Dynamical Systems Applied to Asymptotic Geometry.

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Abstract

In the paper we discuss two fairly difficult questions about smooth expanding dynamical systems on the circle. (i) We characterize the sequences of asymptotic length ratios which occur for systems with Hölder continuous derivative. The sequence of asymptotic length ratios are precisely those given by a positive Hölder continuous function s on the Cantor set C of d-adic integers satisfying a functional equation called the matching condition. The functional equation for the 2-adic integer Cantor set is

$$1 + s(2x+2) = \frac{s(x)}{s(2x+1)} \left(1 + \frac{1}{s(2x)} \right).$$

(ii) We calculate the precise maximum level of smoothness possible for a representative of the system up to diffeomorphism in terms of the functions s and $cr(x) = (1+s(x))/(1+(s(x+1))^{-1})$. For example, in the Lipschitz structure on C determined by s, the maximum smoothness is $C^{1+\alpha}$ for $0 < \alpha < 1$ if and only if s is α-Hölder continuous. The maximum smoothness is $C^{1+\alpha}$ for $1 < \alpha < 2$ if and only if cr is α-Hölder. The two boundary cases correspond to smoothness of class first derivative Lipschitz or first derivative Zygmund, respectively. A curious connection with Mastow type rigidity is provided by the fact that s must be constant if it is α-Hölder for $\alpha > 1$, and cr must be constant if it is α-Hölder for $\alpha > 2$.

Contents

1 Introduction.

One could say that this paper is about the space $A(2)$ of sequences $\{a_0, a_1, a_2, \ldots\}$ of positive real numbers which satisfy

(i) a_n/a_m is exponentially near 1 if $n-m$ is divisible by a high power of two, and

(ii) $a_2, a_4, a_6, \ldots$ is constructed from $a_0, a_1, a_3, a_5, \ldots$ by the recursion

$$1 + a_{2n+2} = \frac{a_n}{a_{2n+1}} \left(1 + \frac{1}{a_{2n}}\right). \tag{1}$$

The only explicit element in $A(2)$ that we know is $\{1, 1, 1, \ldots\}$.

However, it will follow from the first part of this paper that $A(2)$ is a dense subset of a separable infinite dimensional complex Banach manifold of [23], because we will show that $A(2)$ is canonically isomorphic to

A) the set of all possible affine structures on the leaves of the dyadic solenoid $\tilde{S}(2)$ which are transversally Hölder continuous and invariant by the natural dynamics $\tilde{E}(2) : \tilde{S}(2) \to \tilde{S}(2)$. (The solenoid and its dynamics are defined below.)

B) the set of C^r structures on the circle S invariant by the "doubling the angle" dynamics $E(2) : S \to S$, $r > 1$.

C) the set of positive Hölder continuous functions s on the Cantor set C of 2-adic integers satisfying

$$1 + s(2x+2) = \frac{s(x)}{s(2x+1)} \left(1 + \frac{1}{s(2x)}\right). \tag{2}$$

The connection between the sequences of $A(2)$ and C) is direct. One merely restricts s in C) to the dense subset of natural numbers in the Cantor set of 2-adic integers.

The connection between the sequences of $A(2)$ and A) is also direct. One dense leaf in the solenoid is provided with a binary grid and is expanded by the dynamics in a manner combinatorially like $x \to 2x$ acting on $\{n/2^k\} \subset$ reals. The sequence $\{a_0, a_1, a_2, \ldots\}$ is used to define ratios of consecutive lengths between integral points of the grid. The functional equation makes the doubling map look affine between the integral grid and its double. The 2-adic continuity allows the complete affine structure induced by pullback to impress itself on the other leaves of the solenoid $\tilde{S}(2)$.

The passage from A) to B) is also direct. The solenoid $\tilde{S}(2)$ with its dynamics is the inverse limit system associated to the diagram

$$\begin{array}{cccccc} \cdots \xrightarrow{E(2)} & S & \xrightarrow{E(2)} & S & \xrightarrow{E(2)} & S \\ & \uparrow E(2) & & \uparrow E(2) & & \uparrow E(2) \\ \cdots \xrightarrow{E(2)} & S & \xrightarrow{E(2)} & S & \xrightarrow{E(2)} & S \end{array}$$

$\{1,1,1,\ldots\}$. The point is that a homeomorphism of $\mathbf{R}$ which is locally projective must be affine. In [5] the possible difference between these 2 cases requires a magical argument.

The fifth line of Table 1 suggests a formal proof of other known rigidity results related to expanding dynamics. If C and its metric were replaced by a 1-dimensional continuum and a metric associated to a Riemannian metric then (zygmund 1^{st} derivative) $\overset{\text{Table1}}{\Longleftrightarrow}$ (cr is 1-Hölder) $\overset{\text{(calculus)}}{\Longrightarrow}$ cr is constant $\overset{\text{(by definition)}}{\Longrightarrow}$ locally projective $\Longrightarrow$ affine (usually).

The completion of the set in C) correspond to (uaa) structures on the circle S (called symmetric structures in [6]) invariant by the "doubling the angle" dynamics $E(2) : S \to S$ so that all branches of $E(2)^{-n}$ are (uaa) (see definition in [23]). In [23] is also studied the case corresponding to analytic structures on the circle S invariant by the dynamics $E(2) : S \to S$ and leaves the C^r case, $r > 1$, to be studied in this paper.

This theory is giving rise to a deeper understanding of flexibility and rigidity results for transitive non linear hyperbolic dynamics, and it is also giving new relations between asymptotic geometry and Gibbs theory (cf. [13], [15] and [17]).

2 $C^{1+H\ddot{o}lder}$ structures U for the expanding circle map E.

The *expanding circle map* $E = E(d) : S \to S$ with degree $d \geq 2$ is given by $E(z) = z^d$ in complex notation. Let $p \in S$ be one of the fixed points of the expanding circle map E. The *Markov intervals of the expanding circle map* E are the adjacent closed intervals $I_0, \ldots, I_{d-1}$ with non empty interior such that their boundaries are contained in the set $\{E^{-1}(p)\}$ of preimages of the fixed point $p \in S$ and $I_{d-1} \cap I_0$ is equal to the fixed point p. Let the *branch expanding circle map* $E_i : I_i \to S$ be the restriction of the expanding circle map E to the Markov interval I_i, for all $0 \leq i < d$. Let the interval $I_{\alpha_1 \ldots \alpha_n}$ be $E_{\alpha_n}^{-1} \circ \ldots \circ E_{\alpha_1}^{-1}(S)$. The n^{th}*-level of the interval partition of the expanding circle map* E is the set of all closed intervals $I_{\alpha_1 \ldots \alpha_n} \in S$.

A function $h : I \to J$, where $I, J \subset \mathbf{R}$, is $C^{1+H\ddot{o}lder}$ if there is $\varepsilon > 0$ such that the map h is $C^{1+\varepsilon}$ smooth. A homeomorphism $h : I \to J$ is *quasisymmetric* if there are constants $b > 0$ and $c > 1$ with the property that for all $x - \delta_1, x, x + \delta_2 \in I$, such that $c^{-1} < \delta_2/\delta_1 < c$, we have

$$\left| \log \frac{h(x+\delta_2) - h(x)}{h(x) - h(x-\delta_1)} \frac{\delta_1}{\delta_2} \right| < b.$$

Definition 1 *The expanding circle map $E : S \to S$ is $C^{1+H\ddot{o}lder}$ with respect to a structure U on the circle S* if for every finite cover U' of U, (i) there is an $\varepsilon > 0$ with the property that for all charts $u : I \to \mathbf{R}$ and $v : J \to \mathbf{R}$ contained in U' and for all intervals $K \subset I$ such that $E(K) \subset J$, the maps $v \circ E \circ u^{-1}|u(K)$ are $C^{1+\varepsilon}$ and their $C^{1+\varepsilon}$ norms are bounded away from zero and infinity; (ii) for every chart $u : I \to \mathbf{R}$ contained in U' and for every map $u_{iso} : I \to \mathbf{R}$, which is an isometry with respect to the lengths on the circle $S \subset \mathbf{R}^2$ determined by the Euclidean norm on $\mathbf{R}^2$, the composition $u_{iso} \circ u^{-1}$ is a quasisymmetric homeomorphism.

Lemma 1 The expanding circle map $E : S \to S$ is $C^{1+H\ddot{o}lder}$ with respect to a structure U if, and only if, for every finite cover U' of U, there are constants $0 < \mu < 1$ and $b > 1$ with the property that for all charts $u : J \to \mathbf{R}$ and $v : K \to \mathbf{R}$ contained in U' and for all adjacent intervals $I_{\alpha_1\ldots\alpha_n}$ and $I_{\alpha'_1\ldots\alpha'_n}$ at level n of the interval partition such that $I_{\alpha_1\ldots\alpha_n}, I_{\alpha'_1\ldots\alpha'_n} \subset J$ and $E(I_{\alpha_1\ldots\alpha_n}), E(I_{\alpha'_1\ldots\alpha'_n}) \subset K$, we have that

$$b^{-1} < \frac{|u(I_{\alpha_1\ldots\alpha_n})|}{|u(I_{\alpha'_1\ldots\alpha'_n})|} < b \quad \text{and} \quad \left|\log \frac{|u(I_{\alpha_1\ldots\alpha_n})|\ |v(E(I_{\alpha'_1\ldots\alpha'_n}))|}{|u(I_{\alpha'_1\ldots\alpha'_n})|\ |v(E(I_{\alpha_1\ldots\alpha_n}))|}\right| \le O(\mu^n). \tag{3}$$

Lemma 1 follows from Theorem 3 in Section 6.

Lemma 2 Let $E : S \to S$ be an expanding circle map, $C^{1+Hölder}$ with respect to a structure U. For every finite cover U' of U, there is an $\varepsilon > 0$, with the property that for all charts $u : J \to \mathbf{R}$ and $v : K \to \mathbf{R}$ contained in U' and for all adjacent intervals I and I', such that $I, I' \subset J$, $E(I), E(I') \subset K$, we have

$$\left|\log \frac{|u(I)||v(E(I'))|}{|u(I')||v(E(I))|}\right| \le O(|u(I) \cup u(I')|^{\varepsilon}). \tag{4}$$

Lemma 2 follows from the Mean Value Theorem.

3 The solenoid ($\tilde{\mathbf{E}}$,$\tilde{\mathbf{S}}$).

The sequence $\mathbf{x} = (\ldots, x_3, x_2, x_1, x_0)$ is *an inverse path of the expanding circle map* E if $E(x_n) = x_{n-1}$, for all $n \ge 1$. The *topological solenoid* $\tilde{S}$ consists of all inverse paths $\mathbf{x} = (\ldots, x_3, x_2, x_1, x_0)$ of the circle expanding map E with the product topology. The topological solenoid is a compact set. The *solenoid map* $\tilde{E}$ is the bijective map defined by

$$\tilde{E}(\mathbf{x}) = (\ldots x_0, E(x_0)).$$

The *projection map* $\pi = \pi_S : \tilde{S} \to S$ is defined by $\pi(\mathbf{x}) = x_0$. A *fiber* over $x_0 \in S$ is the set of all points $\mathbf{x} \in \tilde{S}$ such that $\pi(\mathbf{x}) = x_0$. A fiber is topologically a Cantor set $\{0, \ldots, d-1\}^{\mathbf{Z}_{\ge 0}}$. A *leaf* $\mathcal{L} = \mathcal{L}_{\mathbf{z}}$ is the set of all points $\mathbf{w} \in \tilde{S}$ path connected to the point $\mathbf{z} \in \tilde{S}$. A *local leaf* $\mathcal{L}'$ is a path connected set. The *monodromy map* $\tilde{M} : \tilde{S} \to \tilde{S}$ is defined such that the local leaf starting on $\mathbf{x}$ and ending on $\tilde{M}(\mathbf{x})$ after being projected by π is an anti-clockwise arc starting on x_0, going around the circle once, and ending on the point x_0. All leaves $\mathcal{L}$ of the solenoid $\tilde{S}$ are dense, since the orbit of any point $\mathbf{x} \in \tilde{S}$ under M is dense on its fiber (see Lemma 5 in Section 4). The topological solenoid $\tilde{S}$ is a twist product of the circle S with the Cantor set $\{0, \ldots, d-1\}^{\mathbf{Z}_{\ge 0}}$.

Definition 2 *The solenoid* $(\tilde{E}, \tilde{S})$ *is transversally Hölder continuous affine (thca)* if (i) every leaf L has an affine structure; (ii) the solenoid map $\tilde{E}$ preserves the affine structure on the leaves; and (iii) the ratio between adjacent leaves determined by their affine structure changes Hölder continuously along transversals.

