Geometric Topology, part I
Localization, Periodicity, and Galois Symmetry

by

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This compulsion to localize began with the author's work on invariants of combinatorial manifolds in 1965-67. It was clear from the beginning that the prime 2 and the odd primes had to be treated differently.

This point arises algebraically when one looks at the invariants of a quadratic form[1]. (Actually for manifolds only characteristic 2 and characteristic zero invariants are considered.)

The point arises geometrically when one tries to see the extent of these invariants. In this regard the question of representing cycles by submanifolds comes up. At 2 every class is representable. At odd primes there are many obstructions. (Thom).

The invariants at odd primes required more investigation because of the simple non-representing fact about cycles. The natural invariant is the signature invariant of M – the function which assigns the "signature of the intersection with M" to every closed submanifold of a tubular neighborhood of M in Euclidean space.

A natural algebraic formulation of this invariant is that of a canonical K-theory orientation

$$\triangle_M \in K\text{-homology of } M\,.$$

In section 6 we discuss this situation in the dual context of bundles. This (Alexander) duality between manifold theory and bundle theory depends on transversality and the geometric technique of surgery. The duality is sharp in the simply connected context.

Thus in this work we treat only the dual bundle theory – however motivated by questions about manifolds.

The bundle theory is homotopy theoretical and amenable to the arithmetic discussions in the first sections. This discussion concerns the problem of "tensoring homotopy theory" with various rings. Most notable are the cases when $\mathbb{Z}$ is replaced by the rationals $\mathbb{Q}$ or the p-adic integers $\hat{\mathbb{Z}}_p$.

These localization processes are motivated in part by the 'invariants discussion' above. The geometric questions do not however motivate going as far as the p-adic integers.[2]

One is led here by Adams' work on fibre homotopy equivalences between vector bundles – which is certainly germane to the manifold questions above. Adams finds that a certain basic homotopy relation should hold between vector bundles related by his famous operations ψ^k.

Adams proves that this relation is *universal* (if it holds at all) – a very provocative state of affairs.

Actually Adams states infinitely many relations – one for each prime p. Each relation has information at every prime not equal to p.

At this point Quillen noticed that the Adams conjecture has an analogue in characteristic p which is immediately provable. He suggested that the etale homotopy of mod p algebraic varieties be used to decide the topological Adams conjecture.

Meanwhile, the Adams conjecture for vector bundles was seen to influence the structure of piecewise linear and topological theories.

The author tried to find some topological or geometric understanding of Adams' phenomenon. What resulted was a reformulation which can be proved just using the existence of an algebraic construction of the finite cohomology of an algebraic variety (etale theory).

This picture which can only be described in the context of the p-adic integers is the following – in the p-adic context the theory of vector bundles *in each dimension* has a natural group of symmetries.

These symmetries in the $(n-1)$ dimensional theory provide canonical fibre homotopy equivalence in the n dimensional theory which more than prove the assertion of Adams. In fact each orbit of the action has a well defined (unstable) fibre homotopy type.

The symmetry in these vector bundle theories is the Galois symmetry of the roots of unity homotopy theoretically realized in the 'Čech nerves' of algebraic coverings of Grassmannians.

The symmetry extends to K-theory and a dense subset of the symmetries may be identified with the "isomorphic part of the Adams operations". We note however that this identification is not essential in the development of consequences of the Galois phenomena. The fact that certain complicated expressions in exterior powers of vector bundles give good operations in K-theory is more a testament to Adams' ingenuity than to the ultimate naturality of this viewpoint.

The Galois symmetry (because of the K-theory formulation of the signature invariant) extends to combinatorial theory and even topological theory (because of the triangulation theorems of Kirby-Siebenmann). This symmetry can be combined with the periodicity of geometric topology to extend Adams' program in several ways –

i) the homotopy relation implied by conjugacy under the action of the Galois group holds in the topological theory and is also *universal* there.

ii) an explicit calculation of the effect of the Galois group on the topology can be made –

for vector bundles E the signature invariant has an analytical description,

$$\triangle_E \text{ in } K_C(E)\,,$$

and the topological type of E is measured by the effect of the Galois group on this invariant.

