$\sum$-models and String Topology

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Abstract

Gromov's homology class of J-holomorphic curves is shown to satisfy at the chain level the quantum master equation $\triangle e^S = 0$ in a background described by algebraic topology and transversality. This provides a mathematical interpretation of Witten's discovery of the corresponding Gromov homology periods as the correlations of a quantum field theory. The basic idea is due to Zwiebach.

1 Short Introduction

Associated to any oriented smooth target manifold M of even dimension d there is a modular operad structure (C_{ij}, S_{ij}) on one version and a related BV algebra structure [1] $((\delta, \{,\})$ on another version of an equivariant singular chain complex associated to the smooth mapping spaces $\{\Sigma \to M\}$ of the Σ-model with target M. These structures are in the unbounded sense Section 3. Here Σ runs over diffeomorphism types of closed oriented connected surfaces of genus g with n marked points. After an even shift of gradings by $(d-6)(1-g)+2n+c(\beta), \beta \epsilon H_2 M$, c linear and even, the degrees of the chain complex ∂ operator and transversal glueing operations C_{ij}, S_{ij}, δ, $\triangle$, and $\{,\}$ are all equal to -1. A solution S in degree zero of the quantum master equation or $\triangle S + 1/2\{S,S\} = 0$ in the BV algebra, defines multilinear pairings on the cohomology of a dual cochain complex. This provides a mathematical definition of physical correlations associated to values of the Feynman path integral with insertions $< \int e^S \varphi_1 \varphi_2 ... \varphi_n >$. In this case the multilinear pairings, as opposed to just linear pairings, are defined because the BV operator $\triangle$ is also a coderivation for a natural coalgebra structure.

When M is almost complex the dimension shift $(d-6)(1-g)+2n+2c_1(\beta)$ gives the formal real dimension of that piece of the moduli space corresponding to J-holomorphic curves of genus g and n marked points in the homology class of β in $H_2 M$.

[1] Used by Batalin Vilkovisky [17] in an algebraic quantization scheme for classical field theories with constraints. In the Σ model the defining equations of the target M could be considered to be constraints. A BV algebra is a graded commutative algebra with an operator $\triangle$ of degree -1 so that $\triangle \cdot \triangle = 0$ and the deviation of $\triangle$ from being a derivation, $\{,\}$, is itself a derivation in each variable. It follows $\{,\}$ is a Lie bracket of degree -1 and $\triangle$ is a derivation of $\{,\}$.

When M is closed symplectic the compactified by energy and Σ complex structure cut off moduli spaces of perturbed J-holomorphic curves regarded as chains with boundary yield a degree zero solution of the quantum master equation in the BV algebra using the dual pictures of Gromov [1] and Sen-Zwiebach [2]. The associated multilinear pairings determine Gromov-Witten invariants.

The transversality and parametrized surgery defining the unbounded structure maps are just like those in String Topology [4]. Underlying both cases, here and [3], is the structure of an unbounded modular operad with $\triangle$ and $\{,\}$ added in. See the longer introduction below for the definition of modular operad and more detailed descriptions.

The discussion here for general M may be viewed as a chain level or off shell background for the closed string A-model or Gromov Witten theory, defined when the target M is closed and symplectic, but which may be formulated more generally in terms of the quantum master equation of the BV algebra.

Whereas the above discussion uses only closed surfaces Σ with finite subsets I of interior marked points, there is also a construction for surfaces with boundary mapping to M where pieces of the boundary between marked boundary points could be required to land in various submanifolds of M. Using parametrized interior and ∂ connected sum one obtains an (unbounded) generalized modular operad structure which combines into one package the higher genus version of string topology [3] mixed together with backgrounds for various discussions of symplectic topology. The latter beyond what is in the present paper is being developed and will appear elsewhere [12],[15],[16].

2 Longer Introduction

A singular chain complex of the smooth mapping spaces $\Sigma \to M$ of closed oriented surfaces Σ into a target manifold M of dimension d even taken altogether as Σ varies and equivariantly with respect to diffeomorphism groups of the Σ's (Subsection 3.1) has an algebraic structure called unbounded modular operad. This means [7] there are non negative graded chain complexes $(C(I), \partial)$ with an additional direct sum decomposition over the genus $C(I) = \bigoplus_{g=0}^{\infty} C(g, I)$ functorially attached to finite sets $I, J, ...$ with unbounded chain maps

