A free differential Lie algebra for the interval [1]

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The history behind the title and the formula is explained below after the following definition, statement of theorem and proof.

Recall that a DGLA is an algebra A over a field k with grading $A = \oplus_{n\in\mathbb{Z}} A_n$ along with a bilinear map $[.,.]\colon A\times A\longrightarrow A$ (bracket) and a linear map $\partial\colon A\longrightarrow A$ (differential) for which $\partial^2=0$ while

- (1) (symmetry of bracket) $[b,a]=-(-1)^{|a||b|}[a,b]$;
- (2) (Jacobi identity) $[[a,b],c]=[a,[b,c]]-(-1)^{|a||b|}[b,[a,c]]$;
- (3) (Leibnitz rule) $\partial[a,b]=[\partial a,b]+(-1)^{|a|}[a,\partial b]$.

Note that the three properties above are valid only for homogeneous elements a, b and c of A (that is an element of $\bigcup_n A_n$), and $|a|\in\mathbb{Z}$ denotes the grading. The bracket and differential are required to respect the grading, in that for homogeneous elements, $|[a,b]|=|a|+|b|$ while $|\partial a|=|a|-1$. The adjoint action of A on itself is given by $\mathrm{ad}_e(a)=[e,a]$ and acts on the grading by $\mathrm{ad}_e\colon A_n\longrightarrow A_{n+|e|}$. In this notation (2) and (3) can be rewritten as

- (2′) (Jacobi identity) $\mathrm{ad}_{[a,b]}=[\mathrm{ad}_a,\mathrm{ad}_b]$
- (3′) (Leibnitz rule) $[\partial,\mathrm{ad}_a]=ad_{\partial a}$

in which the brackets on the right-hand side refer to the (signed) commutator of operators defined by $[x,y]=xy-(-1)^{|x||y|}yx$ where the product is composition of operators and the grading $|x|$ of a (homogeneous) operator is the shift in grading which x induces. Thus the gradings of ad_e and ∂ are $|e|$ and -1, respectively.

An element x of A_{-1} is said to be *flat* iff $\partial x+\frac{1}{2}[x,x]=0$. Any $v\in A_0$ defines a flow on $u\in A_{-1}$ by

$$\frac{du}{dt}=\partial v-\mathrm{ad}_v(u)\,.$$

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The flow actually moves elements in the completion of A_{-1} defined by formal series in powers of ad_e, and we work in the corresponding completion of the free lie algebra for the statement of Theorem 1. Note that such a flow is also defined for any v in the completion of A_0.

Theorem 1 *There is a unique completed free differential graded Lie algebra, A, with generating elements a, b and e, in degrees -1, -1 and 0 respectively, for which a and b are flat while the flow generated by e moves from a to b in unit time. The differential is specified by,*

$$\partial e = (\mathrm{ad}_e)b + \sum_{i=0}^{\infty} \frac{B_i}{i!} (\mathrm{ad}_e)^i (b-a) \, ,$$

where B_i denotes the i^{th} Bernoulli number.

Proof.

Let A be the free graded Lie algebra generated by a, b and e. For any $x \in A_{-1}$, define a map $\partial_x \colon A \longrightarrow A$ by its action on the generators

$$\partial_x a = -\frac{1}{2}[a,a], \qquad \partial_x b = -\frac{1}{2}[b,b], \qquad \partial_x e = x \, ,$$

extended to the whole of A via linearity and Leibnitz' formula. (Here ∂_x is well-defined since derivation by Leibnitz' formula preserves relations (1) and (2).)

The flow generated by e has $\frac{du}{dt} = x - \mathrm{ad}_e(u)$ (note that such a flow is well-defined for any constant term, without needing to know that $\partial^2 = 0$). For the particular solution with $u(0) = a$, differentiating repeatedly gives $\frac{d^n u}{dt^n}|_{t=0} = (-\mathrm{ad}_e)^{n-1}(x) + (-\mathrm{ad}_e)^n(a)$, so that the solution is given by

$$u(t) = e^{-t\mathrm{ad}_e} a + \frac{e^{-t\mathrm{ad}_e} - 1}{(-\mathrm{ad}_e)} x \, ,$$

where the operator acting on x is defined by its series expansion $\sum\limits_{n=1}^{\infty} \frac{t^n}{n!} (-\mathrm{ad}_e)^{n+1}$. The condition that $u(1) = b$ is precisely that

$$x = \frac{(-\mathrm{ad}_e)}{e^{-\mathrm{ad}_e} - 1} \left(b - e^{-\mathrm{ad}_e} a \right) = (\mathrm{ad}_e) b + \frac{ad_e}{e^{\mathrm{ad}_e} - 1} (b-a) \, ,$$

namely the value of ∂e given in the proposition. This proves uniqueness.

It remains only to verify existence, that is show that $\partial_x^2 = 0$ for this particular value of x. From the Leibnitz property for ∂_x, it follows that for all $p, q \in A$, $\partial_x^2[p,q] = [\partial_x^2 p, q] + [p, \partial_x^2 q]$, so that it is only necessary to check that $\partial_x^2 = 0$ on the generators. For the generator a, we have

$$\partial_x^2(a) = \partial_x(-\frac{1}{2}[a,a]) = [a, \partial_x a] = [a, -\frac{1}{2}[a,a]] = 0\,,$$

the final equality following from the Jacobi identity. Similarly $\partial_x^2(b) = 0$.