One consequence is that two different vector bundles which are fixed by elements of finite order in the Galois group are also topologically distinct. For example, at the prime 3 the torsion subgroup is generated by complex conjugation – thus any pair of non isomorphic vector bundles are topologically distinct at 3.

The periodicity alluded to is that in the theory of fibre homotopy equivalences between PL or topological bundles (see section 6 - Normal Invariants).

For odd primes this theory is isomorphic to K-theory, and geometric periodicity becomes Bott periodicity. (For non-simply connected manifolds the periodicity finds beautiful algebraic expression in the surgery groups of C. T. C. Wall.)

To carry out the discussion of section 6 we need the works of the first five chapters.

The main points are contained in chapters 3 and 5.

In chapter 3 a description of the p-adic completion of a homotopy type is given. The resulting object is a homotopy type with the extra structure[3] of a compact topology on the contravariant functor it determines.

The p-adic types one for each p can be combined with a rational homotopy type (section 2) to build a classical homotopy type.

One point about these p-adic types is that they often have symmetry which is not apparent or does not exist in the classical context. For example in section 4 where p-adic spherical fibrations are discussed, we find from the extra symmetry in $\mathbb{C}P^\infty$, p-adically completed, one can construct a theory of principal spherical fibrations (one for each divisor of $p-1$).

Another point about p-adic homotopy types is that they can be naturally constructed from the Grothendieck theory of etale cohomology in algebraic geometry. The long chapter 5 concerns this etale theory which we explicate using the Čech like construction of Lubkin. This construction has geometric appeal and content and should yield many applications in geometric homotopy theory.[4]

To form these p-adic homotopy types we use the inverse limit technique of section 3. The arithmetic square of section 3 shows what has to be added to the etale homotopy type to give the classical homotopy type.[5]

We consider the Galois symmetry in vector bundle theory in some detail and end with an attempt to analyze "real varieties". The attempt leads to an interesting topological conjecture.

Section 1 gives some algebraic background and preparation for the later sections. It contains the examples of profinite groups in topology and algebra that concern us here.

In part II we study the prime 2 and try to interpret geometrically the structure in section 6 on the manifold level. We will also pursue the idea of a localized manifold – a concept which has interesting examples from algebra and geometry.

Finally, we acknowledge our debt to John Morgan of Princeton University – who mastered the lion's share of material in a few short months with one lecture of suggestions. He prepared an earlier manuscript on the beginning sections and I am certain this manuscript would not have appeared now (or in the recent future) without his considerable efforts.

Also, the calculations of Greg Brumfiel were psychologically invaluable in the beginning of this work. I greatly enjoyed and benefited from our conversations at Princeton in 1967 and later.

Notes

1 Which according to Winkelnkemper "... is the basic discretization of a compact manifold."

2 Although the Hasse-Minkowski theorem on quadratic forms should do this.

3 which is "intrinsic" to the homotopy type in the sense of interest here.

4 The study of homotopy theory that has geometric significance by geometrical qua homotopy theoretical methods.

5 Actually it is a beginning.

Section 1. Algebraic Constructions

We will discuss some algebraic constructions. These are localization and completion of rings and groups. We consider properties of each and some connections between them.

Localization

Unless otherwise stated rings will have units and be integral domains.

Let R be a ring. $S \subseteq R - \{0\}$ is a multiplicative subset if $1 \in S$ and $a, b \in S$ implies $a \cdot b \in S$.

Definition 1.1 *If $S \subseteq R - \{0\}$ is a multiplicative subset then*

$$S^{-1}R\,, \quad \text{“}R \textit{ localized away from } S\text{”}$$

is defined as equivalence classes

$$\{r/s \mid r \in R, s \in S\}$$

where

$$r/s \sim r'/s' \quad \textit{iff} \quad rs' = r's\,.$$

$S^{-1}R$ is made into a ring by defining

$$[r/s] \cdot [r'/s'] = [rr'/ss'] \text{ and}$$

$$[r/s] + [r'/s'] = \left[\frac{rs' + sr'}{ss}\right]\,.$$

The *localization homomorphism*

$$R \to S^{-1}R$$

sends r into $[r/1]$.