$$C(I) \otimes C(J) \xrightarrow{C_{ij}} C(I \cup J - \{i, j\})$$

with $i \epsilon I$, $j \epsilon J$ and unbounded chain maps $C(I) \xrightarrow{S_{ij}} C(I - \{i, j\})$ with $\{i, j\} \subset I$. C_{ij} is a parametrized connected sum at i and j of different surfaces along the locus where the map transversally identifies i and j. S_{ij} is a parametrized self connected sum at i and j along the locus where the map transversally identifies i and j. These structure maps act on bidegrees by $(g_1, k) \otimes (g_2, l) \to (g_1 + g_2, k + l - d + 1)$ and $(g, k) \to (g + 1, k - d + 1)$ where d =dim M, g = genus Σ, and k, l are the geometric dimensions of the equivariant chains. The structure maps satisfy the relations that any compositions are associative and anti commute, (Subsection 3.3). The structure maps C_{ij} and S_{ij} are unbounded in the sense they are only defined on dense core domains which are subcomplexes of $\otimes_\alpha C(I_\alpha)$. These subcomplexes satisfy inclusion relations and have isomorphic homology to the entire complexes, Subsection 3.2. These unbounded (or partial) operations are sufficient for our homological or derived category purposes because there are functorial quasiisomorphic completion constructions (in two senses, see [13] motivated by [8]). The core domains are defined by transversality conditions and the structure maps are defined by parametrized surgery both analogous to String Topology [4]. The geometric operations C_{ij} and S_{ij} for d odd will not be discussed further here.

There is a further direct sum decomposition of $C(g, I) = \oplus_\beta C(\beta, g, I)$ where β ranges over all elements in H_2M which may be represented by maps of a genus g surface into M.

Theorem 1 *When d is even, the chain complexes $C(I) = \oplus_g \oplus_\beta C(g, \beta, I)$ form a modular operad (in the unbounded sense above). After an even grading shift down by $\oplus_g \oplus_\beta [(d-6)(1-g) + 2n + c(\beta)]$ where c is linear in β and even, all the operations ∂, C_{ij}, and S_{ij} have degree -1.*

Now we add more structure to the modular operad. We define $\{,\}$ and δ (Subsection 3.4) by the compositions

$$C(\emptyset) \otimes C(\emptyset) \xrightarrow{M_1 \otimes M_1} C(\{i\}) \otimes C(\{j\}) \xrightarrow{C_{ij}} C(\emptyset)$$

and

$$C(\emptyset) \xrightarrow{M_2} C\{i, j\} \xrightarrow{S_{ij}} C(\emptyset)$$

by adding marked points via operators M_1 and M_2. An anomaly appears in the definition of δ which is resolved by reducing $C(\emptyset)$ somewhat. (See Remark 3.10).(Similarly for $I \neq \emptyset$).

Theorem 2 *Now let ∇ be the difference $\delta - \partial$. When d is even and c is even, $(C(\emptyset), \{,\}, \nabla)$ is defined and is an unbounded differential Lie algebra of degree -1. (See Subsection 3.4)*

Remark 2.1. $(C, \nabla, \{\})$ is a Lie algebra of degree -1 means $[a, b] = (-1)^{|a|}\{a, b\}$ is the bracket of a differential Lie algebra (of degree zero) on the graded space C shifted by one. □

We can consider the Maurer Cartan equation $\nabla S + 1/2\{S, S\} = 0$ where S has degree zero. A solution here implies a solution to the equation $\triangle(e^S) = 0$ in the graded symmetric algebra ΛC generated by C. Here $\triangle$ is the (2nd order) derivation on ΛC obtained by adding together both ∇ and $\{,\}$ extended to be coderivations of ΛC. The equation $\triangle \cdot \triangle = 0$ summarizes the differential Lie algebra properties of

$(C, \{\}, \nabla)$. The 2nd order nature and nilpotence of $\triangle$ comes by viewing $\triangle$ as the extension of ∇ plus $\{,\}$ to all disconnected surfaces by summing over the possible glueings (i, j). Then since $a \wedge b$ can be viewed in terms of disjoint union of surfaces and maps over the product of the bases for a and for b,

$$\triangle(a \wedge b) = (\triangle a) \wedge b + (-1)^{|a|} a \wedge \triangle b + \{a, b\},$$

where $\{a, b\}$ denotes the extension of $\{,\}$ on $C = C(\emptyset)$ to all of ΛC to a binary operation which is a graded derivation in each variable. The equation for S in terms of $\triangle$ is called the quantum master equation and can be written either as $\triangle S + 1/2\{S, S\} = 0$ or $\triangle e^S = 0$. We have, $C = C(\emptyset)$

