(\Lambda^{k-1}/\Lambda_Z^{k-1}) \cong H^k(Z)$ we see

$$B : H^k(Z) \times H^l(Z) \to H^{k+l-1}(R/Z).$$

By the naturality assumptions on $*$ and $\dagger$, we see that B is a natural transformation satisfying the hypotheses of Fact 1.2 below. Thus $B \equiv 0$ and $* = \dagger$. $\diamondsuit$

Fact 1.2: Any natural (homotopy invariant) cohomology operation on $\mathcal{C}$ which assigns to a pair of integral cohomology classes (a,b) in dimensions (i,j) an R/Z cohomology class $(a \cdot b)$ in dimension $i+j-1$, so that $(0 \cdot b) = (a \cdot 0) = 0$ must be identically zero.

Proof:

We will tacitly replace finite skeleta of the spaces below that are sufficient for the calculation by manifold thickenings as we did in the proof of Fact 1.1. Apply the hypothetical operation to the universal classes mentioned in the proof of Fact 1.1. $u(k)$ and $u(j)$ pulled back by the respective projections to the cartesian product $K(Z, k) \times K(Z, j)$. We obtain a class p in the product $K(Z, k) \times K(Z, j)$ in degree $k+j-1$ with coefficients in R/Z. To compute the hypothetical operation on a pair of classes we use the pair of classes to build a map into $K(Z, k) \times K(Z, j)$ and then pull back p. By our hypothesis that $a \cdot 0 = 0$ we get that p restricted to the second factor of $K(Z, k) \times K(Z, j)$ is zero. Similarly we deduce p restricted to the first factor is zero. It follows from the exact sequence of a pair that p comes from the smash product obtained by collapsing the two axes in $K(Z, k) \times K(Z, j)$ to a point. The smash product of a $j-1$ connected space and a $k-1$ connected space is $k+j-1$ connected. So the $j+k-1$ cohomology with any coefficients of the smash product is zero. Therefore p is zero and Fact 1.2 is proved. $\diamondsuit$

By Theorem 1.1 we may use Φ to pull back $*$ to obtain a ring structure on $\hat{G}$. We may thus combine Theorems 1.1 and 1.2 as

Theorem 1.3: Any character functor admits exactly one ring structure satisfying 1.16) - 1.21). Equipped with this ring structure, any two character functors are naturally equivalent by a unique equivalence consistent with the Character Diagram.

Characters over R/Γ

Character Functors, together with a unique ring structure, may be defined over R/Γ, where Γ is any proper subring of R. All the above definitions and results are true in this more general setting, and the proofs are identical.

§2 - Geometric Homology Background

Fact 2.1: Let K in M denote the compact image of a smooth singular k-chain in M. Then every neighborhood of K contains a smaller neighborhood whose integral cohomology vanishes above k. (We call these **k-good** neighborhoods.)

Proof of 2.1:

i) Choose a Riemannian metric on M. Since maps satisfying the Lipschitz condition,

$$\text{distance } (f(x), f(y)) \leq L \cdot \text{distance}(x, y),$$

do not increase Hausdorf dimension [2], we see the Hausdorf dimension of K is at most k. In particular the Hausdorf $(k+1)$ measure of K is zero.

ii) Since the Hausdorf $k+1$ measure of K is zero any local orthogonal projection of a compact piece of K into a $k+1$ plane is a closed set of measure zero and so misses an open dense set.

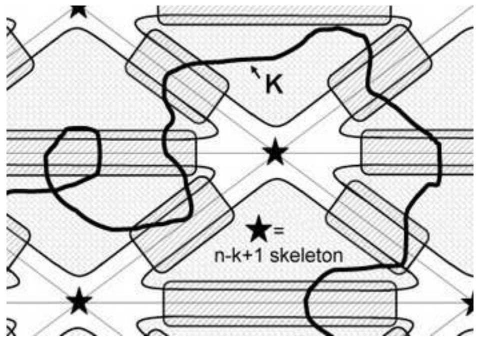


Figure 1:

Thus we can choose a small triangulation so that K misses the $n-k-1$ skeleton. We do this by first choosing one small triangulation and then performing local translations in $k+1$ directions orthogonal to the top cells of the $n-k-1$ skeleton into the open dense sets that miss the local projections of K. Then K is contained in the handlebody neighborhood of the dual k skeleton of the perturbed triangulation which is the complement of the handlebody neighborhood of the $n-k+1$ skeleton which avoids K. (This proof was directly inspired by a conversation with Chris Bishop at Stony Brook University.)

iii) Now dual cells to a triangulation thicken to handles with two properties

   a) the handles have small diameters if the mesh of the triangulation is small.
   b) the handles associated to the dual k-skeleton are convex and only intersect k+1 at a time. See Figure 1.

iv) Taking the dual handles that actually touch K yields small neighborhoods of K by iii a) whose cohomology vanishes in dimensions above k by iii b). These are the k-good neighborhoods.