EXAMPLE 1 If $p \subset R$ is a prime ideal, $R-p$ is a multiplicative subset. Define

$$R_p\,, \quad \text{“}R \text{ localized at } p\text{”}$$

as $(R-p)^{-1}R$.

In R_p every element outside p is invertible. The localization map $R \to R_p$ sends p into the unique maximal ideal of non-units in R_p.

If R is an integral domain 0 is a prime ideal, and R localized at zero is the field of quotients of R.

The localization of the ring R extends to the theory of modules over R. If M is an R-module, define the localized $S^{-1}R$-module, $S^{-1}M$ by

$$S^{-1}M = M \otimes_R S^{-1}R\,.$$

Intuitively $S^{-1}M$ is obtained by making all the operations on M by elements of S into isomorphisms.

Interesting examples occur in topology.

EXAMPLE 2 (P. A. Smith, A. Borel, G. Segal) Let X be a locally compact polyhedron with a symmetry of order 2 (involution), T.

What is the relation between the homology of the subcomplex of fixed points F and the “homology of the pair (X,T)”?

Let S denote the (contractible) infinite dimensional sphere with its antipodal involution. Then $X \times S$ has the diagonal fixed point free involution and there is an equivariant homotopy class of maps

$$X \times S \to S$$

(which is unique up to equivariant homotopy). This gives a map

$$X_T \equiv (X \times S)/T \to S/T \equiv \mathbb{R}\,P^\infty$$

and makes the “equivariant cohomology of (X,T)”

$$H^*(X_T; \mathbb{Z}\,/2)$$

into an R-module, where

$$R \;=\; \mathbb{Z}_2\,[x] = H^*(\mathbb{R}\,P^\infty;\mathbb{Z}\,/2)\,.$$

In R we have the maximal ideal generated by x, and the cohomology of the fixed points with coefficients in the local ring, $R_x = R$ localized at x is just the localized equivariant cohomology,

$$H^*(F;R_x) \cong H^*(X_T;\mathbb{Z}\,/2) \text{ localized at } x \equiv H^*(X_T;\mathbb{Z}\,/2) \otimes_R R_x\,.$$

For most of our work we do not need this general situation of localization. We will consider most often the case where R is the ring of integers and the R-modules are arbitrary Abelian groups.

Let ℓ be a set of primes in $\mathbb{Z}$. We will write "$\mathbb{Z}$ localized at ℓ"

$$\mathbb{Z}_\ell = S^{-1}\,\mathbb{Z}$$

where S is the multiplicative set generated by the primes *not* in ℓ.

When ℓ contains only one prime $\ell = \{p\}$, we can write

$$\mathbb{Z}_\ell = \mathbb{Z}_p$$

since $\mathbb{Z}_\ell$ is just the localization of the integers at the prime ideal p.

Other examples are

$$\mathbb{Z}_{\{\text{all primes}\}} = \mathbb{Z} \;\text{ and }\; \mathbb{Z}_\emptyset = \mathbb{Q} = \mathbb{Z}_0\;.$$

In general, it is easy to see that the collection of $\mathbb{Z}_\ell$'s

$$\{\mathbb{Z}_\ell\}$$

is just the collection of subrings of $\mathbb{Q}$ with unit. We will see below that the tensor product over $\mathbb{Z}$,

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell'} \cong \mathbb{Z}_{\ell\cap\ell'}$$

and the fibre product over $\mathbb{Q}$

$$\mathbb{Z}_\ell \times_{\mathbb{Q}} \mathbb{Z}_{\ell'} \cong \mathbb{Z}_{\ell\cup\ell'}\;.$$

We localize Abelian groups at ℓ as indicated above.