Theorem 3 *When d is even and $c(\beta)$ is even, $(\Lambda C, \triangle)$ is defined and is an unbounded Batalin-Vilkovisky algebra attached to the spaces of maps of surfaces into M modulo diffeomorphisms of the source underlying the Σ model in physics.* [2] *$\triangle$ is a 2nd order derivation and a coderivation. Given a solution S to the quantum master equation the analogues of Feynman path integral correlations are the periods of e^S relative to products of cohomology classes φ_i of the dual cochain complex to $(\Lambda C, \triangle)$ denoted $< \int e^S \varphi_1 \varphi_2 ... \varphi_n >$.*

(See Subsection 3.4 for the proof).

Application As mentioned above, the even integer $(d-6)(1-g) + 2n + c(\beta)$ used to shift the chain complexes $C(g, I)$ above when $c(\beta) = 2c_1(\beta)$ yields the formal dimension of the moduli space of J-holomorphic maps of Σ into an almost complex manifold (M, J) representing the homology class β in $H_2(M)$, $n =$ cardinality of I.

If the target manifold M is a closed symplectic manifold with 2-form ω, and J is a compatible almost complex structure ($\omega(x, Jy)$ is symmetric and positive definite), Gromov studied the J-holomorphic mappings $\Sigma \xrightarrow{f} M$ of Riemann surfaces into M and analyzed the non-compactness of the space of these in the given homology class β in $H_2 M$. He found exactly the familiar non compactness in the complex structure of Σ corresponding to pinching curves, together with Freed-Uhlenbeck bubbling off of 2-spheres in the map into M. Bubbling off or pinching off an essential separating curve in the complex structure of Σ will be seen to be inverse to an operation like C_{ij}. Similarly, pinching off a nonseparating curve is inverse to an operation like S_{ij}. Thus if one introduces cut off inequalities to prevent these pinching and bubbling phenomenon from happening (Section 4) one obtains for each $(C(\beta, g)$ a compact family of J-holomorphic curves. Using Q coefficients and perturbing the equation the (relative) cycle is defined

[2] In the physics discussions the ordinary maps of surfaces Σ into the target M are augmented by (contractible) odd components. A (formal) symplectic manifold of odd degrees appears where functions define (formally) an odd version of Poisson algebra with bracket derived from a BV operator [16]. The functions are closely related to differential forms on the original mapping spaces. New variables are added to form the equivariant forms relative to diffeomorphisms of Σ, the BV algebra structure extends formally and a solution of the quantum master equation is sought where the leading term is a classical action in the original superfields. The structure is formal.

[9], [10] and has a boundary essentially described by a sum over the operations $\{,\}$ and δ. In this way (after shifting each $C(\beta, g)$ down by $(d-6)(1-g)+2c_1(\beta)$ we obtain a solution S to the quantum master equation of degree zero, where S assigns to each β and each genus of Σ the oriented Gromov moduli space compactified by cut off, plus small collars.

Theorem 4 *(Gromov, Eliashberg, Fukaya, Hofer, Kontsevich, Li, MacDuff, Ono, Ruan, Salomon, Sullivan, Tian,...,Zwiebach) Let $k = 0, 1, 2, \ldots$. For a closed symplectic manifold M the set of all cut off oriented moduli spaces of perturbed J-holomorphic curves with a fixed number k of nodes define Q-chains which provide a degree zero solution to the quantum master equation $\partial S = \delta S + 1/2\{S, S\}$ in an unbounded BV algebra attached to the Σ-model of mapping spaces with target M. Equivalently, $\triangle e^S = 0$.*

(See Section 4 for the proof and a more specific statement.)

Note The...in the theorem refers to anyone omitted who has contributed to the formidable task of making the picture of the Gromov chain or homology class into rigorous mathematics.

Writing the quantum master equation in Zwiebach's form [2], which directly inspired this paper,

$$\partial S = \delta S + \frac{1}{2}\{S, S\}, \text{ (Zwiebach's form)}$$

we see the geometry of Gromov's virtual cycle where the first term on the right hand side corresponds to approaching the ∂ (of the left hand side) by pinching a non separating curve while the 2nd term on the right hand side corresponds to approaching the ∂ (of the left hand side) by either pinching off an essential separating curve in the complex structure or by bubbling off a two sphere in the map.

The invariants of S, namely the periods of the total class of e^S in $(\Lambda C, \triangle)$ relative to the cohomology of the dual cochain complex determine Gromov-Witten invariants.