Fact 2.2: If dim M is at least k, a smooth singular $(k-1)$ cycle is homologous in every neighborhood of its image to the fundamental cycle of a piecewise smoothly embedded oriented $(k-1)$ pseudomanifold. (See proof for a recall of the definition of pseudomanifold.)

Remark: Here we are considering (as did Poincaré [1] in his original paper) smooth curvilinear nondegenerate simplices in M defined by smooth equations and smooth inequalities. We consider M as decomposed into such simplices, further arbitrarily small diameter subdivisions, after Munkres [3], and perturbations of these as in Figures 2 and 3.

If $k-1$ were the dimension of M, and M were a closed oriented manifold, the theorem would only be true for the generators of homology. Multiples of generators would not admit embedded representations.

Proof:

i) Let W be a smooth manifold with boundary, chosen to be a neighborhood of the image of our $(k-1)$ singular cycle inside the given neighborhood. We can choose a piecewise smooth triangulation of M in which W is a subcomplex, cf. [6].

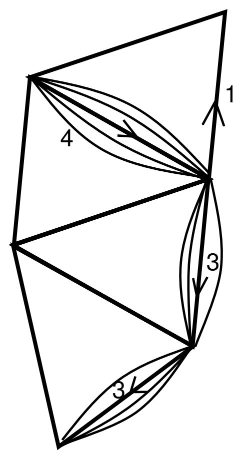
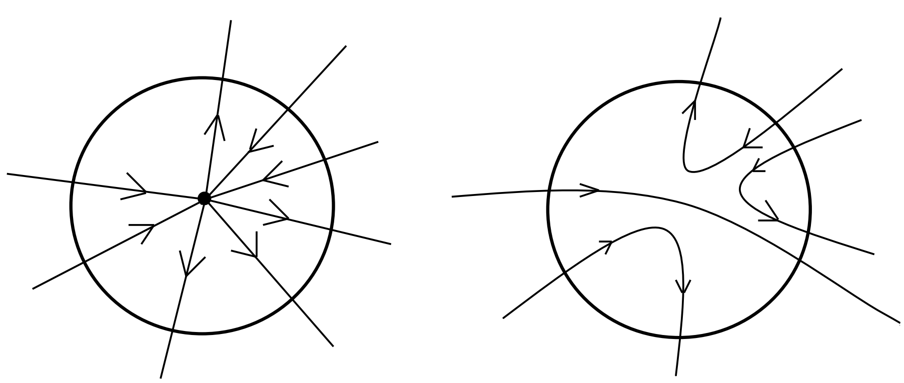


Figure 2: Splitting



3a

3b

Figure 3: Resolving

ii) The smooth singular homology of W is naturally isomorphic (by the inverse of the inclusion) to the cellular homology of W for this triangulation. Thus our cycle is homologous in W (by a smooth singular homology) to a cellular $k-1$ cycle in W.

iii) Now we orient the $(k-1)$ cells of the support of this cycle so that the cycle is written with positive coefficients.

iv) If a coefficient of a cell is an integer n greater than one we replace that cell by n deformed copies, as in Figure 2. This uses the hypothesis $(k-1) <$ dimension M.

v) Now transverse to the $(k-2)$ cells of the deformed cycle we have a picture like Figure 3a. We now perform the deformation indicated by Figure 3b.

vi) The deformations of iv) and v) provide a further homology of our cycle to one which is carried by the cell sum of a $(k-1)$-dimension oriented curvilinear polyhedron so that each $(k-2)$ cell is the face of exactly two $(k-1)$ cells with cancelling orientation. Note: This explains the phrase "the fundamental cycle of an oriented embedded pseudomanifold".

Fact 2.3: Suppose dim $M \geq k$. If the fundamental cycle ξ of an embedded oriented $(k-1)$ pseudomanifold is homologous to zero in M, then ξ is also bounds in some $(k-1)$-good neighborhood of its support.

Proof:

i) Extend a cellular decomposition of the cycle to a piecewise smooth cell decomposition of M. Then $\xi = \partial w''$ for some k-chain in this subdivision.

ii) Orient the k cells of w'' to get positive integer coefficients. If the dimension $M = k$ and ξ bounds w'' in M, then $|\xi|$ separates M and w'' picks out a compact part of the complement which is oriented and has oriented boundary equal to ξ. If the dimension $M \geq k+1$, apply the deformation iv) of the previous proof, indicated in Figure 2, to get a new cellular homology w' with all coefficients equal to 1. We still have $\partial w' = \xi$.

iii) Continuing in the case dimension $M \geq k+1$, apply the deformation of v) of the previous proof to $(k-1)$ cells of $|w'|$ in the "interior" i.e. those not in $|\xi'|$ on the boundary of $|w'|$. At a boundary cell we perform a variant of this deformation at each $(k-1)$ cell indicated by Figure 4, where the $(k-1)$ cell of ξ is depicted as a point.