Definition 1.2 *If G is an Abelian group then the localization of G with respect to a set of primes ℓ, G_ℓ is the $\mathbb{Z}_\ell$-module*

$$G \otimes \mathbb{Z}_\ell \,.$$

The natural inclusion $\mathbb{Z} \to \mathbb{Z}_\ell$ induces the "localization homomorphism"

$$G \to G_\ell \,.$$

We can describe localization as a direct limit procedure.

Order the multiplicative sets $\{s\}$ of products of primes not in ℓ by divisibility.

Form a directed system of groups and homomorphisms by directed set $= \{s\}$; $G_s \equiv G$; $G_s \xrightarrow{\text{multiplication by } s/s'} G_{s'}$ if $s \leqslant s'$.

Proposition 1.1

$$\varinjlim_s G_s \cong G \otimes \mathbb{Z}_\ell \equiv G_\ell \,.$$

Proof: Define compatible maps

$$G_s \to G \otimes \mathbb{Z}_\ell$$

by $g \mapsto g \otimes 1/s$. These determine

$$\varinjlim_s G_s \to G \otimes \mathbb{Z}_\ell \,.$$

In case $G = \mathbb{Z}$ this map is clearly an isomorphism. (Each map $\mathbb{Z} \to \mathbb{Z}_\ell$ is an injection thus the direct limit injects. Also a/s in $\mathbb{Z}_\ell$ is in the image of $\mathbb{Z} = G_s \to \mathbb{Z}_\ell$.)

The general case follows since taking direct limits commutes with tensor products.

Lemma 1.2 *If ℓ and ℓ' are two sets of primes, then $\mathbb{Z}_\ell \otimes \mathbb{Z}_{\ell'}$ is isomorphic to $\mathbb{Z}_{\ell\cap\ell'}$ as rings.*

Proof: Define a map on generators

$$\mathbb{Z}_\ell \otimes \mathbb{Z}_{\ell'} \xrightarrow{\rho} \mathbb{Z}_{\ell\cap\ell'}$$

by $\rho(a/b \otimes a'/b') = aa'/bb'$. Since b is a product of primes outside ℓ and b' is a product of primes outside ℓ', bb' is a product of primes outside $\ell \cap \ell'$ and ρ is well defined.

To see that ρ is onto, take r/s in $\mathbb{Z}_{\ell\cap\ell'}$ and factor $s = s_1 s_2$ so that "s_1 is outside ℓ" and "s_2 is outside ℓ'." Then $\rho(1/s_1 \otimes r/s_2) = r/s$.

To see that ρ is an embedding assume

$$\sum_i a_i/b_i \otimes c_i/d_i \xrightarrow{\rho} 0\,.$$

Then $\sum_i a_i c_i/b_i d_i = 0$, or

$$\sum_i a_i c_i \prod_{i\neq j} b_j d_j \;=\; 0\,.$$

This means that

$$\begin{aligned}
\sum_i a_i/b_i \otimes c_i/d_i &= \sum_i a_i c_i (1/b_i \otimes 1/d_i) \\
&= \sum_i \big(\prod_{i\neq j} b_j d_j a_i c_i\big)\big(1/\prod_h b_h \otimes 1/\prod_h d_h\big) \\
&= 0
\end{aligned}$$

and ρ has no kernel.

Lemma 1.3 *The $\mathbb{Z}$-module structure on an Abelian group G extends to a $\mathbb{Z}_\ell$-module structure if and only if G is isomorphic to its localizations at every set of primes containing ℓ.*

Proof: This follows from Proposition 1.1.