We say that $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *is a triple*, if the points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are distinct and are contained in the same leaf $\mathcal{L}$ of $\tilde{S}$. Let T be the set of all triples $(\mathbf{x}, \mathbf{y}, \mathbf{z})$. The function $r : T \to \mathbf{R}^+$ is *invariant by the action of the solenoid map* $\tilde{E}$ if and only if, for all triples $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in T$, we have $r(\mathbf{x}, \mathbf{y}, \mathbf{z}) = r(\tilde{E}(\mathbf{x}), \tilde{E}(\mathbf{y}), \tilde{E}(\mathbf{z}))$. The function $r : T \to \mathbf{R}^+$ *varies Hölder continuously along fibers* if and only if there are constants $c > 0$ and $0 < \mu < 1$ with the property that for all triples $(\mathbf{x}, \mathbf{y}, \mathbf{z}), (\mathbf{x}', \mathbf{y}', \mathbf{z}') \in T$ such that $x_n = x'_n$, $y_n = y'_n$ and $z_n = z'_n$, we have

$$|\log(r(\mathbf{x}, \mathbf{y}, \mathbf{z})) - \log(r(\mathbf{x}', \mathbf{y}', \mathbf{z}'))| \le O(\mu^n).$$

Definition 3 A *Hölder leaf ratio function* $r : T \to \mathbf{R}^+$ is a continuous function, which varies Hölder continuously along fibers, is invariant by the action of the solenoid map $\tilde{E}$, and satisfies the following *matching condition*: for all triples $(\mathbf{x}, \mathbf{w}, \mathbf{y}), (\mathbf{w}, \mathbf{y}, \mathbf{z}) \in T$,

$$r(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{r(\mathbf{x}, \mathbf{w}, \mathbf{y}) r(\mathbf{w}, \mathbf{y}, \mathbf{z})}{1 + r(\mathbf{x}, \mathbf{w}, \mathbf{y})}.$$

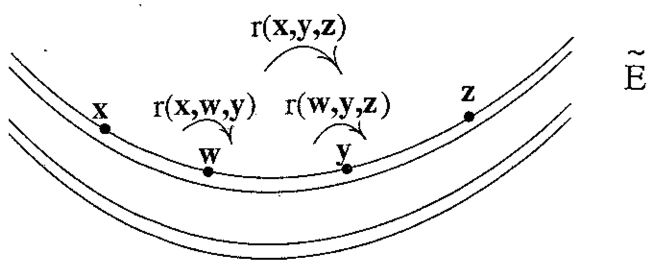


Figure 1: The matching condition for the leaf ratio function r.

Lemma 3 There is a one-to-one correspondence between (thca) solenoids $(\tilde{E}, \tilde{S})$ and Hölder leaf ratio functions $r : T \to \mathbf{R}^+$.

Proof: The affine structures on the leaves of the solenoid $\tilde{S}$ determine a function $r : T \to \mathbf{R}^+$ which varies continuously along leaves and satisfies the matching condition. The converse is also true. Moreover, (i) the solenoid map $\tilde{S}$ preserves the affine structure on the leaves if and only if the function $r : T \to \mathbf{R}^+$ is invariant by the action of the solenoid map $\tilde{E}$ and (ii) the ratio between adjacent leaves determined by their affine structure changes Hölder continuously along transversals if and only if the function $r : T \to \mathbf{R}^+$ varies Hölder continuously along fibers. ∎

Lemma 4 Let $E : S \to S$ be an expanding circle map $C^{1+Hölder}$ with respect to a structure U. Then $E : S \to S$ generates a Hölder leaf ratio function $r_{E,U} : T \to \mathbf{R}^+$.

Proof: Let U' be a finite cover of U. For every triple $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in T$ and n large enough, let $u_n : J_n \to \mathbf{R}$ be a chart contained in U' such that $x_n, y_n, z_n \in J_n$. Define $r_{E,U}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ by

$$r_{E,U}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \lim_{n \to \infty} \frac{|u_n(y_n) - u_n(z_n)|}{|u_n(x_n) - u_n(y_n)|}.$$

By Lemma 2, $r_{E,U} : T \to \mathbf{R}^+$ is a leaf ratio function. ∎

4 The solenoid function $s : C \to \mathbf{R}^+$.

Let $\sum_{i=-\infty}^{\infty} a_i d^i$ be a *d-adic number*. The d-adic numbers

$$\sum_{i=-\infty}^{n-1} (d-1)d^i + \sum_{i=n}^{\infty} a_i d^i \quad \text{and} \quad (a_n + 1)d^n + \sum_{i=n+1}^{\infty} a_i d^i$$

such that $a_n + 1 < d$ are *d-adic equivalent. The d-adic set* $\tilde{\Omega}$ is the topological Cantor set $\{0, \ldots, d-1\}^{\mathbf{Z}}$ corresponding to all d-adic numbers modulo this d-adic equivalence. The *product map* $d\times : \tilde{\Omega} \to \tilde{\Omega}$ corresponds naturally to the multiplication by d of the d-adic numbers. The *add* 1 *map* $1+ : \tilde{\Omega} \to \tilde{\Omega}$ corresponds naturally to the sum of 1 to the d-adic numbers and the *add d map* $d+ : \tilde{\Omega} \to \tilde{\Omega}$ is d compositions of the add 1 map.

Let the map $\tilde{\omega} : \tilde{\Omega} \to \tilde{S}$ be the homeomorphism between the d-adic set $\tilde{\Omega}$ and the solenoid $\tilde{S}$ defined as follows. For every d-adic number $\sum_{i=-\infty}^{\infty} a_i d^i$, the point $\tilde{\omega}(\sum_{i=-\infty}^{\infty} a_i d^i)$ is $\mathbf{x} = (\ldots, x_1, x_0) \in \tilde{S}$, such that $x_i \in I_{a_{i-1}}$ and $E^i(x_0) \in I_{a_{-(i+1)}}$, for all $i \geq 0$, where I_{a_i} are the Markov intervals of the expanding circle map E. By construction, the solenoid map $\tilde{E} : \tilde{S} \to \tilde{S}$ is topologically conjugate to the product map $d\times : \tilde{\Omega} \to \tilde{\Omega}$; and the monodromy map $\tilde{M} : \tilde{S} \to \tilde{S}$ is topologically conjugate to the add d map $d+ : \tilde{\Omega} \to \tilde{\Omega}$, by the map $\tilde{\omega} : \tilde{\Omega} \to \tilde{S}$.

Lemma 5 Every orbit of the monodromy map is dense on its fiber.

Proof: Since the add d map $d+ : \tilde{\Omega} \to \tilde{\Omega}$ is dense on the image $\tilde{\omega}^{-1}(F)$ of every fiber F of the solenoid $\tilde{S}$, the lemma follows.

The set Ω is the topological Cantor set $\{0, \ldots, d-1\}^{\mathbf{Z}_{<0}}$ corresponding to all d-adic numbers of the form $\sum_{i=-\infty}^{-1} a_i d^i$ modulo the d-adic equivalence. *The projection map* $\pi_\Omega : \tilde{\Omega} \to \Omega$ is defined by $\pi_\Omega \left(\sum_{i=-\infty}^{\infty} a_i d^i\right) = \sum_{i=-\infty}^{-1} a_i d^i$. The map $\omega : \Omega \to S$ is defined by means of the Markov intervals as follows: $\omega(\sum_{i=-\infty}^{-1} a_i d^i) = x$ where $E^{-i}(x) \in I_{a_{i-1}}$. By construction,

$$\omega \circ \pi_\Omega \left(\sum_{i=-\infty}^{\infty} a_i d^i \right) = \pi_S \circ \tilde{\omega} \left(\sum_{i=-\infty}^{\infty} a_i d^i \right),$$

for all $\sum_{i=-\infty}^{\infty} a_i d^i \in \tilde{\Omega}$.

The set C is the topological Cantor set $\{0, \ldots, d-1\}^{\mathbf{Z}_{\geq 0}}$ corresponding to all d-adic integers of the form $\sum_{i=0}^{\infty} a_i d^i$.

Definition 4 The *solenoid function* $s : C \to \mathbf{R}^+$ is a continuous function satisfying the following *matching condition*, for all $a \in C$:

$$1 + s(da+2) = \frac{s(a)}{s(da+1)} \left(1 + \frac{1}{s(da)}\right). \tag{5}$$

Lemma 6 The Hölder leaf ratio function $r : T \to \mathbf{R}^+$ determines a Hölder solenoid function $s_r : C \to \mathbf{R}^+$.

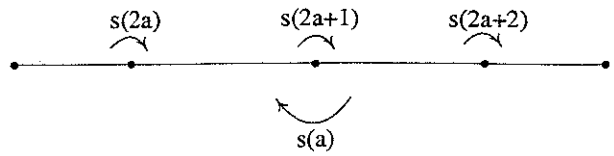


Figure 2: The matching condition for the solenoid function ($d = 2$).

Proof: For all $\sum_{i=0}^{\infty} a_i d^i \in C$, define

$$s_r\left(\sum_{i=0}^{\infty} a_i d^i\right) = r\left(\tilde{\omega}\left(\sum_{i=0}^{\infty} a_i d^i - d\right), \tilde{\omega}\left(\sum_{i=0}^{\infty} a_i d^i\right), \tilde{\omega}\left(\sum_{i=0}^{\infty} a_i d^i + d\right)\right).$$

The matching condition and the Hölder continuity of the leaf ratio function $r : T \to \mathbf{R}^+$ implies the matching condition and the Hölder continuity of the solenoid function $s_r : C \to \mathbf{R}^+$, respectively. ■

Lemma 7 There is a one-to-one correspondence between Hölder solenoid functions $s : C \to \mathbf{R}^+$ and sequences $\{r_0, r_1, r_2, \ldots\} \in A(d)$ of positive real numbers which satisfy

(i) $r_n/r_m \leq O(\mu^i)$ if $n - m$ is divisible by d^i, where $0 < \mu < 1$, and

(ii) $r_2, r_{d+2}, r_{2d+2}, \ldots$ is constructed by the recursion

$$1 + r_{di+2} = \frac{r_i}{r_{di+1}}\left(1 + \frac{1}{r_{di}}\right). \tag{6}$$

A more geometric interpretation of the sequences contained in the set $A(d)$ is given by the d-quasiperiodic fixed grids in Section 8.

Proof: Given a Hölder solenoid function $s : C \to \mathbf{R}^+$ for all $i = \sum_{j=0}^{j_i} a_j d^j \in \mathbf{Z}_{\geq 0}$ define r_i by

$$r_i = s\left(\sum_{j=0}^{j_i} a_j d^j\right).$$

The matching condition of the solenoid function $s : C \to \mathbf{R}^+$, implies that the ratios $r_2, r_{d+2}, r_{2d+2}, \ldots$ satisfy the recursion (6). The Hölder continuity of the solenoid function $s : C \to \mathbf{R}^+$ implies condition (i).

Conversely, for every d-adic integer $a = \sum_{i=0}^{\infty} a_i d^i \in C$, let $\underline{a}_n \in \mathbf{Z}_{\geq 0}$ be equal to $\sum_{i=0}^{n} a_i d^i$. Define the value $s(a)$ by

$$s(a) = \lim_{n \to \infty} r_{\underline{a}_n}.$$

By condition (i) the limit is well defined and the function $s : C \to \mathbf{R}^+$ is Hölder continuous. By condition (ii) the function $s : C \to \mathbf{R}^+$ satisfies the matching condition. ■

5 Solenoidal charts for the $C^{1+Hölder}$ expanding circle map E.

Let $\mathcal{L}$ be a local leaf with an affine structure which projects by $\pi_{\mathcal{L}} = \pi_S|\mathcal{L}$ homeomorphically onto an interval J of the circle S. Let $\phi : \mathcal{L} \to \mathbf{R}$ be a map which preserves the affine structure of the leaf $\mathcal{L}$. A *solenoidal chart* $u_{\mathcal{L}} : J \to \mathbf{R}$ *on the circle* S is defined by $u_{\mathcal{L}} = \phi \circ \pi_{\mathcal{L}}^{-1}$ (see Figure 3).

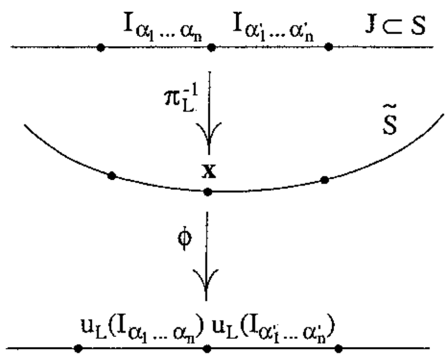


Figure 3: The solenoidal chart.

Lemma 8 The solenoidal charts determined by a (thca) solenoid $(\tilde{E}, \tilde{S})$ produce a structure U for which the expanding circle map E is $C^{1+Hölder}$.