Conclusion: In either case dimension $M = k$ or dimension $M \geq k+1$ we obtain an embedded oriented pseudomanifold with boundary ξ.

iv) Now the argument that a connected oriented triangulated k-dimensional manifold with boundary deformation retracts to a $(k-1)$ dimensional subcomplex goes by pushing in k-cells starting from the boundary until none are left. There are always exposed $(k-1)$ faces to push in on until we get to a subcomplex of the $(k-1)$ skeleton because if not there would be a non-trivial k-cycle. The latter is impossible because we started with a connected oriented manifold with boundary. (The same argument works for unoriented manifolds using mod 2 homology considerations.)

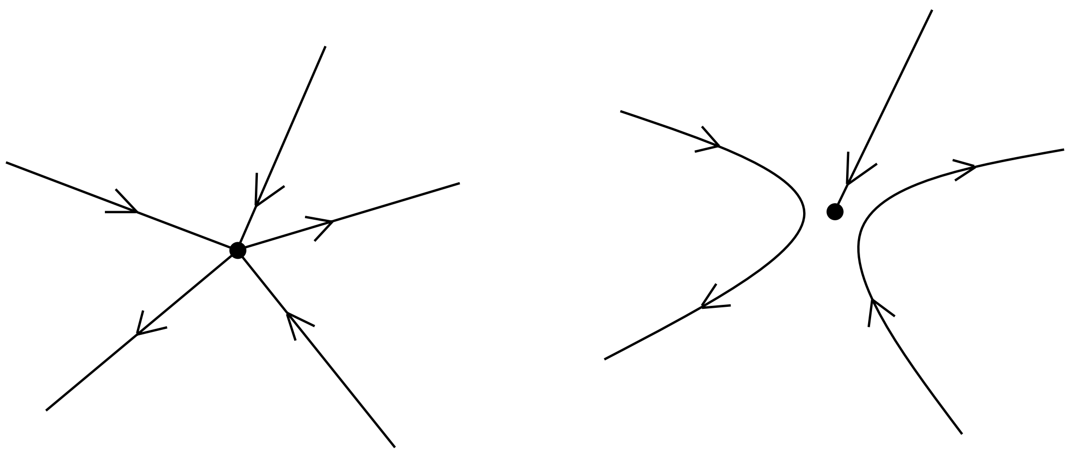


Figure 4:

v) The above arguments work word for word the same for irreducible k-pseudomanifolds with boundary (i.e. the dual 1-skeleton is connected) to show that such an object deformation retracts to a $(k-1)$ dimensional subcomplex for some curvilinear triangulation.

vi) Now a regular neighborhood of the k-pseudomanifold with, boundary has the same homotopy type and we have proved Fact 2.3.

References

1. Poincaré, Henri. "Complement a l'Analysis Situs". Oeuvres Completes Tome VI. Gauthier Villars Paris. 1954. p. 291.

2. Hurewicz, Witold and Wallman, Henry. "Dimension Theory". Princeton University Press. 1941.

3. Munkres, J.R. "Elementary Differential Topology". Annals of Mathematics Studies. Number 54. 1969. p. 79.

4. Sullivan, Dennis. "Singularities in Spaces". Proceedings of the Liverpool Conference on Singularities Symposium II, Liverpool, England. Springer Lecture Notes (209) pp. 196-206.

5. Chern, S.S. and Simons, James. "Characteristic Forms and Geometric Invariants". Annals of Mathematics. Vol. 99. No. 1. January 1974. pp. 48-69.

6. Simons, James. "Characteristic Forms and Transgression: Characters Associated to a Connection". Stony Brook University Preprint. 1972.

7. Cheeger, Jeff. "Multiplication of Differential Characters". Instituto Nazionale di Alta Mathematica, Symposia Mathematica XI. 1973. pp. 441-445.

8. Cheeger, Jeff and Simons, James. "Differential Characters and Geometric Invariants". Notes of Stanford Conference 1973, Lecture Notes in Math. No. 1167. Springer-Verlag, New York. 1985. pp. 50-90.

9. Gillet, H. and Soule, C.. "Arithmetic Chow Groups and Differential Characters, in Algebraic K-Theory: Connections with Geometry and Topology". Jardine and Snaith, editors. Kluwer Academic Publishers. 1989. pp. 30-68.

10. Harris, B. "Differential Characters and the Abel-Jacobi map, in Algebraic K-Theory: Connections with Geometry and Topology". Jardine and Snaith, editors. Kluwer Academic Publishers. 1989. pp. 69-86.

11. Brylinski, Jean-Luc. Loop Spaces, Characteristic Classes, and Geometric Quantization. Birkhauser, Boston. 1993.

12. Harvey, Reese; Lawson, Blaine; and Zweck, John. "The de Rham-Federer theory of differential characters and character duality". American Journal of Mathematics. Vol 125. 2003. pp. 791-847.

13. Harvey, Reese and Lawson, Blaine. "From Sparks to Grundles-Differential Characters". Communications in Analysis and Geometry. Vol. 14, No. 1. 2006. pp. 25-58.

14. Hopkins, Michael and Singer, Isadore. "Quadratic functions in geometry, topology,and M-theory". http://arxiv.org/pdf/math/0211216.