Example 3

$$(\mathbb{Z}\,/p^n)_\ell \equiv \mathbb{Z}\,/p^n \otimes \mathbb{Z}_\ell \equiv \begin{cases} 0 & p \notin \ell \\ \mathbb{Z}\,/p^n & p \in \ell \end{cases}$$

$$\begin{pmatrix}\text{finitely generated}\\ \text{Abelian group } G\end{pmatrix}_\ell \cong \underbrace{\mathbb{Z}_\ell \oplus \cdots \oplus \mathbb{Z}_\ell}_{\text{rank } G \text{ factors}} \oplus \mathbb{Z}_\ell \oplus l\text{-torsion } G$$

Proposition 1.4 *Localization takes exact sequences of Abelian groups into exact sequences of Abelian groups.*

Proof: This also follows from Proposition 1.1 since passage to a direct limit preserves exactness.

Corollary 1.5 *If $0 \to A \to B \to C \to 0$ is an exact sequence of*

Abelian groups and two of the three groups are $\mathbb{Z}_\ell$-modules then so is the third.

PROOF: Consider the localization diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\otimes \mathbb{Z}_\ell"] & B \arrow[r] \arrow[d, "\otimes \mathbb{Z}_\ell"] & C \arrow[r] \arrow[d, "\otimes \mathbb{Z}_\ell"] & 0 \\ 0 \arrow[r] & A_\ell \arrow[r] & B_\ell \arrow[r] & C_\ell \arrow[r] & 0 \end{tikzcd}$$

The lower sequence is exact by Proposition 1.4. By hypothesis and Lemma 1.2 two of the maps are isomorphisms. By the five lemma the third is also.

COROLLARY 1.6 *If in the long exact sequence*

$$\cdots \to A_n \to B_n \to C_n \to A_{n-1} \to B_{n-1} \to \ldots$$

two of the three sets of groups

$$\{A_n\},\ \{B_n\},\ \{C_n\}$$

are $\mathbb{Z}_\ell$-modules, then so is the third.

PROOF: Apply the five lemma as above.

COROLLARY 1.7 *Let $F \to E \to B$ be a Serre fibration of connected spaces with Abelian fundamental groups. Then if two of*

$$\pi_* F,\ \pi_* E,\ \pi_* B$$

are $\mathbb{Z}_\ell$-modules the third is also.

PROOF: This follows from the exact homotopy sequence

$$\cdots \to \pi_i F \to \pi_i E \to \pi_i B \to \ldots .$$

This situation extends easily to homology.

PROPOSITION 1.8 *Let $F \to E \to B$ be a Serre fibration in which $\pi_1 B$ acts trivially on $\widetilde{H}_*(F; \mathbb{Z}/p)$ for primes p not in ℓ. Then if two of the* integral

$$\widetilde{H}_* F,\ \widetilde{H}_* E,\ \widetilde{H}_* B$$

are $\mathbb{Z}_\ell$-modules, the third is also.

PROOF: $\widetilde{H}_* X$ is a $\mathbb{Z}_\ell$-module iff $\widetilde{H}_*(X; \mathbb{Z}/p)$ vanishes for p not in ℓ. This follows from the exact sequence of coefficients

$$\cdots \to \widetilde{H}_i(X) \xrightarrow{p} \widetilde{H}_i(X) \to \widetilde{H}_i(X; \mathbb{Z}/p) \to \ldots .$$

But from the Serre spectral sequence with $\mathbb{Z}/p$ coefficients we can conclude that if two of

$$\widetilde{H}_*(F;\mathbb{Z}/p),\ \widetilde{H}_*(E;\mathbb{Z}/p),\ \widetilde{H}_*(B;\mathbb{Z}/p)$$

vanish the third does also.

NOTE: We are indebted to D. W. Anderson for this very simple proof of Proposition 1.8.

Let us say that a square of Abelian groups

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "l"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

is a *fibre square* if the sequence

$$0 \to A \xrightarrow{i \oplus j} B \oplus C \xrightarrow{l-k} D \to 0$$

is exact.

LEMMA 1.9 *The direct limit of fibre squares is a fibre square.*

PROOF: The direct limit of exact sequences is an exact sequence.