Proof: Let U' be a finite cover consisting of solenoidal charts. Let $I_{\alpha_1 \ldots \alpha_n}$, $I_{\alpha'_1 \ldots \alpha'_n}$ be adjacent intervals at level n of the interval partition and $u_{\mathcal{L}} : J \to \mathbf{R}$ and $v_{\mathcal{L}'} : K \to \mathbf{R}$ solenoidal charts such that $I_{\alpha_1 \ldots \alpha_n}, I_{\alpha'_1 \ldots \alpha'_n} \subset J$ and $I_{\alpha_2 \ldots \alpha_n}, I_{\alpha'_2 \ldots \alpha'_n} \subset K$. Let $\mathbf{x} \in \mathcal{L}$ and $\mathbf{y} \in \mathcal{L}'$ be the points with

$$\pi(\mathbf{x}) = I_{\alpha_1 \ldots \alpha_n} \cap I_{\alpha'_1 \ldots \alpha'_n} \quad \text{and} \quad \pi(\mathbf{y}) = I_{\alpha_2 \ldots \alpha_n} \cap I_{\alpha'_2 \ldots \alpha'_n}.$$

Since $\tilde{E}$ is affine on leaves, we have

$$s\left(\tilde{\omega}^{-1}(\tilde{E}^n(\mathbf{x}))\right) = \frac{u_{\mathcal{L}}(I_{\alpha_1 \ldots \alpha_n})}{u_{\mathcal{L}}(I_{\alpha'_1 \ldots \alpha'_n})} \quad \text{and} \quad s\left(\tilde{\omega}^{-1}(\tilde{E}^{n-1}(\mathbf{y}))\right) = \frac{v_{\mathcal{L}'}(I_{\alpha_2 \ldots \alpha_n})}{v_{\mathcal{L}'}(I_{\alpha'_2 \ldots \alpha'_n})}, \tag{7}$$

where $s : C \to \mathbf{R}^+$ is the Hölder solenoid function determined by the (thca) solenoid $(\tilde{E}, \tilde{S})$ (see Lemma 6). By Hölder continuity of the solenoid function,

$$\left| \log \frac{s\left(\tilde{\omega}^{-1}(\tilde{E}^n(\mathbf{x}))\right)}{s\left(\tilde{\omega}^{-1}(\tilde{E}^{n-1}(\mathbf{y}))\right)} \right| \leq O(\mu^n), \tag{8}$$

for some $0 < \mu < 1$. By equality (7), inequality (8), and Lemma 1 the expanding circle map E is $C^{1+Hölder}$ with respect to the structure U produced by these solenoidal charts. ∎

Lemma 9 The Hölder solenoid function $s : C \to \mathbf{R}^+$ determines a set of solenoidal charts which produce a structure U such that the expanding circle map E is $C^{1+Hölder}$.

Proof: Every point $a \in C$ determines a triple

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\tilde{\omega}(a - d), \tilde{\omega}(a), \tilde{\omega}(a + d))$$

and the solenoid value $s(a)$ determines a ratio $r(\mathbf{x}, \mathbf{y}, \mathbf{z}) = s(a)$. We define the ratios $r(\tilde{E}^n(\mathbf{x}), \tilde{E}^n(\mathbf{y}), \tilde{E}^n(\mathbf{z}))$ equal to $s(a)$, for all $n \in \mathbf{Z}$. By construction, the ratios are invariant under the solenoid map $\tilde{E}$. Since the solenoid function satisfies a matching condition, these ratios also satisfy a matching condition and determine an affine structure on the leaves of the solenoid. By construction, the solenoidal charts determined by this affine structure on the leaves satisfy equality (7) and inequality (8). By Lemma 1, the expanding circle map E is $C^{1+Hölder}$ with respect to the structure U produced by these solenoidal charts. ■

Theorem 1 There is a one-to-one correspondence between (i) $C^{1+Hölder}$ structures U for the expanding circle map $E : S \to S$; (ii) (thca) solenoids $(\tilde{E}, \tilde{S})$; (iii) Hölder leaf ratio functions $r : T \to \mathbf{R}^+$; (iv) Hölder solenoid functions $s : C \to \mathbf{R}^+$; (v) sequences $\{r_0, r_1, \ldots\} \in A(d)$; (vi) d-quasiperiodic fixed grids.

For the definition of a d-quasiperiodic fixed grid see Section 8.

Proof: It follows from the following diagram, where the implications are detemined by the lemmas indicated by their numbers:

$$\begin{array}{ccccccc}
 & & (i) & \overset{8}{\Longleftarrow} & (ii) & \overset{13}{\Longleftrightarrow} & (vi) \\
 & & 9\Uparrow & \overset{4}{\searrow} & \Updownarrow 3 & & \\
(v) & \overset{7}{\Longleftrightarrow} & (iv) & \overset{6}{\Longleftarrow} & (iii) & &
\end{array}$$

■

5.1 Smooth properties of the solenoidal charts.

Let the expanding circle map E be $C^{1+Hölder}$ with respect to a structure U. By Lemma 3 and Lemma 4, the structure U determines a (thca) solenoid $(\tilde{E}, \tilde{S})$. *The solenoidal cover V of U* is the set of all solenoidal charts determined by the (thca) solenoid $(\tilde{E}, \tilde{S})$.

Theorem 2 Let U' be a finite cover of the $C^{1+Hölder}$ structure U for the expanding circle map E. The smoothness of the expanding circle map E when measured in terms of the cover U' attains its maximum when the cover U' is a subset of the solenoidal cover V.

Proof: Let the expanding circle map $E : S \to S$ be C^r smooth, for some $r > 1$, with respect to a finite cover U' of the structure U. We shall prove that the solenoidal charts $v_{\mathcal{L}} : I \to \mathbf{R}$ are C^r compatible with the charts contained in U', proving the theorem.

Let $\mathcal{L}$ be a local leaf which projects by $\pi_{\mathcal{L}} = \pi_S | \mathcal{L}$ homeomorphically on an interval I contained in the domain J of a chart $u : J \to \mathbf{R}$ of U'. For n large enough, let $u_n : J_n \to \mathbf{R}$ be a chart in U' such that $I_n = \pi_S(\tilde{E}^{-n}(\mathcal{L})) \subset J_n$. Let $\lambda_n : u_n(I_n) \to (0, 1)$ be the restriction to the interval $u_n(I_n)$ of an affine map sending the interval $u_n(I_n)$ onto the interval $(0, 1)$.

Let $e_n : (0, 1) \to \mathbf{R}$ be the C^r smooth map defined by $e_n = u \circ E^n \circ u_n^{-1} \circ \lambda_n^{-1}$ (see Figure 4). The map e_n is the composition of a contraction λ_n^{-1} followed by an expansion $u \circ E^n \circ u_n^{-1}$. Therefore, by the blow-down blow-up technique (see [1] and [12]), the map $e : (0, 1) \to \mathbf{R}$ defined by $e = \lim_{n \to \infty} e_n$ is a C^r homeomorphism. By construction, the map $v_{\mathcal{L}} : I \to \mathbf{R}$ defined by $e^{-1} \circ u$ is a solenoidal chart C^r compatible with the charts contained in U'. ■

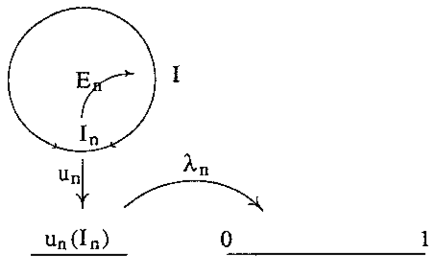


Figure 4: The construction of the solenoidal charts from the $C^{1+Hölder}$ structure U.

6 Smoothness of $h : \mathbf{R} \to \mathbf{R}$ and cross ratio distortion of grids.

Let $h : I \to J$ be a homeomorphism, where $I, J \subset \mathbf{R}$. We are going to analyse the C^r smoothness of the map in terms of the distortion of the cross ratios of a grid (see the following Table 2). Table 2 is an important tool used to prove Table 1.

A *grid* g of an interval $I \subset \mathbf{R}$ is a collection of *grid* intervals $\{I^n_\beta : n \in \mathbf{N} \text{ and } \beta \in \mathbf{Z}\}$ such that, for all $n \geq 1$, (i) the union $\cup_{\beta \in \mathbf{Z}} I^n_\beta$ of all adjacent intervals I^n_β at the same *level* n is equal to the interval I; (ii) the set of all end points of the intervals I^n_β at level n is contained in the set of all end points of the intervals I^{n+1}_β at level $n+1$; (iii) the ratios $r^n_\beta = |I^n_{\beta+1}|/|I^n_\beta|$ between any two adjacent intervals I^n_β and $I^n_{\beta+1}$ are bounded away from zero and infinity, independently of the intervals $I^n_\beta, I^n_{\beta+1}$ considered and of the level n; (iv) the ratios $|I^{n+1}_\beta|/|I^n_\alpha|$ are bounded away from zero and infinity independently of β and of the level n, where $I^{n+1}_\beta \subset I^n_\alpha$.

Let $h : I \subset \mathbf{R} \to J \subset \mathbf{R}$ be a homeomorphism. Let I_β, $I_{\beta'}$ and $I_{\beta''}$ be three adjacent intervals in I where I_β is on the left of $I_{\beta'}$ and $I_{\beta'}$ is on the left of $I_{\beta''}$. For any interval I_β in I, denote by J_β the interval $h(I_\beta)$. (i) Let the *average derivative* d_β be defined by $d_\beta = |J_\beta|/|I_\beta|$. (ii) Let the *ratio* r_β be defined by $|I_{\beta'}|/|I_\beta|$. (iii) Let the *ratio* $r_{h\beta}$ be defined by $|J_{\beta'}|/|J_\beta|$. (iv) Let the *ratio distortion* l_β be defined by

$$l_\beta = \log \frac{r_{h\beta}}{r_\beta} = \log \frac{d_{\beta'}}{d_\beta}. \tag{9}$$

(v) Let I_α be equal to the union $I_\beta \cup I_{\beta'} \cup I_{\beta''}$ of the adjacent intervals I_β, $I_{\beta'}$ and $I_{\beta''}$. The *cross ratio* or equivalently the *Poincare length* $P(I_{\beta'} \subset I_\alpha)$ of $I_{\beta'}$ in $I_\alpha = I_\beta \cup I_{\beta'} \cup I_{\beta''}$ is defined by

$$\begin{aligned} P(I_{\beta'} \subset I_\alpha) &= \log\left(1 + \frac{|I_{\beta'}|}{|I_\beta|}\frac{|I_\beta| + |I_{\beta'}| + |I_{\beta''}|}{|I_{\beta''}|}\right) \\ &= \log\left((1+r_\beta)(1+r^{-1}_{\beta'})\right). \end{aligned}$$

(vi) Let J_α be equal to the union $J_\beta \cup J_{\beta'} \cup J_{\beta''}$. The *cross ratio distortion* c_β is defined by

$$\begin{aligned} c_\beta &= P(J_{\beta'} \subset J_\alpha) - P(I_{\beta'} \subset I_\alpha) \\ &= \log\left(\frac{1+r_{h\beta}}{1+r_\beta}\frac{1+r^{-1}_{h\beta'}}{1+r^{-1}_{\beta'}}\right). \end{aligned} \tag{10}$$

Theorem 3 Let $h : \mathbf{R} \to \mathbf{R}$ be a homeomorphism and g a grid of $\mathbf{R}$. Let I_β, $I_{\beta'}$ and $I_{\beta''}$ be three adjacent grid intervals contained in any bounded interval of $\mathbf{R}$. Then, we have the equivalence shown in the following Table 2:

The smoothness of the map $h : \mathbf{R} \to \mathbf{R}$.	The order of the cross ratio distortion c_β.	The order of the ratio distortion l_β.
$C^{1+\alpha}$	$O(\|I_\beta\|^\alpha)$	$O(\|I_\beta\|^\alpha)$
$C^{1+Zygmund}$	$O(\|I_\beta\|)$	—
$C^{1+zygmund}$	$o(\|I_\beta\|)$	—
$C^{1+Lipschitz}$	—	$O(\|I_\beta\|)$
$C^{2+\alpha}$	$O(\|I_\beta\|^{1+\alpha})$	—
$C^{2+Lipschitz}$	$O(\|I_\beta\|^2)$	—
Affine	$o(\|I_\beta\|^2)$	$o(\|I_\beta\|)$

Table 2.

The Proof of Theorem 3 will follow from Proposition 1 and Proposition 2 below.

By Theorem 3, we obtain the following non-trivial Corollary 1 saying that we do not have to study the ratio distortion or the cross ratio distortion of all adjacent intervals $I_\beta, I_{\beta'}, I_{\beta''}$, but just the ones which are grid intervals.