3 The modular operad and BV algebra structure

3.1 The equivariant chain complex

We first define the equivariant chain complexes $C(I)$ for the mapping spaces, $\Sigma \to M$. Let I be a finite set and for each oriented connected pseudo manifold σ (see Remark 3.1) consider bundles η over σ with fibre the pair (Σ, I) where Σ is a closed connected oriented surface of genus g and $I \subset \Sigma$ is an embedding of I into Σ. We also need piecewise smooth maps $f : \eta \to M$ of the total space of η into the target manifold M. Two such pairs $(\eta, f)$and (η', f') are equivalent iff there is an oriented bundle isomorphism b which is an oriented piecewise diffeomorphism between σ and σ', sends I to I on each fibre by the identity, and relates f to f', namely $f'b = f$. We take the free module $C(g, I)$ over the coefficient ring Q on these equivalence classes for σ

connected. We add the relation (σ, orientation) equals - (σ,opposite orientation). We also assign to the disjoint union of pseudomanifolds, bundles, maps, etc the sum.

Remark 3.1. We work in the piecewise differentiable category of spaces built by glueing together compact manifolds with corners (eg. curvilinear polyhedra) and piecewise smooth maps. A pseudomanifold of dimension k by definition is an object in this category which admits such a decomposition with each $(k-1)$ dimensional part lying in the face of one or two k-dimensional pieces. The ∂ is the $(k-1)$ dimensional part made of pieces with only one k dimensional face. □

Remark 3.2. A self equivalence of (σ, η, f) which is orientation reversing on σ forces that generater to be zero. □

The sum of the oriented geometric boundary components defines the usual boundary operator ∂. The direct sum of all these chain complexes for the equivalence relation above requiring the identity map on I over all genera $g = 0, 1, 2, ..$ is the chain complex $C(I)$ over the ground ring Q functorally attached to the finite set I.

3.2 Core domains of unbounded structures

Now we define core domains $D(I) \subset C(I)$ and $D(I_1, I_2, ..., I_n) \subset \bigotimes_{\alpha=1}^{n} C(I_\alpha)$ by transversality. In each case the constraint is imposed on the connected basis elements and then the core domain is closed under Q-linear combinations. For one generator (η, f) the constraint is that the map $\sigma \to M^I$ which evaluates f at the subset I of the fibre is in general position relative to all the sub diagonals of M^I. For a collection $(\eta_\alpha, f_\alpha, \sigma_\alpha)$ the constraint is that the product evaluation map is in general position relative to all the sub diagonals of the big product, and this is also true for the restriction to all the natural product strata in the cartesian product of the σ_α.

Proposition 3.3. *The core domains $D(I_1, I_2, ..., I_n)$ are subcomplexes and satisfy the two properties (compare [8]):*

i) for each partition $\beta_1, \beta_2, ..., \beta_k$ of $\{1, 2, ..., n\}$ such that $\beta_i = \{j_i, j_i+1, \ldots, j_i+m_i\}$ is a segment

$$D(I_1, I_2, ..., I_n) \subseteq D(\{I_\beta\}_{\beta\in\beta_1}) \otimes \cdots \otimes D(\{I_\beta\}_{\beta\in\beta_k}) \subset \bigotimes_{\alpha=1}^{n} C(I_\alpha).$$

ii) each inclusion in i) induces an isomorphism on homology.

Proof. They are subcomplexes because of the definition of general position. The inclusions of i) exist because more constraints are added by the definitions passing from right to left. To see that any such inclusion is onto in homology observe that any cycle in the range may be perturbed slightly to satisfy the constraints or the further constraints in the inclusions of ii). Injectivity follows because a homology in the range between cycles in general position is also in general position near its boundary and by a small deformation relative to its boundary may be put in general position everywhere. ■

Remark 3.4. The set of generators which are in general position is dense for a natural topology. The "denseness" of the core domain motivates the adjective "unbounded" in describing these structures. □

3.3 The operations of the modular operad

Now we define the operations of the unbounded modular operad. First

$$C(I) \otimes C(J) \xrightarrow{C_{ij}} C(I \cup J - \{i,j\}).$$

On the core domain subcomplex (Subsection 3.2) $D(I,J) \subset C(I) \otimes C(J)$ for each generator $\sigma_I \otimes \sigma_J$ in $D(I,J)$ the preimage of the diagonal by $(f(i), f(j))$(called the locus (i,j)) has a normal bundle by transversality. Thus the normal bundle and then the locus may be oriented when M is oriented to define a relative cycle whose ∂ lies in $\partial(\sigma_I \times \sigma_J)$ in a manner compatible with the natural pieces $\partial\sigma_I \times \sigma_J$ and $\sigma_I \times \partial\sigma_J$. Note this orientation of the normal bundle is independent of the order of i and j because d is even.