PROPOSITION 1.10 *If G is any Abelian group and ℓ and ℓ' are two sets of primes such that*

$$\ell \cap \ell' = \emptyset,\ \ell \cup \ell' = \textit{all primes}$$

then

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G \otimes \mathbb{Z}_\ell \arrow[d] \\ G \otimes \mathbb{Z}_{\ell'} \arrow[r] & G \otimes \mathbb{Q} \end{tikzcd}$$

is a fibre square.

PROOF:

Case 1: $G = \mathbb{Z}$: an easy argument shows

$$0 \to \mathbb{Z} \to \mathbb{Z}_\ell \oplus \mathbb{Z}_{\ell'} \to \mathbb{Q} \to 0$$

is exact.

Case 2: $G = \mathbb{Z}/p^\alpha$, the square reduces to

$$\begin{tikzcd} \mathbb{Z}/p^\alpha \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{Z}/p^\alpha \arrow[r] & 0 \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \mathbb{Z}/p^\alpha \arrow[r, "\cong"] \arrow[d] & \mathbb{Z}/p^\alpha \arrow[d] \\ 0 \arrow[r] & 0\,. \end{tikzcd}$$

Case 3: G is a finitely generated group: this is a finite direct sum of the first two cases.

Case 4: G any Abelian group: this follows from case 3 and Lemma 1.9.

We can paraphrase the proposition "G is the fibre product of its localizations G_ℓ and $G_{\ell'}$ over G_0,"

More generally, we have

Meta Proposition 1.12 *Form the infinite diagram*

$$\begin{tikzcd} G_2 \arrow[dr] & G_3 \arrow[d] & G_5 \ldots \arrow[dl] \\ & G_0 & \end{tikzcd}$$

Then G is the infinite fibre product of its localizations G_2, G_3, … over G_0.

Proof: The previous proposition shows $G_{(2,3)}$ is the fibre product of $G_{(2)}$ and $G_{(3)}$ over $G_{(0)}$. Then $G_{(2,3,5)}$ is the fibre product of $G_{(2,3)}$ and $G_{(5)}$ over $G_{(0)}$, etc. This description depends on ordering the primes; however since the particular ordering used is immaterial the statement should be regarded symmetrically.

Completions

We turn now to completion of rings and groups. As for rings we are again concerned mostly with the ring of integers for which we discuss the "arithmetic completions". In the case of groups we consider profinite completions and for Abelian groups related formal completions.

At the end of the section we consider some examples of profinite groups in topology and algebra and discuss the structure of the p-adic units.

Finally we consider connections between localizations and completions, deriving certain fibre squares which occur later on the CW complex level.

Completion of Rings – the p-adic Integers

Let R be a ring with unit. Let

$$I_1 \supset I_2 \supset \dots$$

be a decreasing sequence of ideals in R with

$$\bigcap_{j=1}^{\infty} I_j = \{0\}\,.$$

We can use these ideals to define a metric on R, namely

$$d(x,y) = e^{-k},\ \ e > 1$$

where $x-y \in I_k$ but $x-y \notin I_{k+l}$, $(I_0 = R)$. If $x-y \in I_k$ and $y-z \in I_l$ then $x-z \in I_{\min(k,\,l)}$. Thus

$$d(x,z) \leqslant \max\bigl(d(x,y), d(y,z)\bigr)\,,$$

a strong form of triangle inequality. Also, $d(x,y) = 0$ means

$$x - y \ \in \ \bigcap_{j=0}^{\infty} I_j = \{0\}\,.$$

This means that d defines a distance function on the ring R.

Definition 1.3 *Given a ring with metric d, define the completion of R with respect to d, $\widehat{R}_d$, by the Cauchy sequence procedure. That is, form all sequences in R, $\{x_n\}$, so that*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0\,.$$ [1]

Make $\{x_n\}$ equivalent to $\{y_n\}$ if $d(x_n,y_n) \to 0$. Then the set of equivalence classes $\widehat{R}_d$ is made into a topological ring by defining

$$[\{x_n\}] + [\{y_n\}] = [\{x_n + y_n\}]\,,$$

$$[\{x_n\}] \cdot [\{y_n\}] = [\{x_n y_n\}]\,.$$

There is a natural completion homomorphism

$$R \xrightarrow{c} \widehat{R}_d$$

sending r into $[\{r, r, \dots\}]$. c is universal with respect to continuous ring maps into complete topological rings.