Corollary 1 Let $h : \mathbf{R} \to \mathbf{R}$ be a homeomorphism and g a grid of $\mathbf{R}$. The grid g measures all the information needed to caracterize the smoothness of the map h, in terms of the order of the ratio distortion or of the order of the cross ratio distortion.

6.1 Proof of Theorem 3.

We first introduce several definitions and lemmas necessary for the proofs of Proposition 1 and Proposition 2.

Let $h : \mathbf{R} \to \mathbf{R}$ be a homeomorphism and g a grid of $\mathbf{R}$. The *level n ratio distortion $l(n)$ of an interval K* is the maximum of the ratio distortions l_β, over all the grid intervals I^n_β in K. Similarly, *the level n cross ratio distortion $c(n)$ of an interval K* is the maximum of the cross ratios c_β, over all the grid intervals I^n_β in K.

Lemma 10 If the cross ratio distortion $c(n)$ on some open interval K is bounded independently of n, then the map h is quasisymmetric in every closed interval $L \subset K$. Conversely, if the map h is quasisymmetric in an interval L then the cross ratio distortion $c(n)$ is bounded on L, independently of n.

The definition of a quasisymmetric homeomorphism h is given in Section 2.

Proof of Lemma 10. Let I_β, $I_{\beta'}$, $I_{\beta''}$ and $I_{\beta'''}$ be four adjacent intervals at level n in K. Let S be the smallest interval between $J_{\beta'}$ and $J_{\beta''}$. Suppose $S = J_{\beta''}$. Since the cross ratio distortion c_β is bounded

$$1 \leq \frac{|J_{\beta'}|}{|J_{\beta''}|} < \frac{|J_{\beta'}|}{|J_{\beta''}|} \frac{|J_\beta| + |I_{\beta'}| + |J_{\beta''}|}{|J_\beta|} < b.$$

Therefore, the ratio $r_{h\beta'}$ is bounded. Since the ratio $r_{\beta'}$ is also bounded, we have that the ratio distortion $l_{\beta'}$ is bounded. Similarly, if $S = J_{\beta'}$ then $1 < r_{h\beta'}^{-1} < b$, which implies that the modulus of the ratio distortion $l_{\beta'}$ is bounded.

Let $c > 0$ be a constant. For all points $x - \delta_1, x, x + \delta_2 \in L$ such that $c^{-1} < \delta_2/\delta_1 < c$, there are adjacent grid intervals $I_1, \ldots, I_g$ at some level n such that (i) the interval $[x - \delta_1, x + \delta_2]$ is contained in the union $\cup_{i=1}^{g} I_i$; (ii) there is at least one interval $I_d \subset [x - \delta_1, x]$ such that the points $x - \delta_1, x$ are not contained in I_d, where $1 \leq d \leq g$; (iii) there is at least one interval I_e such that $x \in I_e$, where $d < e \leq g$; (iv) there is at least one interval $I_f \subset [x, x + \delta_2]$ such that the points $x, x + \delta_2$ are not contained in I_f, where $e < f \leq g$; (v) the constant g just depends on the constant c; (see Figure 5).

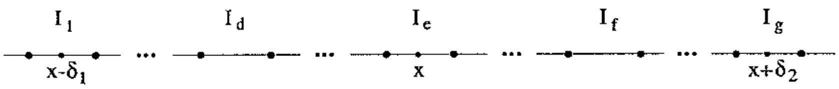


Figure 5: The adjacent grid intervals $I_1, \ldots, I_g$.

Since the modulus of the ratio distortion is bounded and the constant g just depends on c, the following ratio is bounded:

$$\left| \log \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)} \frac{\delta_1}{\delta_2} \right| < b$$

Conversely, if the ratio distortion is bounded, then by equality (10) the cross ratio distortion is bounded. ■

A homeomorphism $h : I \to J$ is *uniformly asymptotically affine (uaa)* if there is a constant $c > 0$ and a continuous function $\varepsilon_c : \mathbf{R}_0^+ \to \mathbf{R}_0^+$, with $\varepsilon_c(0) = 0$, with the property that for all $x - \delta_1, x, x + \delta_2 \in I$, such that $c^{-1} < \delta_2 / \delta_1 < c$, we have

$$\left| \log \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)} \frac{\delta_1}{\delta_2} \right| < \varepsilon_c(\delta_1).$$

Lemma 11 If the cross ratio distortion c_β is of order $o(1)$ for all grid intervals I_β contained in some open interval K, then the map h is uniformly asymptotically affine (uaa) in every closed interval $L \subset K$. Conversely, if the map h is uniformly asymptotically affine (uaa) in an interval L, then the cross ratio distortion c_β is of order $o(1)$ for all grid intervals I_β contained in L.

Let a *symmetric triple* $(I_\beta, I_{\beta'}, I_{\beta''})$ be three adjacent intervals $I_\beta, I_{\beta'}, I_{\beta''} \subset I$ of the same length. Let the *standard grid* be a grid (i) where all the intervals at the same level n have the same length and (ii) each grid interval at level n is equal to the union of two grid intervals at level $n + 1$.

We will denote, for simplicity of notation, an interval $[a - \varepsilon, a + \varepsilon]$ by $a \pm \varepsilon$.

Proof: First, we prove that if the cross ratio distortion $c_\alpha \in o(1)$ for all grid intervals I_α, then the cross ratio distortion $c_\beta \in o(1)$ for all symmetric triples $(I_\beta, I_{\beta'}, I_{\beta''})$. Secondly, we prove that the ratio distortion l_β converges to zero when the length of the intervals of the symmetric triples converges to zero. In the third part, we complete the proof.

The first part. For all small $\varepsilon > 0$, there is an $M > 0$ such that for all symmetric triples $(I_\beta, I_{\beta'}, I_{\beta''})$ there are grid intervals $I_{\alpha_1}, \ldots, I_{\alpha_m}$, where $m < M$, such that the cross ratio distortion of c_β is $\varepsilon/2$-close to an analytic formula $c_{\underline{\alpha}}$ of the cross ratio distortions $c_{\alpha_1}, \ldots, c_{\alpha_m}$. By hypotheses, the c_{α_i} converge to zero when the lengths of the intervals I_{α_i} converge to zero. Therefore, there is an $N > 0$ such that for all symmetric triples $(I_\beta, I_{\beta'}, I_{\beta''})$ with length $|I_\beta| < N^{-1}$ the corresponding analytic formula $c_{\underline{\alpha}}$ is $\varepsilon/2$-close to zero. Thus, the cross ratio distortion c_β is ε-close to zero.

The second part. Suppose that there is a $c > 0$ with the property that for all $n \geq 1$, there exists an interval $I_\beta \subset L$, with length $|I_\beta| < 2^{-n}$ and $l_\beta > c$. Therefore there is a constant $c_1 > 0$ such that

$$\frac{1 + r_{h\beta}}{1 + r_\beta} > 1 + c_1. \tag{11}$$

Let I_β, $I_{\beta'}$, $I_{\beta''}$, and $I_{\beta'''}$ be adjacent intervals with the same length. By equality (10) and since the distortion of the cross ratio c_β converges to zero when the length of the interval I_β converges to zero, there is a small $\varepsilon > 0$ such that

$$\frac{1 + r_{h\beta}}{1 + r_\beta} \frac{1 + r_{h\beta'}^{-1}}{1 + r_{\beta'}^{-1}} \in 1 \pm \varepsilon. \tag{12}$$

By (11) and (12), there is a constant $c_2 > 0$ depending only on the constant c, such that if the constant c gets large then c_2 gets large and

$$\frac{r_{h\beta'}^{-1}}{r_{\beta'}^{-1}} < 1 - c_2.$$

By the same argument as above applied to the cross ratio $c_{\beta'}$, there is a constant c_3 depending only on c such that if the constant c gets large, then c_3 gets large, and $l_{\beta''} > c_3$.

The difference $c_\beta - c_{\beta''}$ between the cross ratio distortions c_β and $c_{\beta''}$ is equal to

$$c_\beta - c_{\beta''} = \log \left(\frac{r_{h\beta''}}{r_{\beta''}} \frac{r_{\beta'}}{r_{h\beta'}} \frac{1 + r_{\beta''}}{1 + r_{h\beta''}} \frac{1 + r_{h\beta}}{1 + r_\beta} \right) \in \pm 2\varepsilon.$$

Therefore,

$$l_{\beta''} - l_{\beta'} \in \log \left(\frac{1 + r_{h\beta''}}{1 + r_{\beta''}} \right) - \log \left(\frac{1 + r_{h\beta}}{1 + r_\beta} \right) \pm 2\varepsilon. \tag{13}$$

Let the interval I_α be equal to the union $I_\beta \cup I_{\beta'}$ of the adjacent intervals $I_\beta, I_{\beta'}$; and the interval $I_{\alpha'}$ be equal to the union $I_{\beta''} \cup I_{\beta'''}$ of the adjacent intervals $I_{\beta''}, I_{\beta'''}$. By concatenation, the ratio r_α is equal to

$$r_\alpha = \frac{r_\beta r'_\beta + r_\beta r'_\beta r''_\beta}{1 + r_\beta}.$$

Therefore,

$$l_\alpha = l_\beta + l_{\beta'} + \log \frac{1 + r_{h\beta''}}{1 + r_{\beta''}} - \log \frac{1 + r_{h\beta}}{1 + r_\beta} \tag{14}$$

By (13) and (14), for some constant $e > 0$

$$l_\alpha \geq l_\beta + l_{\beta''} - 2\varepsilon > c + c_3 - 2\varepsilon > c + e. \tag{15}$$

Let I_{β_m} and $I_{\beta'_m}$ be adjacent intervals of the same length such that (i) $I_{\beta_{m-1}} = I_{\beta_m} \cup I_{\beta'_m}$; (ii) $I_{\beta_n} = I_\beta$ and $I_{\beta'_n} = I_{\beta'}$; (see Figure 6). By inequality (15) and by induction on m, the ratio distortion $l_{\beta_m} \geq c + (n - m)e$. Thus, the ratio distortion l_{β_m} is not bounded when $n - m$ and n tend to infinity, which is absurd by Lemma 10. Therefore, the ratio distortion l_β converges to zero when the length of the interval I_β converges to zero.

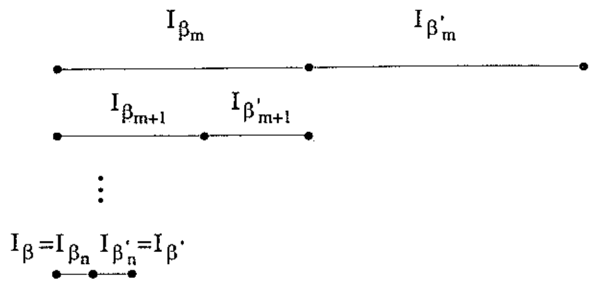


Figure 6: The adjacent intervals I_β and $I_{\beta'}$.

The third part. In particular, we have proved for the standard grid that the ratio distortion $l(n)$ converges to zero when n tends to infinity. Let $c > 0$ and $\varepsilon > 0$ be constants. For all points $x - \delta_1, x, x + \delta_2 \in L$ such that $0 < \delta_1, \delta_2 < \delta$ and $c^{-1} < \delta_2/\delta_1 < c$, there are adjacent grid intervals $I_1, \ldots, I_e$ at some level n with the property that (i) the point $x - \delta_1 \in I_1$, the point $x \in I_d$, and the point $x + \delta_2 \in I_e$, for some $1 < d < e$; (ii) the ratio $(\sum_{i=1}^{d-1} |I_i|)/\delta_1 \in 1 \pm \varepsilon$ and the ratio $(\sum_{i=d}^{e} |I_i|)/\delta_2 \in 1 \pm \varepsilon$, for some $\varepsilon > 0$; (iii) the bound on the number e of intervals I_i just depends on $c > 0$ and $\varepsilon > 0$. The level n of the grid intervals $I_1, \ldots, I_e$ tends to infinity when δ converges to zero. Since the ratio distortion $l(n)$ converges to zero when n tends to infinity, for all small $\varepsilon > 0$ there is $\delta > 0$ small enough such that

$$\left| \log \frac{h(x+\delta_2) - h(x)}{h(x) - h(x-\delta_1)} \frac{\delta_1}{\delta_2} \right| \in 1 \pm \varepsilon.$$

Conversely, by equality (10), if the ratio distortion $l(n)$ converges to zero when n tends to infinity, then the cross ratio distortion $c(n)$ converges to zero when n tends to infinity. ∎

Lemma 12 Let I_β and $I_{\beta'}$ be adjacent intervals at level m. The cross ratio c_β is equal to

$$c_\beta = \frac{l_\beta}{1 + r_\beta^{-1}} - \frac{l_{\beta'}}{1 + r_{\beta'}}, \tag{16}$$

up to terms of order $O(l_\beta^2, l_{\beta'}^2)$. Let I_α and $I_{\alpha'}$ be adjacent intervals at level $m - 1$ such that $I_\beta \cup I_{\beta'} \subset I_\alpha \cup I_{\alpha'}$. Let $I_{\beta'''}$ be any interval at level m such that $I_{\beta'''} \subset I_\alpha \cup I_{\alpha'}$. The ratio distortion l_α is equal to

$$l_\alpha = \frac{|I_\alpha| + |I_{\alpha'}|}{|I_\beta| + |I_{\beta'}|} l_\beta \tag{17}$$

up to terms of order $O(l_{\beta'''}^2, c_{\beta'''})$.