Along the locus (i,j) we have a well defined mapping of the one point union of the fibre over σ_I and the fibre over σ_J into M. Now for each surface we replace the marked point by the set of directions at that point and glue these together along the boundary by a rigid map. The ambiguity in such a map is a circle parametrizing the set of orientation reversing isometries between these two boundaries (after choosing metrics on the boundaries). We can map any of the connected sums to the one point union by collapsing the glued up circles to the one common point.

Combining all this we get a fibration of connected sum surfaces over a circle bundle (of all rigid glueings) over the (i,j) locus and a canonical map into M of the total space of the surface bundle. Up to our equivalence the result is a well defined map into M. We combine the natural orientation of the circle (Remark 3) with the orientation of the locus (i,j) to orient the circle bundle. This is the output of the operation between two elements C_{ij} which commutes with the ∂ operator.

The self connected sum construction of $C(I) \xrightarrow{S_{ij}} C(I - \{i,j\})$ on the domain $D(I) \subset C(I)$ is done essentially the same as in the case of C_{ij}.

Remark 3.5. (Orientation of the circle) The surfaces Σ are oriented. Thus each small circle around a marked point is oriented. In the glueing an orientation reversing isometry between boundary circles is used so that the glued up surfaces has a natural orientation. There is a circle of such orientation reversing isometries and this circle needs to be given an orientation.

The family of all the glued up surfaces is a standard Dehn twist family over the circle of glueings. We orient that base circle by declaring the monodromy around that direction to be a right Dehn twist. □

Proposition 3.6. *When d is even S_{ij} only depends on the unordered pair $\{i,j\}$. Also C_{ij} is graded symmetric. The glueing operations have odd degree, and compositions are associative and anticommute.*

Proof. This follows from the construction and Remark 3.5. ■

Now let us shift down the grading in the (g,β) component of $C(I)$, n= cardinality of I, by $(d-6)(1-g)+2n+c(\beta)$ where c is linear in β. In the new grading we have

Proposition 3.7. *When d is even and $c(\beta)$ is even, the dimension shift is even and ∂, C_{ij} and S_{ij} each has degree -1.*

Proof. We check the degrees. Before C_{ij} sent $(g_1,n_1,k_1)\otimes(g_2,n_2,k_2)\to(g_1+g_2,n_1+n_2-2,k_1+k_2-d+1)$ and S_{ij} sent $(g,n,k)\to(g+1,n-2,k-d+1)$. Using $c(\beta_1+\beta_2)=c(\beta_1)+c(\beta_2)$ the degree of C_{ij} in the new grading is thus

$$[(d-6)(1-g_1)+2n_1+(d-6)(1-g_2)+2n_2]+[-d+1]-[(d-6)(1-g_1-g_2)+2(n_1+n_2-2)]=-1$$

by direct calculation.

Similarly, the degree of S_{ij} in the new grading is $[(d-6)(1-g)+2n]+[-d+1]-[(d-6)(1-g-1)+2(n-2)]=-1$. ■

Remark 3.8. The number $(d-6)(1-g)+2n+c(\beta)$ used to shift the grading in the above proposition when $c(\beta)=2c_1(\beta)$ gives the formal (real) dimension of the moduli space of J-holomorphic curves of genus g with n marked points in the homology class β of any almost complex manifold (M,J) of real dimension d. In other words the formal complex dimension is $(d_{\mathbb{C}}-3)\ (1-g)+n+c_1\beta$ where $d_{\mathbb{C}}$ is the complex dimension of (M,J). Thus if $c_1=0$ (eg. Calabi Yau) $d_{\mathbb{C}}=3$ (or $d=6$) is the critical dimension when there is a discrete number of robust J-holomorphic curves of every genus. □

3.4 The operations of the BV algebra

We define δ and $\{,\}$ on the chain complex $C(\emptyset)$of Subsection 3.1 and on the enlargement $NC(\emptyset)$ which includes maps of surfaces with nodes. There is a map $C(\emptyset)\stackrel{M_1}{\to}C(\{*\})$ which adds a marked point in all possible positions to the surfaces in the family defining an element in $C(\emptyset)$. M_1 has degree $+2$ because the base of $M(x)$ in $C(\{*\})$ is the total space of x in $C(\emptyset)$. The bracket $\{,\}$ in $C(\emptyset)$ is the composition

$$C(\emptyset)\otimes C(\emptyset)\stackrel{M_1\otimes M_1}{\longrightarrow}C(\{i\})\otimes C(\{j\})\stackrel{C_{ij}}{\longrightarrow}C(\emptyset)$$

where i is the marked point in the first factor and j on the second. This bracket has degree $4-d+1=-d+5$ and is analogous to the generalized Goldman bracket or string bracket of [4] of degree $-d+2$.