EXAMPLE 1 Let $I_j = (p^j) \subseteq \mathbb{Z}$. The induced topology is the p-adic topology on $\mathbb{Z}$, and the completion is the ring of p-adic integers, $\widehat{\mathbb{Z}}_p$.

The ring $\widehat{\mathbb{Z}}_p$ was constructed by Hensel to study Diophantine equations. A solution in $\widehat{\mathbb{Z}}_p$ corresponds to solving the associated Diophantine congruence modulo arbitrarily high powers of p.

Solving such congruences for all moduli becomes equivalent to an infinite number of independent problems over the various rings of p-adic numbers.

Certain non-trivial polynomials can be completely factored in $\widehat{\mathbb{Z}}_p$, for example

$$x^{p-1} - 1$$

(see the proof of Proposition 1.16.)

Thus here and in other situations we are faced with the pleasant possibility of studying independent p-adic projections of familiar problems over $\mathbb{Z}$ armed with such additional tools as $(p-1)^{\text{st}}$ roots of unity.

EXAMPLE 2 Let ℓ be a non-void set of primes $(p_1, p_2, \dots)$. Define

$$I_j^l = (p_1^j p_2^j \dots p_j^j)\,.$$

The resulting topology on $\mathbb{Z}$ is the ℓ-adic topology and the completion is denoted $\widehat{\mathbb{Z}}_\ell$.

If $\ell' \subset \ell$ then $I_j^\ell > I_j^{\ell'}$ and any Cauchy sequence in the ℓ-adic topology is Cauchy in the ℓ'-adic topology. This gives a map

$$\widehat{\mathbb{Z}}_\ell \to \widehat{\mathbb{Z}}_{\ell'}\,.$$

PROPOSITION 1.13 *Form the inverse system of rings* $\{\mathbb{Z}/p^n\}$, *where* $\mathbb{Z}/p^n \to \mathbb{Z}/p^m$ *is a reduction mod* p^m *whenever* $n \geqslant m$. *Then there is a natural ring isomorphism*

$$\widehat{\mathbb{Z}}_p \underset{\cong}{\xrightarrow{\widehat{\rho}_p}} \varprojlim \{\mathbb{Z}/p^n\}\,.$$

Proof: First define a ring homomorphism

$$\widehat{\mathbb{Z}}_p \underset{\rho_n}{\longrightarrow} \mathbb{Z}/p^n\,.$$

If $\{x_i\}$ is a Cauchy sequence in $\mathbb{Z}$, the p^n residue of x_i is constant for large i so define

$$\widehat{\rho}_n[\{x_i\}] = \text{stable residue } x_i\,.$$

If $\{x_i\}$ is equivalent to $\{y_i\}$, p^n eventually divides every $x_i - y_i$, so ρ_n is well defined.

The collection of homomorphisms ρ_n are clearly onto and compatible with the maps in the inverse system. Thus they define

$$\widehat{\mathbb{Z}}_p \underset{\widehat{\rho}_p}{\longrightarrow} \underset{\leftarrow}{\lim}\, \mathbb{Z}/p^n\,.$$

$\widehat{\rho}_p$ is injective. For $\widehat{\rho}_p\{x_i\} = 0$, means p^n eventually divides x_i for all n. Thus $\{x_i\}$ is eventually in I_n for every n. This is exactly the condition that $\{x_i\}$ is equivalent to $\{0, 0, 0, \dots\}$.