Proof: Since r_β and $r_{h\beta}$ are positive,

$$\left| \log \frac{1 + r_{h\beta}}{1 + r_\beta} \right| \le \left| \log \frac{r_{h\beta}}{r_\beta} \right| = |l_\beta|.$$

By Taylor series expansion of the logarithmic function,

$$\log \frac{1 + r_{h\beta}}{1 + r_\beta} \in \frac{l_\beta}{1 + r_\beta^{-1}} \pm O(l_\beta^2). \tag{18}$$

Similarly,

$$\log\frac{1+r_{h\beta}^{-1}}{1+r_{\beta}^{-1}} \in -\frac{l_\beta}{1+r_\beta} \pm O(l_\beta^2).$$

By (18), we get equality (16). Let the interval I_α be equal to the union $I_{\beta_1}\cup\ldots\cup I_{\beta_M}$. Let the interval $I_{\alpha'}$ be equal to the union $I_{\beta_{M+1}}\cup\ldots\cup I_{\beta_{N-1}}$. By equality (16),

$$\frac{l_{\beta_i}}{|I_{\beta_i}|+|I_{\beta_{i+1}}|} \in \frac{l_{\beta_j}}{|I_{\beta_j}|+|I_{\beta_{j+1}}|} \pm O(l_{\beta_i}^2,\ldots,l_{\beta_j}^2,c_{\beta_i},\ldots,c_{\beta_{j-1}}).$$

Therefore, we just have to prove equality (17) for $\beta=\beta_1$ and $\beta'=\beta_2$. Let a_j be the product $r_{\beta_1}\ldots r_{\beta_j}$ and a_{hj} the product $r_{h\beta_1}\ldots r_{h\beta_j}$. Let R be the sum $1+\sum_{j=1}^{M-1}a_j$ and R_h be the sum $1+\sum_{j=1}^{M-1}a_{hj}$. Let R' be the sum $1+\sum_{j=M}^{N-1}a_j$ and R'_h the sum $1+\sum_{j=M}^{N-1}a_{hj}$. By definition of the ratio distortion l_α,

$$l_\alpha=\log\frac{R'_h}{R'}\frac{R}{R_h}.$$

The ratios $r_{h\beta_i}$ are equal to $r_{\beta_i}(1+l_{\beta_i})\pm O(l_{\beta_i}^2)$. Therefore, the sum R_h is equal to $R+E$, where

$$E\in\sum_{j=1}^{M-1}a_j\left(\sum_{l=1}^{j}l_{\beta_l}\right)\pm O(l_{\beta_1}^2,\ldots,l_{\beta_{M-1}}^2).$$

The sum R'_h is equal $R'+E'$, where

$$E'\in\sum_{j=1}^{N-1}a_j\left(\sum_{l=1}^{j}l_{\beta_l}\right)\pm O(l_{\beta_1}^2,\ldots,l_{\beta_{N-1}}^2).$$

Therefore, the ratio distortion l_α satisfies

$$l_\alpha\in\frac{E'}{R'}-\frac{E}{R}\pm O(l_{\beta_1}^2,\ldots,l_{\beta_{N-1}}^2). \tag{19}$$

By equality (16),

$$\sum_{l=1}^{j}l_{\beta_l}\in\frac{l_{\beta_1}}{1+r_{\beta_1}}\left(1+a_1+\sum_{l=2}^{j}(a_{l-1}\ldots a_j)\right)\pm O(l_{\beta_1}^2,\ldots,l_{\beta_j}^2,c_{\beta_1},\ldots,c_{\beta_{j-1}}).$$

Therefore,

$$E\in\frac{l_{\beta_1}}{1+r_{\beta_1}}(1+a_1+\ldots+a_{M-1})(a_1+\ldots+a_{M-1})\pm O(l_{\beta_i}^2,c_{\beta_i}),$$

for all $i=1,\ldots,M-1$. Similarly,

$$E'=\frac{l_{\beta_1}}{1+r_{\beta_1}}(1+2a_1+\ldots+2a_{M-1}+a_M+\ldots+a_{N-1})(a_M+\ldots+a_{N-1}),$$

up to terms of order $O(l_{\beta_i}^2,c_{\beta_i})$, for all $i=1,\ldots,N-1$. By (19),

$$l_\alpha\in\frac{|I_\alpha|+|I_{\alpha'}|}{|I_{\beta_1}|+|I_{\beta_2}|}l_{\beta_1}\pm O(l_{\beta_i}^2,c_{\beta_i}),$$

for all $i=1,\ldots,N-1$. ∎

Proposition 1 For $0 < r < 1$ or $1 < r < 2$, if the cross ratio distortion c_β is of order $O(|I_\beta|^r)$ for all grid intervals I_β contained in some open interval K, then the map h is C^{r+1} smooth. If the cross ratio distortion c_β is of order $O(|I_\beta|^2)$ or $o(|I_\beta|^2)$, then the map $\log dh$ is $C^{1+Lipschitz}$, or the map h is Möbius, respectively. Conversely, for $0 < r < 1$ or $1 < r < 2$, if the map h is C^{r+1} smooth in a closed interval $L \subset K$ then the cross ratio distortion $c_\beta \in O(|I_\beta|^r)$, for all grid intervals I_β contained in L. If the map $\log dh$ is $C^{1+Lipschitz}$ or the map h is Möbius then the cross ratio distortion c_β is of order $O(|I_\beta|^2)$ or $o(|I_\beta|^2)$, respectively.

Proof: *Case* $0 < r < 1$. First, we prove that the cross ratio distortion $c_\beta \in \pm O(|I_\beta|^r)$, for all intervals I_β^r contained in K, implies the ratio distortion $l_\beta \in \pm O(|I_\beta|^r)$. Secondly, we prove that if $l_\beta \in \pm O(|I_\beta|^r)$ then the map h is C^{r+1} smooth. Finally, we prove that if the map h is C^{r+1} smooth then $c_\beta \in \pm O(|I_\beta|^r)$.

Let the cross ratio distortion $c_\beta \in \pm O(|I_\beta|^r)$, for all intervals I_β. By Lemma 11, the ratio distortion l_β converges to zero when the level n of the interval I_β tends to infinity. Let us suppose, that the ratio distortion $l_\beta \in \pm O(|I_\beta|^s)$, for some $0 < s < r$, and that for all $N \geq 1$, there is $n > N$ and some interval $I_{\beta_n} \subset L$ at level n such that $l_{\beta_n} = O(|I_{\beta_n}|^s)$. We will prove by contradiction, that $l_\beta \in O(|I_{\beta_n}|^r)$. A similar argument applies, if we consider even slower speeds of convergence for the ratio distortion l_β and a sequence l_{β_n} converging at the possible slowest speed.

For all $1 \leq m \leq n$, let $\{I_{\beta_m} \subset I_{\beta_{m-1}}\}$ be a nested sequence of intervals. By induction on equality (17),

$$\frac{l_{\beta_n}}{|I_{\beta_n}|} \leq O(|I_{\beta_n}|^{r-1}, |I_{\beta_n}|^{2s-1}) < O(|I_{\beta_n}|^{s-1}),$$

which is absurd, since by hypotheses,

$$\frac{l_{\beta_n}}{|I_{\beta_{n-1}}|} \geq O(|I_{\beta_n}|^{s-1}).$$

Therefore, $l_\beta \in \pm O(|I_\beta|^r)$ for all intervals I_β.

Conversely, if the ratio distortion $l_\beta \in \pm O(|I_\beta|^r)$, for all intervals $I_\beta \subset L$, then by equality (16), the cross ratio distortion $c_\beta \in \pm O(|I_\beta|^r)$.

For all points $P \neq P'$ contained in L, take the highest possible value of m for which there are adjacent intervals $I_\beta, I_{\beta'}$ at level m with the property that $P \in I_\beta$, $P' \in I_{\beta'}$, $P \notin I_{\beta'}$ and $P' \notin I_\beta$. Let the interval I_α at level $m-1$ contain the union $I_\beta \cup I_{\beta'}$. By construction of the interval I_α and bounded geometry, the ratio $|P' - P|/|I_\alpha|$ is bounded away from zero and infinity independently of the points P and P'. Since $l_\beta \in O(|I_\alpha|^r)$, we obtain

$$d_\beta - d_{\beta'} \in \pm O(|I_\alpha|^r). \tag{20}$$

By definition of average derivative d_α and (20),

$$d_\alpha \in d_\beta \pm O(|I_\alpha|^r) \subset d_{\beta'} \pm O(|I_\alpha|^r). \tag{21}$$

For all $n > m$, take a nested sequence of intervals $\{I_{\beta_n} \subset I_\beta\}$ at level $n+m$ and a nested sequence of intervals $\{I_{\beta'_n} \subset I_{\beta'}\}$ at level $n+m$, such that the sequence $\{I_{\beta_n}\}$ converges to the point P and the sequence $\{I_{\beta'_n}\}$ converges to the point P' (see Figure 7).

By definition of average derivative and (20), we obtain that

$$d_{\beta_n} \in d_{\beta_{n+1}} \pm O(|I_{\beta_n}|^r). \tag{22}$$

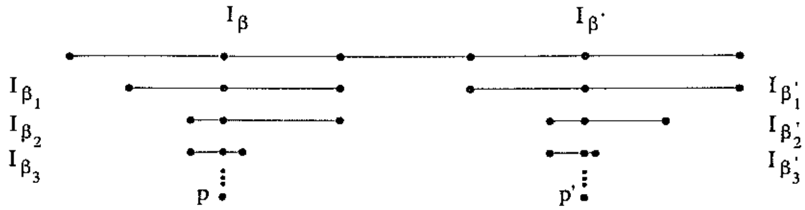


Figure 7: The nested sequence of intervals I_{β_n} and $I_{\beta'_n}$.

Therefore, the average derivative d_{β_n} converges to a limit d_P. Similarly, the average derivative $d_{\beta'_n}$ converges to a limit $d_{P'}$, when n tends to infinity. The limits d_P and $d_{P'}$ are contained in $d_\alpha \pm O(|I_\alpha|^r)$. By (20) and (22),

$$|d_{P'} - d_P| \leq O(|P' - P|^r). \tag{23}$$

We will prove that the homeomorphism h is differentiable at the points P and P' with derivatives $dh(P) = d_P$ and $dh(P') = d_{P'}$.

For every point $p \in L$, take any interval I_θ such that the point $p \in I_\theta$. Let I_{γ_1} and I_{γ_2} be adjacent intervals at the highest possible level L' for which the interval I_θ is contained in the union $I_{\gamma_1} \cup I_{\gamma_2}$. Let I_γ at level $L'-1$ contain the union $I_{\gamma_1} \cup I_{\gamma_2}$. By construction of I_γ, the ratio $|I_\gamma|/|I_\theta|$ is bounded away from zero and infinity independently of the interval I_θ.

Take the smallest sequence of adjacent intervals $I_{\gamma_1^l}, \ldots, I_{\gamma_{i_l}^l}$ at level $L'+l$ such that I_θ is contained in the union $I_{\gamma_1^l} \cup \ldots \cup I_{\gamma_{i_l}^l}$ (see Figure 8).

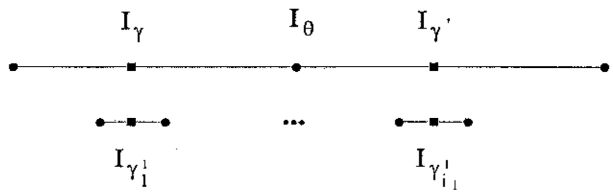


Figure 8: The interval I_θ.

By definition of the average derivative d_θ,

$$d_\theta = \lim_{l\to\infty} \sum_{i=1}^{i_l} \frac{|I_{\gamma_i^l}|}{|I_\theta|} d_{\gamma_i^l}. \tag{24}$$

By (22) and definition of d_p, the average derivative $d_{\gamma_i^l}$ is contained in $d_p \pm O(|I_\theta|^r)$. By equality (24), the average derivative d_θ is contained in $d_p \pm O(|I_\theta|^r)$. Therefore, by inequality (23), the map h is C^{1+r} smooth and $dh(p) = d_p$.

Conversely, we prove that if the homeomorphism h is C^{r+1} smooth on L, then the cross ratio distortion $c_\beta \in \pm O(|I_\beta|^r)$ for all intervals $I_\beta \subset L$ and $0 < r < 1$.