We will define δ analogously as the composition $C(\emptyset) \xrightarrow{M_2} C(\{i,j\}) \xrightarrow{S_{ij}} C(\emptyset)$ where M_2 replaces a family of surfaces with no marked points with the same family with a pair of distinguished distinct points in all possible positions. The base of $M_2(x)$ is the total space of the fibrewise 2 point configuration space over the base of x. There is a compactness issue which is discussed in Remark 3.10. Since M_2 has degree 4 the composition δ also has degree $4 - d + 1 = -d + 5$.

Now form the graded symmetric algebra coalgebra generated by $C(\emptyset)$, $\Lambda C(\emptyset)$ and extend $-\partial$, δ, and $\{,\}$ to coderivations of $\Lambda C(\emptyset)$. The sum of these is called $\triangle$. Extend $\{,\}$ to a bracket on $\Lambda C(\emptyset)$ so that Leibniz holds and denote it $\{,\}$.

Theorem 3 *$\Lambda C(\emptyset)$ with its multiplication and the operator $\triangle$ is a BV algebra with derived Lie bracket $\{,\}$ of odd degree. Thus $\triangle \cdot \triangle = 0$ and the deviation of $\triangle$ from being a derivation is $\{,\}$. Furthermore $\triangle$ is a coderivation of the natural coalgebra structure on $\Lambda C(\emptyset)$.*

Proof. A point on the locus where each of the two pairs is identified in M contributes to two points (by definition) in the chain representing the output of $\triangle' \cdot \triangle'$. Here $\triangle'$ is $\triangle$ take away -∂. Since the circles of glueings for each pair is taken in opposite orders for the two points the orientations at the two points are opposite. Thus the involution interchanging the two points is orientation reversing and this output chain is equivalent to zero by Remark 3.2. The rest follows directly from the definitions, the glueing picture, and Remark 3.10 which shows ∂ and $\triangle'$ commute. Thus $\triangle \cdot \triangle = 0$ which is equivalent as well to the statement of Theorem 2 of Section 2. ■

Remark 3.9. We may extend all the above to the chain complex $NC(\emptyset)$ corresponding to maps of connected nodal surfaces into M- namely disconnected surfaces glued together at distinct pairs of points to make the entire collection connected. These are called garlands in Chernov-Rudyak [14] which studied a factor of $\{,\}$. Now there is also a compactness issue for $\{,\}$, when a point of a glueing pair we add comes close to a nodal point. This also concerns δ as well as the issue for δ of the two points of the glueing pair approaching each other in the surface. Again see Remark 3.10.

The same argument gives Theorem 3 for $NC(\emptyset)$. □

Remark 3.10. We treat the non compactness of the 2 point configuration space appearing in the definition of M_2 by first blowing up the diagonal in the 2 point configuration space of Σ by replacing each point on the diagonal by the circle of directions at that point.

Then as a pair of points in a fiber Σ of η approaches l in the direction of l' we pull out a tiny two sphere with marked points i and j. More precisely, we form the connected sum of the two sphere with marked points i, j, k to Σ attaching the direction blow up of k in the two sphere to the direction blow up of l in Σ so that the segment on the two sphere between i and j lines up with l' in Σ. We also allow a small bubble to appear in the map f at l. More precisely we use the original map f of Σ for each pair of points in the configuration space outside a small neighborhood of the diagonal, while on the small neighborhood of the diagonal we alter the map f so that it continuously incorporates at $f(p)$, p on the diagonal, the map of the two sphere at $f(p)$ from a fixed family e_{ijk} of maps of the two sphere parametrized by the points of M.

The d-chain of maps $e_{ijk}$is parametrized by M so that for $m \in M$ it maps the two sphere with labeled points i, j, k near to m so that k goes to m and i and j are mapped differently except at a finite number of points in M where they coincide transversally. Then M_2 is defined by using these maps of Σ over the compactified by blow up configuration space of two points on Σ, namely the original map f over most of the open part of the configuration space, then pulling out a small bubble near the diagonal blown up using the bubble d-chain e_{ijk} to guide the process.