To prove that $\partial_x^2(e) = 0$, consider the flow u generated by e for which $u(0) = a$ as above. By our choice of x, this flow also has $u(1) = b$. Consider the function $f(t) = \partial_x u + \frac{1}{2}[u,u]$, taking values in A_{-2} (the curvature). Its derivative is

$$\begin{aligned}
\frac{df}{dt} &= \partial_x \frac{du}{dt} + \big[u, \frac{du}{dt}\big] \\
&= \partial_x(x - \mathrm{ad}_e u) + [u, x - \mathrm{ad}_e u] \\
&= \partial_x^2(e) - \big(\mathrm{ad}_{\partial_x e}(u) + \mathrm{ad}_e \partial_x u\big) + ([u,x] - [u, \mathrm{ad}_e u]) \\
&= \partial_x^2(e) - \mathrm{ad}_e(f(t))
\end{aligned}$$

where we have used Leibnitz' rule and that $[u, \mathrm{ad}_e u] = \frac{1}{2}\mathrm{ad}_e[u,u]$ from the Jacobi identity. Thus f satisfies a first-order linear differential equation with constant (operator) coefficients of the same form as that satisfied by u where now x is replaced by $\partial_x^2 e$, while $f(0) = f(1) = 0$ (since a and b are flat). It follows that $\partial_x^2 e = 0$, as required. □

From the last calculation in the proof, it can be seen that in a DGLA (where $\partial^2 = 0$), the flow defined by an arbitrary element of A_0 on A_{-1} preserves flatness of elements.

Background and history

The possibility of such a differential on the free Lie algebra on three generators corresponding to the unit interval arose for one of us about ten years ago in conversations with Maxim Kontsevich at the IHES. At first it was considered to be a remarkable combinatorial miracle involving Bernoulli numbers. Later it was regarded by several workers to be an obvious consequence of the homotopy commutativity and associativity of the diagonal map for the chains on the unit interval.

The situation can be understood partially in terms of algebra, combinatorial topology and Quillen's rational homotopy theory.

In algebra there is the notion of a differential graded cocommutative coalgebra which is infinitely homotopy coassociative. Besides the differential and the comultiplication, there are multilinear co-operations with three outputs, four outputs, etc. These satisfy symmetry relations and quadratic relations which are succinctly expressed by extending each operation to a derivation of the the free Lie algebra generated by the graded space shifted down by one, and writing that the formal sum of all these derivations is a differential. This is the infinity structure of the title.

Quillen employed such free differential Lie algebras, graded in positive degrees, as algebraic models of rational homotopy types with one zero-cell and no one-cells [1]. The infinity structure interpretation of Quillen's models began to be suggested in the 90's. This interpretation uses combinatorial topology and chain approximations of the diagonal map of a cell complex X. Over $\mathbb{Q}$, the induced chain maps can be symmetrized but any associativity is destroyed. (Actually Quillen's work was motivated by the problem of constructing some commutative and associative cochain functor.) However because the diagonal approximations stay near the diagonal it is intuitively obvious that homotopies restoring coassociativity and subsequent homotopies of homotopies etc. can also be constructed near the diagonals. (For one formal argument see the appendix to Tradler and Zeinalian [2] by one of the authors.) The combinatorial argument of that appendix shows the existence of a free differential Lie algebra associated to any cell complex so that closures of cells are contractible. There is one generator for each cell, with the vertices in degree -1, edges in degree 0, 2-cells in degree 1 etc.

The formula of this paper can be interpreted as an explicit construction of the infinity cocommutative coassociative coalgebra structure on the the rational chains of the unit interval, namely a derivation of square zero on the free Lie algebra generated by its cells a, b in degree -1 and e in degree 0.

The authors worked out in Israel the proof that the formula for ∂e that takes the flat element a to the flat element b by the flow associated to the infinitesimal gauge group action leads to a derivation of square zero.

Problems

There are three open problems.

- Show that the two formulae agree for the unit interval, namely the one given here and the specialization of that given in [2].

- Explain the topological meaning of the free differential Lie algebra for a general cell complex. Note that a regular cell complex has multiple vertices, so there are multiple free Lie generators in degrees $-1, 0, 1, 2, 3, \ldots$ one for each cell.

- Perform the construction and topological interpretation for regular cell complexes with a maximal tree in the one-skeleton collapsed to a point. Then a reduced version of the free Lie algebra with differential would be concentrated in non-negative degrees with the degree zero component mod the image of the differential playing the rôle of the Lie algebra of the fundamental group.

References

[1] D.G. Quillen, *Rational homotopy theory*, Ann. of Math.(2) **90** (1969) 205–295.

[2] T. Tradler, M. Zeinalian, *Infinity structure of Poincaré duality spaces*, preprint, `math.AT/0309455` (revised 2006) .