Let I_β, $I_{\beta'}$ and $I_{\beta''}$ be adjacent intervals at the same level n. By the Mean Value Theorem, there are $x, y \in I_\beta \cup I_{\beta'}$ such that $l_\beta \in \pm O(|y-x|^r) \subset \pm O(|I_\beta|^r)$. Similarly, $l_{\beta'} \in \pm O(|I_{\beta'}|^r)$. By equality (16), the cross ratio distortion $c_\beta \in \pm O(|I_\beta|^r)$.

Case $1 < r \leq 2$. By the above, the map $h : L \to L$ has a C^s smooth derivative dh, for every $0 < s < 1$. Let I_β and $I_{\beta'}$ be adjacent intervals at level m. Let $\phi : L \to L$ be the map defined by $\phi(x) = \int_{x_0}^{x} \log dh(y)dy$. The *average derivative of the map* $\phi(x)$ is defined by

$$\phi_\beta = \frac{|\int_{I_\beta} \log dh(y)dy|}{|I_\beta|}.$$

Using the Taylor series of the logarithmic function, and since the map h has a C^s smooth derivative, $\phi_\beta = \log(d_\beta)$ up to terms of order $O(|I_\beta|^{2s})$, where $0 < s < 1$. By definition of ϕ_β and by equality (9),

$$l_\beta \in \phi_{\beta'} - \phi_\beta \pm O(|I_\beta|^{2s}). \tag{25}$$

For all points $P \neq P'$ contained in L, take the highest possible value m for which there are two pairs of adjacent intervals $I_\beta, I_{\beta'}$ and $I_\psi, I_{\psi'}$ at level m with the property that (i) there is a pair of adjacent intervals $I_\alpha, I_{\alpha'}$, at level $m-1$, which contains the union $I_\beta \cup I_{\beta'} \cup I_\psi \cup I_{\psi'}$; (ii) the point $P \in I_\beta \cup I_{\beta'}$, the point $P' \in I_\psi \cup I_{\psi'}$, the point $P \notin I_\psi \cup I_{\psi'}$, and the point $P' \notin I_\beta \cup I_{\beta'}$.

For all $n > m$, take a nested sequence of adjacent intervals $\{I_{\beta_n}, I_{\beta'_n} \subset I_\beta \cup I_{\beta'}\}$ at level n, such that the point $P \in I_{\beta_n} \cup I_{\beta'_n}$. Similarly, take a nested sequence of adjacent intervals $\{I_{\psi_n}, I_{\psi'_n} \subset I_\psi \cup I_{\psi'}\}$, at level n, such that the point $P' \in I_{\psi_n} \cup I_{\psi'_n}$. By (17) and (25),

$$\frac{\phi_{\psi'} - \phi_\psi}{|I_{\psi'}| + |I_\psi|} - \frac{\phi_{\beta'} - \phi_\beta}{|I_{\beta'}| + |I_\beta|} \leq O(|I_\beta|^{r-1}, |I_\psi|^{r-1}). \tag{26}$$

By (17) and (25),

$$\frac{\phi_{\beta'_{n-1}} - \phi_{\beta_{n-1}}}{|I_{\beta'_{n-1}} \cup I_{\beta_{n-1}}|} \in \frac{\phi_{\beta'_n} - \phi_{\beta_n}}{|I_{\beta'_n} \cup I_{\beta_n}|} \pm O(|I_{\beta_n}|^{r-1}). \tag{27}$$

By (27), the ratio $(\phi_{\beta'_n} - \phi_{\beta_n})/|I_{\beta'_n} \cup I_{\beta_n}|$ converges to a limit d_P^2, when $I_{\beta'_n} \cup I_{\beta_n}$ tends to the point P. The limit d_P^2 is contained in $(\phi_{\beta'} - \phi_\beta)/|I_{\beta'} \cup I_\beta| \pm O(|I_\beta|^{r-1})$. Similarly, the ratio $(\phi_{\psi'_n} - \phi_{\psi_n})/|I_{\psi'_n} \cup I_{\psi_n}|$ converges to a limit $d_{P'}^2$, when $I_{\psi'_n} \cup I_{\psi_n}$ tends to the point P'. The limit $d_{P'}^2$ is contained in $(\phi_{\psi'} - \phi_\psi)/|I_{\psi'} \cup I_\psi| \pm O(|I_\psi|^{r-1})$. By inequality (26),

$$|d_{P'}^2 - d_P^2| < O(|P' - P|^{r-1}). \tag{28}$$

We now prove that the diffeomorphism ϕ is C^{r+1} smooth at the points P and P', that the second derivative $d^2\phi(P)$ is equal to $2d_P^2$, and that the second derivative $d^2\phi(P')$ is equal to $2d_{P'}^2$. For every point $p \in L$, take any two small adjacent intervals I_γ and $I_{\gamma'}$, at some level L', such that p is contained in $I_\gamma \cup I_{\gamma'}$. For all points $x, y \in I_\gamma \cup I_{\gamma'}$ and for all $l > L'$, take the smallest sequence of adjacent intervals $I_{\gamma_1^l}, \ldots, I_{\gamma_{i_l}^l}$ which contain the points x and y, at level $l + L'$. By (25),

$$|d\phi(y) - d\phi(x)|/|y - x| = \lim_{l \to \infty} \frac{\sum_{i=1}^{i_l - 1} l_{\gamma_i^l}}{\sum_{i=1}^{i_l - 1} |I_{\gamma_i^l}|}. \tag{29}$$

By (27) and definition of d_p^2, we have

$$\frac{l_{\gamma_i^l}}{|I_{\gamma_i^l}| + |I_{\gamma_{i+1}^l}|} \in \frac{d_p^2}{|I_\gamma| + |I_{\gamma'}|} \pm O(|I_\gamma|^{r-1}).$$

By equality (29), we obtain $|d\phi(y) - d\phi(x)|/|y-x| \in 2d_p \pm O(|I_\gamma|^{r-1})$. Therefore, by inequality (28) the map $\phi : L \to L$ is C^{r+1} smooth and $d^2\phi(p) = 2d_p^2$. If $r = 2$, then the map $\phi : L \to L$ is $C^{1+Lipschitz}$. If the cross ratio distortion $c_\beta \in \pm o(|I_\beta|^2)$, then the Schwarzian derivative of the map $h : L \to L$ is equal to zero.

Conversely, we prove that if the homeomorphism h is C^{r+1} smooth on L then the cross ratio distortion $c_\gamma \in \pm O(|I_\gamma|^r)$, for all interval I_γ contained in L.

Let I_γ, $I_{\gamma'}$ and $I_{\gamma''}$ be adjacent intervals at the same level n with endpoints x, y, z, w. By Taylor series,

$$l_\gamma \in (|I_\gamma| + |I_{\gamma'}|)\frac{d^2\phi(y)}{2} \pm O(|I_\gamma|^r, |I_{\gamma'}|^r)$$

and

$$l_{\gamma'} \in (|I_{\gamma'}| + |I_{\gamma''}|)\frac{d^2\phi(z)}{2} \pm O(|I_{\gamma'}|^r, |I_{\gamma''}|^r).$$

Therefore, by equality (16), the cross ratio distortion $c_\gamma \in \pm O(|I_\gamma|^r)$. ∎

The adjacent intervals I_γ, $I_{\gamma'}$ and $I_{\gamma''}$ form a *triangle triple* if $I_\gamma \cup I_{\gamma'} \cup I_{\gamma''}$ is contained in the union of three grid intervals at some level n and each interval $I_\gamma, I_{\gamma'}$ and $I_{\gamma''}$ is equal to a union of grid intervals at level $n+1$. The intervals $I_\gamma, I_{\gamma'}, I_{\gamma''}$ are the *triangle* intervals. Given an interval I_γ, let m_γ be the middle point of the interval I_γ. The *middle set* M of a grid is equal to the disjoint union of all middle points m_γ of the triangle intervals I_γ. A function $f : M \to L$ is *triangle Zygmund* if and only if for all triangle triples $(I_\gamma, I_{\gamma'}, I_{\gamma''})$,

$$\left| \frac{(|I_\gamma| + |I_{\gamma'}|)f(m_{\gamma''}) + (|I_{\gamma'}| + |I_{\gamma''}|)f(m_\gamma)}{|I_\gamma| + 2|I_{\gamma'}| + |I_{\gamma''}|} - f(m_{\gamma'}) \right| \le O(|I_\gamma|).$$

For all pair of adjacent intervals $I_\alpha, I_{\alpha'}$ contained in the union $I_\gamma \cup I_{\gamma'} \cup I_{\gamma''}$ of a triangle triple, let the interval $I_{\alpha,\alpha'}$ be equal to $[m_\alpha, m_{\alpha'}]$. Let $f_{\alpha,\alpha'} : I_{\alpha,\alpha'} \to L$ be the *middle affine map* defined by $f_{\alpha,\alpha'}(m_\alpha) = f(m_\alpha)$ and $f_{\alpha,\alpha'}(m_{\alpha'}) = f(m_{\alpha'})$ (see Figure 9).

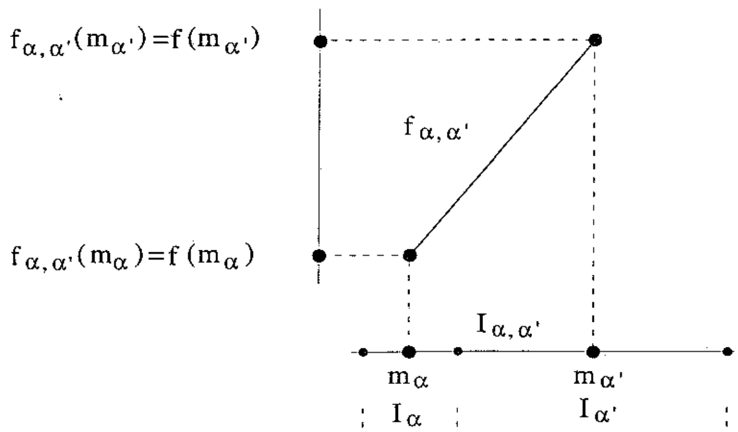
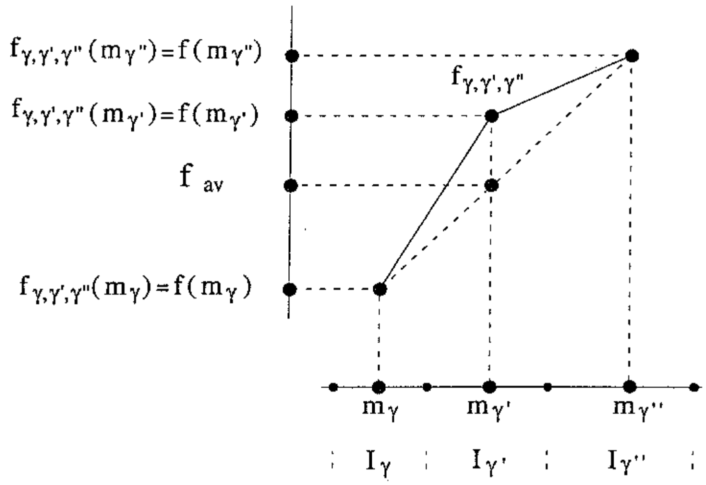


Figure 9: The map $f_{\alpha,\alpha'}$.

Define the *middle piecewise affine map* $f_{\gamma,\gamma',\gamma''} : I_{\gamma,\gamma'} \cup I_{\gamma',\gamma''} \to L$ by

$$f_{\gamma,\gamma',\gamma''}(x) = \begin{cases} f_{\gamma,\gamma'}(x), & \text{if } x \in I_{\gamma,\gamma'} \\ f_{\gamma',\gamma''}(x), & \text{if } x \in I_{\gamma',\gamma''} \end{cases}$$

(see Figure 10).



$$f_{av} = \frac{(|I_\gamma|+|I_{\gamma'}|)f_{\gamma,\gamma',\gamma''}(m_{\gamma''}) + (|I_{\gamma'}|+|I_{\gamma''}|)f_{\gamma,\gamma',\gamma''}(m_\gamma)}{|I_\gamma|+2|I_{\gamma'}|+|I_{\gamma''}|}$$

Figure 10: The map $f_{\gamma,\gamma',\gamma''}$.

The map $h : I \to J$ is *Zygmund* if for all points $x - \delta_1, x, x + \delta_2 \in I$, such that $\delta_1, \delta_2 > 0$, we have

$$\left| \frac{\delta_1 h(x+\delta_2) + \delta_2 h(x-\delta_1)}{\delta_1 + \delta_2} - h(x) \right| \leq O(\delta_1 + \delta_2)$$

(see Figure 11). The map $h : I \to J$ is *zygmund* if the bound in the inequality above is replaced by $o(\delta_1 + \delta_2)$.