For a dense set of generators x, $M_2(x)$ will be in the domain of S_{ij} because e_{ijk} is chosen that way. By construction we have

$$(\partial M_{\{i,j\}} - M_{\{i,j\}}\partial)(x) = C_{kl}(e_{ijk} \otimes M_l(x)).$$

Here the notation for M_2 and M_1 has been slightly expanded. Now further assume e_{ijk} is the d-chain which for each point m in the target manifold M^d maps a fixed 2-sphere with 3-marked points $\{i, j, k\}$ to a small segment between the images of i and j near the point m. Assuming transversality holds, we can calculate,

$$(\partial\delta - \delta\partial)(x) = S_{ij} \cdot C_{kl}(e_{ijk} \otimes M_l(x)) = -C_{kl}((S_{ij}e_{ijk}) \otimes M_l(x)) = 0$$

using $\delta = S_{ij} \cdot M_{\{ij\}}$ and the anticommutativity of modular operad operations.

The term $S_{ij}e_{ijk}$ is only non zero at the finite set of points of M where for the corresponding map of S^2 i and j coincide. At such a point it is represented by a Dehn twist family of labeled torii mapping to the point.

Assuming the construction of e_{ijk} is done this way we can make $\partial\delta - \delta\partial$ zero by passing to the reduced complex where the generators corresponding to Dehn twist torus families of constant maps are set to zero. □

Corollary 3.11. *δ and ∂ commute if the compactification of $M_2 = M_{\{ij\}}$ via the choice of e_{ijk} is done as above and we pass to the reduced complex.*

Exactly the same kind of considerations occur in the complexes $NC(\emptyset)$ in compactifying M_i and M_{ij} to achieve $[\partial, \delta] = 0$ and $[\partial, \{,\}] = 0$.

Proposition 3.12. *The bracket operation $\{,\}$ passes to the reduced complex.*

Proof. We have to show $\{t, x\} = 0$ for any x in $C(\emptyset)$ and any torus family of constant maps t which is set to zero in passing to the reduced complex. But $M_1(t)$ is already equivalent to zero because it has an orientation reversing self equivalence. ■

3.5 The quantum master equation and correlations

Now we consider the Maurer Cartan equation or the quantum master equation associated to $(C, \nabla, \{,\})$, the differential Lie algebra of degree -1, where $C = C(\emptyset)$. Let S denote an object of shifted grading zero with components in possibly infinitely many of the genus and β gradings $g = 0, 1, 2, \ldots$. The MC or QM equation is by definition,

$$\nabla S + \frac{1}{2}\{S, S\} = 0 \text{ or } \Delta S + \frac{1}{2}\{S, S\} = 0,$$

which is the same equation since $\nabla = \Delta$ for elements like S in monomial degree one. These equations can be checked component wise in the genus and β grading.

Recall the complex $(\Lambda C, \triangle)$ generated by $(C, \nabla, \{,\})$ as in Subsection 3.4.

Proposition 3.13. *A solution S to MC or QM equation above implies a solution to the equation in ΛC, $\triangle e^S = 0$, where e^S is interpreted component wise for the new monomial grading of ΛC.*

Proof. Since $\triangle$ is 2nd order (Section 2) one computes formally that

$$\triangle e^S = (\triangle S + 1/2\{S, S\})e^S.$$

Thus $\triangle S + 1/2\{S, S\} = 0$ implies $\triangle e^S = 0$, and conversely. ∎

Corollary 3.14. *A solution S of the quantum master equation in C implies there are multi linear functionals on the cohomology of the dual complex which we write $< \int e^S \varphi_1 \varphi_2 \ldots \varphi_n >$.*

Proof. Consider the dual $(\Lambda C)^*$ of the coalgebra ΛC with dual differential $\triangle^*$. Now the dual of a coalgebra (like ΛC) is an algebra and the dual of a coderivation (like $\triangle$) is a derivation. Thus cocycles representing φ_i can be multiplied in $(\Lambda C)^*$ to determine cocycles there. The latter may be evaluated on e^S yielding $< \int e^S \varphi_1 \ldots \varphi_n >$. ∎

4 The Gromov Chain of J-holomorphic curves

4.1 Perturbed J-holomorphic curves

We will refer to the formalism of [10] especially chapter 1 section 6 for discussing the Gromov chain associated to J-holomorphic curves in a symplectic almost complex manifold (M, J, ω). Fixing a stable combinatorics and homology data (Σ, β) there is a set of J-holomorphic curves with this type which is compact if we add the J-holomorphic curves of all the finitely many stable combinatorial types (Σ', β') obtained by degenerating. (see Subsection 4.2 and Subsection 4.3 below). This compactness is the achievement of Gromov to which many have added. See [9], [10] and the introduction and references. This compact set admits a finite stratification into types but the dimension and the regularity of the various strata is not necessarily what we want because the solutions of equation: (complex structure in Σ so that $\overline{\partial}$ (map)=0) may not be transversal.