Proposition 2 If the cross ratio distortion c_β is of order $O(|I_\beta|)$ or of order $o(|I_\beta|)$ for all grid intervals I_β contained in some open interval K, then the map $\log dh$ is Zygmund or zygmund, respectively, for every closed interval $L \subset K$. Conversely, if the map $\log dh$ is Zygmund or zygmund in an interval L then the cross ratio distortion c_β is of order $O(|I_\beta|)$ or of order $o(|I_\beta|)$, respectively, for all grid intervals I_β contained in L.

Proof: By Proposition 1 and since for all grid triples $I_\beta, I_{\beta'}, I_{\beta''}$ the cross ratio $c_\beta \in \pm O(|I_\beta|)$, and the map $h : L \to L$ is C^{1+s} smooth, for every $0 < s < 1$. Let $\psi : L \to L$ be the map defined by $\psi(x) = \int_{x_0}^{x} \log dh(y)dy$. The *average derivative of the map* $\psi(x)$ is defined by

$$\phi_\beta = \frac{|\int_{I_\beta} \log dh(x)dx|}{|I_\beta|}.$$

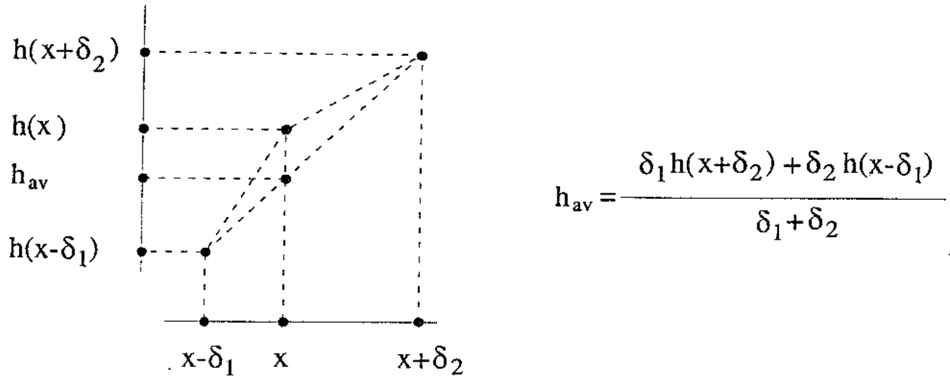


Figure 11: The graph of the map h at points $x-\delta_1$, x and $x+\delta_2$.

Using the Taylor series of the logarithmic function, and since the map h is smooth for all triples $I_\beta, I_{\beta'}, I_{\beta''}$,

$$l_\beta \in \phi_{\beta'} - \phi_\beta \pm O(|I_\beta|^{2s}). \tag{30}$$

The *average derivative function* $\phi : M \to L$ is defined by $\phi(m_\gamma) = \phi_\gamma$. Since for all triangle intervals I_γ the cross ratio distortion c_γ is an analytic function of the cross ratio distortions c_β of a bounded number of grid intervals I_β, and by assumption the cross ratio distortions $c_\beta \in \pm O(|I_\beta|)$, we have $c_\gamma \in O(|I_\gamma|)$. By (16) and (30), for all triangle triples $(I_\gamma, I_{\gamma'}, I_{\gamma''})$, the average derivative function $\phi : M \to L$ is triangle Zygmund; i. e.

$$\left| \frac{(|I_\gamma| + |I_{\gamma'}|)\phi_{\gamma''} + (|I_{\gamma'}| + |I_{\gamma''}|)\phi_\gamma}{|I_\gamma| + 2|I_{\gamma'}| + |I_{\gamma''}|} - \phi_{\gamma'} \right| \le O(|I_\gamma|). \tag{31}$$

Let $(I_\omega, I_{\omega'}, I_{\omega''})$ and $(I_{\omega'}, I_{\omega''}, I_{\omega'''})$ be two triangle triples such that

$$I_{\omega'} \cup I_{\omega''} = I_{\gamma'} \quad \text{and} \quad I_\omega \cup I_{\omega'} \cup I_{\omega''} \cup I_{\omega'''} = I_\gamma \cup I_{\gamma'} \cup I_{\gamma''}.$$

Let $\phi_{\omega,\omega'} : I_{\omega,\omega'} \to L$ be a middle affine map (see Figure 9) and $\phi_{\gamma,\gamma',\gamma''} : I_{\gamma,\gamma'} \cup I_{\gamma',\gamma''} \to L$ be a middle piecewise affine map (see Figure 10). Define the map $\phi_{\omega,\omega',\omega'',\omega'''} : I_{\omega,\omega'} \cup I_{\omega',\omega''} \cup I_{\omega'',\omega'''} \to L$ by

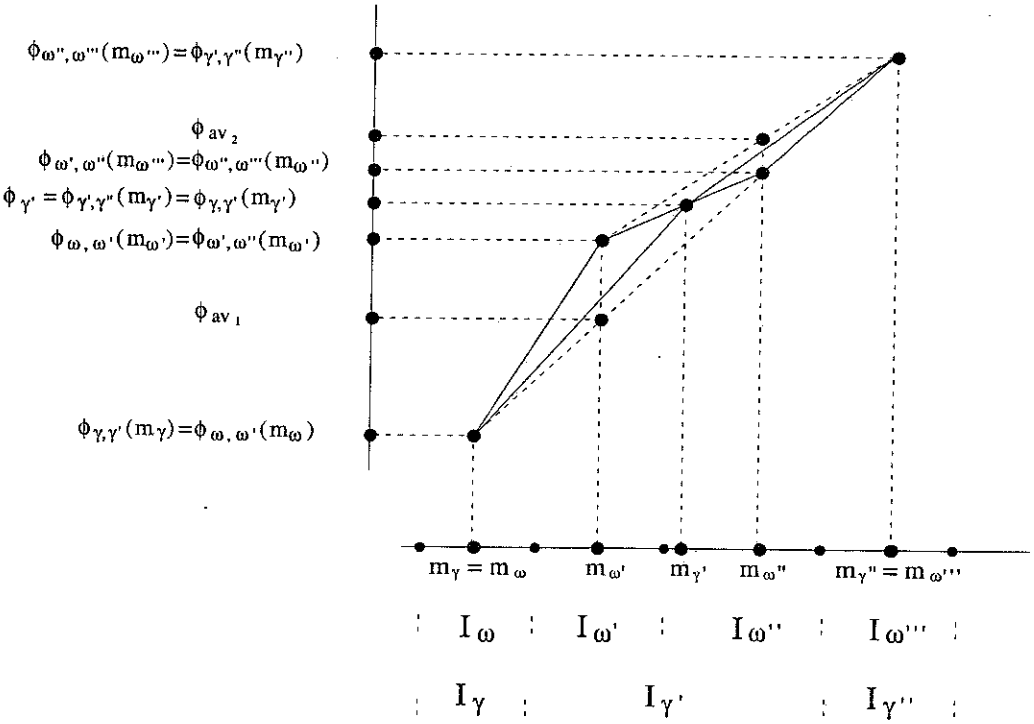
$$\phi_{\omega,\omega',\omega'',\omega'''}(x) = \begin{cases} \phi_{\omega,\omega'}(x), & \text{if } x \in I_{\omega,\omega'} \\ \phi_{\omega',\omega''}(x), & \text{if } x \in I_{\omega',\omega''} \\ \phi_{\omega'',\omega'''}(x), & \text{if } x \in I_{\omega'',\omega'''} \end{cases}$$

(see Figure 12). By definition of average derivative and since $I_{\omega'} \cup I_{\omega''} = I_{\gamma'}$, we have that

$$\phi_{\gamma'} = \frac{|m_{\omega'} - m_{\gamma'}|\phi_{\omega''} + |m_{\omega''} - m_{\gamma'}|\phi_{\omega'}}{|m_{\omega''} - m_{\omega'}|}.$$

By the triangle Zygmund property for the triangle triples $(I_\omega, I_{\omega'}, I_{\omega''})$ and $(I_{\omega'}, I_{\omega''}, I_{\omega'''})$, drawing the graphics of the maps $\phi_{\gamma,\gamma',\gamma''}$ and $\phi_{\omega,\omega',\omega'',\omega'''}$ (see Figure 12 and Figure 13), we obtain geometrically that, for all $x \in I_{\gamma,\gamma'} \cup I_{\gamma',\gamma''}$,

$$|\phi_{\omega,\omega',\omega'',\omega'''}(x) - \phi_{\gamma,\gamma',\gamma''}(x)| \le O(|I_\gamma|). \tag{32}$$



$$\phi_{av_1} = \frac{(|I_\omega| + |I_{\omega'}|)\,\phi_{\omega',\omega''}(m_{\omega''}) + (|I_{\omega'}| + |I_{\omega''}|)\,\phi_{\omega,\omega'}(m_\omega)}{|I_{\omega''}| + 2\,|I_{\omega'}| + |I_\omega|}$$

$$\phi_{av_2} = \frac{(|I_{\omega'}| + |I_{\omega''}|)\,\phi_{\omega'',\omega'''}(m_{\omega'''}) + (|I_{\omega''}| + |I_{\omega'''}|)\,\phi_{\omega',\omega''}(m_{\omega'})}{|I_{\omega'''}| + 2\,|I_{\omega''}| + |I_{\omega'}|}$$

Figure 12: The map $\phi_{\omega,\omega',\omega'',\omega'''}$ and the map $\phi_{\gamma,\gamma',\gamma''}$.

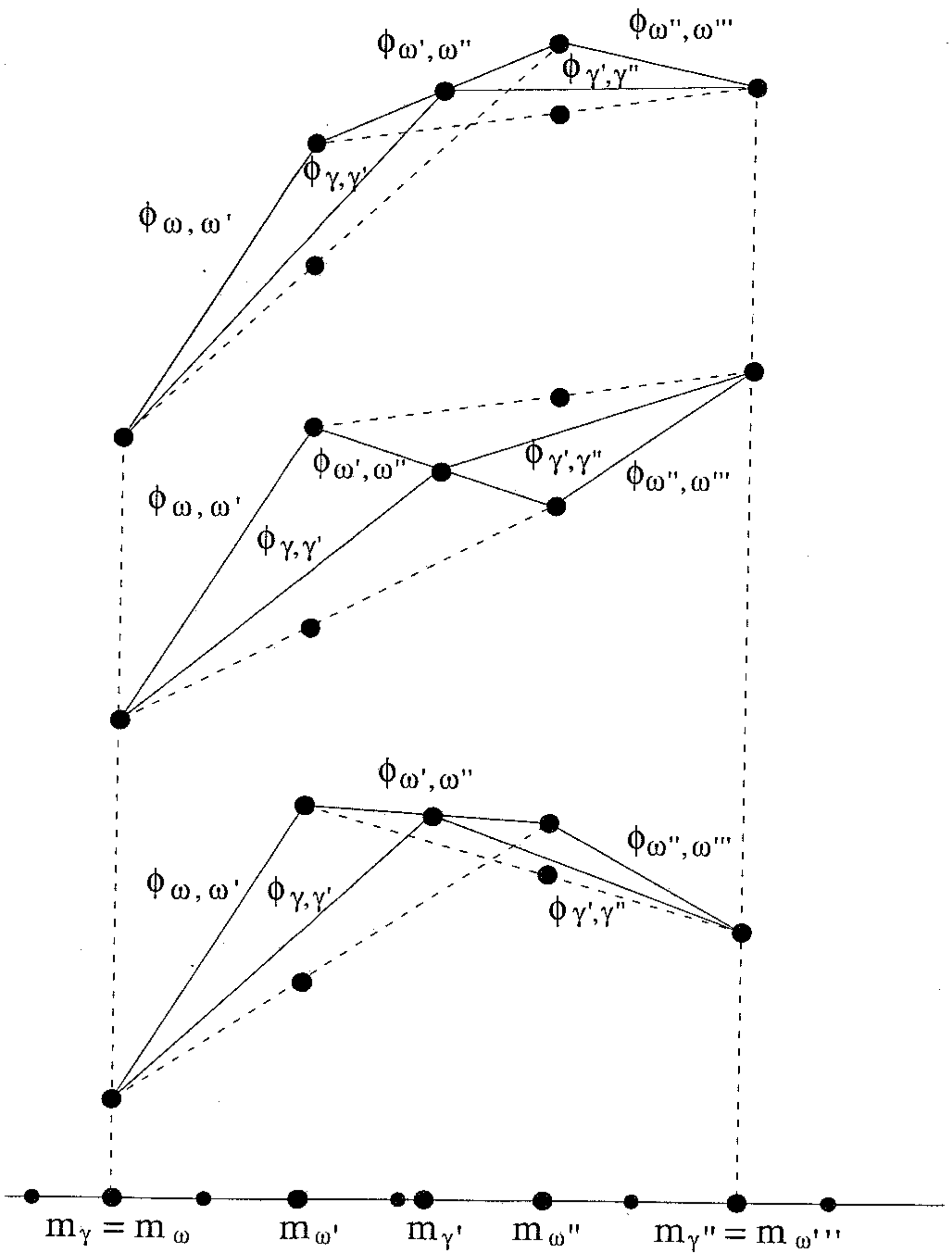


Figure 13: Graphics of the map $\phi_{\omega,\omega',\omega'',\omega'''}$ and of the map $\phi_{\gamma,\gamma',\gamma''}$.