The idea is to locally perturb the equation near the compact set and consider the transversal zeros which will be in some neighborhood of the compact set. Because of the presence of finite automorphism groups orbifolds, orbibundles, and multisections must be used, see [10] section 6, chapter 1, to construct the Gromov chain with Q coefficients. It is a tour de force challenge and achievement. All we need to know here is that the objects are produced by local perturbations and transversality, and that <u>all</u> perturbed J-holomorphic curves are included.

4.2 Combinatorics

Let us discuss the combinatorics of (Σ, β). By Σ we mean a finite collection of connected closed oriented surfaces together with nodal data- a finite subset of Σ (up to isotopy) with a fixed point free involution so that glueing related points yields a connected "nodal surface". By β we mean an integral homology class assigned to each component of Σ. It is important that the set of homology classes realized by "perturbed J-holomorphic" curves lies in a sharp cone in $H_2(M, R)$. This follows from the hypothesis that M is a closed symplectic manifold, see [18] which contains a more general result.Thus when representations of a realizable class degenerate into components there are only finitely many possibilities for their homology classes.

4.3 Stable combinatorics and degeneration

The combinatorics is stable if each component with $\beta = 0$ has Euler characteristic strictly less than the number of nodes on that component. Now let us study the stratum or part of the Gromov chain of (perturbed by η) J-holomorphic curves corresponding to a certain combinatorics (Σ, β) of "nodal" curves and homological position.

This part of the Gromov chain has an open part with the above combinatorics and a codimension two part where more degeneration takes place. If we impose inequalities on the complex structure and on the local energy of the maps we can carve out a small (generically tubular) neighborhood of the codimension two part where degeneracies happen. The homological boundary of what is left after carving out is essentially a sum over boundaries of tubes around strata where one degeneration happens. Fix one of those terms in the boundary of the cut off Gromov chain. Along that stratum one sees new combinatorics.

One component of the nodal surface we started with has either had a handle pinched off or been pinched into two components. The homology class β in the latter case splits into two parts and the genus splits into two parts. Zero homology class and zero genus can occur but the Euler characteristic condition above still holds.

4.4 The main result

Now we can present the main result. Let S' denote the cut off Gromov chain (Subsection 4.3) associated to all stable combinatorial types of connected surfaces with k nodes. So S' defines an element in $NC(\emptyset)$. Set $NC(\emptyset) = N$. We will modify S' to S by adding small collars as small corrections in the argument below. Then in (a completion of) $\Lambda N, e^S$ can be defined and it may be construed as the corrected cutoff Gromov chain of perturbed J-holomorphic curves associated to all stable combinatorial types of possibly disconnected surfaces with each component having exactly k nodes.

Theorem 5 *In* ΛN,

$$\partial S = \delta S + \frac{1}{2}\{S, S\}$$

In other words, $\triangle e^S = 0$ *.*

Proof. The ∂ of the total connected Gromov chains is made out of boundary of tubular neighborhoods of the strata corresponding to one additional degeneration. These are approximately described by applying the operation δ (if the degeneration doesn't disconnect) to the piece of the Gromov chain corresponding to the new combinatorics obtained by pulling apart the new node and erasing the points. If the degeneration disconnects, the boundary of the tube is approximately described by applying $\{,\}$ to the two pieces obtained by pulling apart the created node and erasing the points.

To replace approximate by exact we add small collars. Note we are assuming the (η perturbed) J-holomorphic curves satisfy all the transversality required for our operations to be defined. Thus η generic is required.

We find the equation $\partial S = \delta S + \frac{1}{2}\{S, S\}$, or equivalently $\triangle e^S = 0$.

■

Remark 4.1. We can always add n marked points labelled by I to the above discussion. Then the argument of Remark 3.10 when discussing M_1 and M_2 must be used again for the non compactness created by the added point or points approaching I. □

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