Define the sequence $(\phi_n)_{n \in \mathbf{Z}}$ of piecewise affine maps $\phi_n : L \to \mathbf{R}$ by $\phi_n | I_{\alpha,\alpha'} = \phi_{\alpha,\alpha'}$, for all adjacent grid intervals I_γ, $I_{\gamma'}$ at level n. By definition of average derivative, the sequence of maps $\phi_n : L \to L$ converges to the map $\log dh$ in the C^0 topology, when n tends to infinity.

For all triples of points (x, y, z) in L, choose the smallest $n > 0$ such that there are adjacent grid intervals I_γ, $I_{\gamma'}$, $I_{\gamma''}$, at level n, with the property that $x, y, z \in I_{\gamma,\gamma'} \cup I_{\gamma',\gamma''}$. By the triangle Zygmund condition, the map $\phi_n | I_{\gamma,\gamma'} \cup I_{\gamma',\gamma''}$ satisfies the Zygmund condition for the points x, y, z; i. e.

$$\left| \frac{|y-x|\phi_n(z) + |z-y|\phi_n(x)}{|z-x|} - \phi_n(y) \right| \leq O(|z-x|).$$

By inequality (32) and replacing the map ϕ_n by the limiting map $\log dh$ in the Zygmund condition, we obtain that the map $\log dh : L \to L$ is Zygmund; i. e.

$$\left| \frac{|y-x| \log dh(z) + |z-y| \log dh(x)}{|z-x|} - \log dh(y) \right| \leq O(|z-x|).$$

Conversely, for all symmetric triples $(I_\beta, I_\gamma, I_\alpha)$ let $x, y, z \in L$ be the corresponding middle points. Let $x', x'' \in L$ be the endpoints of the interval I_β. Since the map $\log dh : L \to L$ is Zygmund, the integral

$$\begin{aligned} \int_{x'}^{x''} \log dh(t) dt &= \int_{x'}^{x} (\log dh(t) + \log dh(-t + 2x)) dt \\ &\in (\log dh(x) \pm O(|I_\beta|))|I_\beta|. \end{aligned}$$

Therefore,

$$\phi_\beta = \log dh(x) \pm O(|I_\beta|). \tag{33}$$

Similarly, the average derivative $\phi_\gamma \in \log dh(y) \pm O(|I_\gamma|)$ and the average derivative $\phi_\alpha \in \log dh(z) \pm O(|I_\alpha|)$. By (16), (30), and (33),

$$\begin{aligned} 2c_\beta &= \phi_\alpha - 2\phi_\gamma + \phi_\beta \\ &= O(|I_\beta|). \end{aligned}$$

The proof follows similarly for zygmund. ∎

7 Proof of Table 1.

Proof of Table 1. Let $\mathcal{L}$ and $\mathcal{L}'$ be two local leaves and $u : J = \pi_S(\mathcal{L}) \to \mathbf{R}$ and $v : J' = \pi_S(\mathcal{L}') \to$ $\mathbf{R}$ the corresponding solenoidal charts. If $J \cap J' \neq \emptyset$, let $I_{\beta_1 \ldots \beta_n} \subset J \cap J'$ be any interval at any level n of the interval partition. Let the points $\mathbf{x} \in \mathcal{L}$ and $\mathbf{y} \in \mathcal{L}'$ be such that $\pi_S(\mathbf{x}) = \pi_S(\mathbf{y}) \in S$ is the right extreme point of the interval $I_{\beta_1 \ldots \beta_n}$. Let a be the point $\tilde{\omega}(\tilde{E}^n(\mathbf{x})) \in C$ and b the point $\tilde{\omega}(\tilde{E}^n(\mathbf{y})) \in C$. By definition of the metric $|r|$, the distance $|r|(a, b)$ is equal to the supremum of $|z(I_{\beta_1 \ldots \beta_n})|$ over all solenoidal charts $z : S \setminus \{e_i\} \to (0, 1)$, where e_i is an endpoint of a Markov interval. By Lemma 2, the overlap maps $z \circ u^{-1}$ and $z \circ v^{-1}$ are $C^{1+Hölder}$ smooth. Therefore, by Lemma 1 the ratios

$$\frac{|r|(x,y)}{|u(I_{\beta_1 \ldots \beta_n})|} \quad \text{and} \quad \frac{|r|(x,y)}{|v(I_{\beta_1 \ldots \beta_n})|} \tag{34}$$

are bounded away from zero and infinity, independently of the interval $I_{\beta_1 \ldots \beta_n} \subset J \cap J'$ at level n of the interval partition and of $n \geq 1$.

Let $I_{\beta'_1\ldots\beta'_n}$ and $I_{\beta''_1\ldots\beta''_n}$ be adjacent intervals at level n of the interval partition, such that $I_{\beta'_1\ldots\beta'_n}$ is also adjacent to $I_{\beta_1\ldots\beta_n}$. By proof of Lemma 9,

$$s(a) = \frac{u(I_{\beta'_1\ldots\beta'_n})}{u(I_{\beta_1\ldots\beta_n})}, \qquad s(a+1) = \frac{u(I_{\beta''_1\ldots\beta''_n})}{u(I_{\beta'_1\ldots\beta'_n})},$$

and

$$s(b) = \frac{v(I_{\beta'_1\ldots\beta'_n})}{v(I_{\beta_1\ldots\beta_n})}, \qquad s(b+1) = \frac{v(I_{\beta''_1\ldots\beta''_n})}{v(I_{\beta'_1\ldots\beta'_n})}.$$

The interval partition of the expanding circle map E generates a grid g_u in the set $u(J \cap J')$. Therefore, Table 2 implies that the overlap map $h = v \circ u^{-1} : u(J \cap J') \to v(J \cap J')$ satisfies Table 1. ■

8 d-Quasiperiodic fixed grids g_d.

The d-quasiperiodic fixed grids g_d defined below are completely characterized by the same sequence of ratios $\underline{r} = \{r_i\}_{i \in \mathbf{Z}}$ at every level $n \geq 1$ of the grid g_d (see definition of a grid in Section 6). Moreover, the ratios $\{r_i\}_{i \in \mathbf{Z}_{\geq 0}} \in A(d)$. These ratios determine exactly the solenoid function in a dense set of the Cantor set C. By continuity of the solenoid function, the ratios $\{r_i\}_{i \in \mathbf{Z}}$ determine the solenoid function completely. The grid g_d also corresponds to the affine structure of a leaf $\mathcal{L}$ which is fixed by the solenoid map $\tilde{E}$, i. e. $\tilde{E}(\mathcal{L}) = \mathcal{L}$. By density of the leaf $\mathcal{L}$, the affine structure of the leaf $\mathcal{L}$ determines a (thca) solenoid $(\tilde{E}, \tilde{S})$.

Let $g = g_d$ be a grid of $\mathbf{R}$ such that every interval I^n_α at level n contains exactly d intervals of level $n+1$. The ratios $r^n_m = |I^n_{m+1}|/|I^n_m|$ between any two adjacent intervals I^n_m and I^n_{m+1} determine the grid g, up to an affine transformation. Let the sequences $\underline{r}^n$ at level n of the grid g be equal to $(r^n_m)_{m \in \mathbf{Z}}$ and let B denote the set of all these sequences $\underline{r}^n$, for all $n \geq 1$. The *amalgamation operator* $A : B \to B$ is defined by $A(\underline{r}) = \underline{s}$, where

$$s_i = r_{d(i-1)+1,di} \frac{1 + \sum_{m=di+1}^{d(i+1)-1} r_{di+1,m}}{1 + \sum_{m=d(i-1)+1}^{di-1} r_{d(i-1)+1,m}},$$

for all $i \in \mathbf{Z}$. The amalgamation operator $A : B \to B$ determines the sequence $\underline{r}^{n-1} = A(\underline{r}^n)$ at level $n-1$, from a sequence $\underline{r}^n$ at level n. A grid g_d is up to an affine transformation a point in the inverse limit space $\{\ldots \to A \to A\}$ of amalgamation operators A. A *fixed point of amalgamation* is a sequence $\underline{r}$ such that $A_d(\underline{r}) = \underline{r}$. A sequence $\underline{r} = (r_m)_{m \in \mathbf{Z}}$ is *d-quasiperiodic* if there is $0 < \mu < 1$ such that $|r_j - r_k| \leq O(\mu^i)$, when $(j-k)/d^i$ is an integer.

Definition 5 Let $\underline{r}$ be a d-quasiperiodic sequence which is a fixed point of amalgamation. A *d-quasiperiodic fixed grid* g_d is determined, up to affine transformations, by the sequence $\ldots\underline{r}\underline{r}$.

Lemma 13 There is a one-to-one correspondence between d-quasiperiodic fixed grids g and affine structures on a leaf $\mathcal{L} = \tilde{E}(\mathcal{L})$ contained in a (thca) solenoid $(\tilde{E}, \tilde{S})$.

Proof: Let $\mathcal{L}$ be a leaf of the (thca) solenoid $(\tilde{E}, \tilde{S})$ which contains a fixed point x_0 of the solenoid map $\tilde{E}$. The leaf $\mathcal{L}$ is marked by the points $\ldots, \mathrm{x}_{-1}, \mathrm{x}_0, \mathrm{x}_1, \ldots$, which project on the same point of the circle as the fixed point $\mathbf{x}_0$, and such that there is a local leaf $\mathcal{L}_m$ with extreme points $\mathbf{x}_m$ and $\mathbf{x}_{m+1}$ which does not contain any other point $\mathbf{x}_j$, for $m \neq j \neq m+1$.

The affine structure on the leaf $\mathcal{L}$ determines the ratios $r_m = r(\mathbf{x}_{m-1}, \mathbf{x}_m, \mathbf{x}_{m+1})$ of the leaf ratio function $r : T \to \mathbf{R}^+$, for all $m \in \mathbf{Z}$. For every $m \in \mathbf{Z}$, the ratios r_m coincide with the value of the solenoid function $s(\sum_{n=1}^{\infty} a_n d^n)$, where $m = \sum_{n=1}^{\infty} a_n d^n$. Note that the sequence $r_0, r_1, \ldots \in A(d)$.

Since the solenoid map $\tilde{E}$ is affine and $\tilde{E}(\mathcal{L}) = \mathcal{L}$, the sequence of ratios $\underline{r} = (r_m)_{m\in\mathbf{Z}}$ is fixed by the amalgamation operator A_d (see Figure 14). The Hölder transversality of the solenoid $(\tilde{E}, \tilde{S})$ implies that the sequence $\underline{r}$ is d-quasiperiodic. Therefore, the element $\ldots \underline{rr}$ is a d-quasiperiodic fixed grid $g_d = \ldots \underline{rr}$.

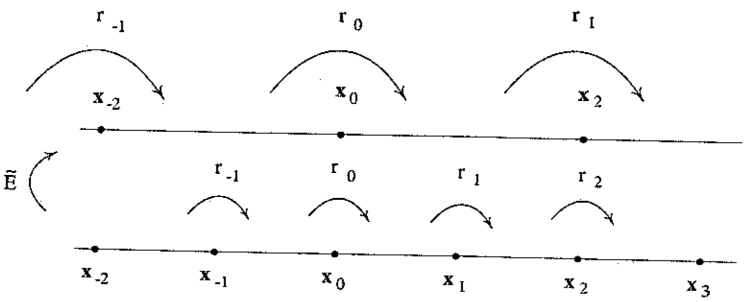


Figure 14: The leaf $\mathcal{L}$ fixed by the solenoid map $\tilde{E}$.

Conversely, a d-quasiperiodic fixed grid $g_d = \ldots \underline{rr}$ determines uniquely the affine structure of a leaf $\mathcal{L}$ which is fixed by the solenoid map $\tilde{E}$. Since $\underline{r}$ is a fixed point of the amalgamation operator A_d, the solenoid map $\tilde{E}$ is affine on the leaf $\mathcal{L}$. By density of the leaf $\mathcal{L}$ on the solenoid $\tilde{S}$ and since the grid g_d is d-quasiperiodic, the affine structure of the leaf $\mathcal{L}$ extends to an affine structure transversally Hölder continuous on the solenoid $\tilde{S}$, such that the solenoid map $\tilde{E}$ leaves the affine structure invariant.

■

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