$\widehat{\rho}_p$ is surjective. If (r_i) is a compatible sequence of residues in $\underset{\leftarrow}{\lim}\, \mathbb{Z}/p^n$, let $\{\widetilde{r}_i\}$ be a sequence of integers in this sequence of residue classes. $\{\widetilde{r}_i\}$ is clearly a Cauchy sequence and

$$\widehat{\rho}_p\{\widetilde{r}_i\} = (r_i) \in \underset{\leftarrow}{\lim}\, \mathbb{Z}/p^n$$

Corollary $\widehat{\mathbb{Z}}_p$ *is compact.*

Proof: The isomorphism $\widehat{\rho}_p$ is continuous with respect to the inverse limit topology on $\underset{\leftarrow}{\lim}\, \mathbb{Z}/p^n$.

Proposition 1.14 *The product of the natural maps $\mathbb{Z}_\ell \to \mathbb{Z}_p$, $p \in \ell$ yields an isomorphism of rings*

$$\widehat{\mathbb{Z}}_\ell \underset{\cong}{\longrightarrow} \prod_{p \in \ell} \widehat{\mathbb{Z}}_p\,.$$

Proof: The argument of Proposition 1.13 shows that $\mathbf{Z}_\ell$ is an inverse limit of finite ℓ-rings

$$\mathbb{Z}/p_1^j \dots p_j^j,\ \ell = \{p_1, p_2, \dots\}\,.$$

But

$$\begin{aligned}
\varprojlim_j \mathbb{Z}/p_1^j \dots p_j^j &= \varprojlim_j \prod_{i=1}^{j} \mathbb{Z}/p_i^j \\
&= \varprojlim_{(n,j)} \prod_{i=1}^{n} \mathbb{Z}/p_i^j \\
&= \varprojlim_n \varprojlim_j \prod_{i=1}^{n} \mathbb{Z}/p_i^j \\
&= \varprojlim_n \prod_{i=1}^{n} \varprojlim_j \mathbb{Z}/p_i^j \\
&= \varprojlim_n \prod_{i=1}^{n} \widehat{\mathbb{Z}}_{p_i} \\
&= \prod_{p\in\ell} \widehat{\mathbb{Z}}_p \,.
\end{aligned}$$

NOTE: $\widehat{\mathbb{Z}}_\ell$ is a ring with unit, but unlike $\widehat{\mathbb{Z}}_p$ it is not an integral domain if ℓ contains more than one prime. Like $\widehat{\mathbb{Z}}_p$, $\widehat{\mathbb{Z}}_\ell$ is compact and topologically cyclic – the multiples of one element can form a dense set.

EXAMPLE 3 $\big(K(\mathbb{R}P^\infty)$, Atiyah$\big)$
Let R be the ring of virtual complex representations of $\mathbb{Z}/2$,

$$R \cong \mathbb{Z}[x]/(x^2-1)\,.$$

$n+mx$ corresponds to the representation

$$\{1,0\} \to \left.\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} m \end{matrix}\right. \,,\quad \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

of $\mathbb{Z}/2$ on $\mathbb{C}^{n+m}$. Let I_j be the ideal generated by $(x-1)^j$. The completion of the representation ring R with respect to this topology is naturally isomorphic to the complex K-theory of $\mathbb{R}P^\infty$,

$$\widehat{R} \cong K(\mathbb{R}P^\infty) \equiv [\mathbb{R}P^\infty, \mathbb{Z}\times BU]\,.$$

It is easy to see that additively this completion of R is isomorphic to the integers direct sum the 2-adic integers.

Example 4 (K(fixed point set), Atiyah and Segal) Consider again the compact space X with involution T, fixed point set F, and 'homotopy theoretical orbit space', $X_T = X \times S^\infty/((x,s) \sim (Tx,-s))$.

We have the Grothendieck ring of equivariant vector bundles over X, $K_G(X)$ – a ring over the representation ring R. $K_G(X)$ is a rather subtle invariant of the geometry of (X,T). However, Atiyah and Segal show that

i) the completion of $K_G(X)$ with respect to the ideals $(x-1)^j K_G(X)$ is the K-theory of X_T.

ii) the completion of $K_G(X)$ with respect to the ideals $(x+1)^j K_G(X)$ is related to $K(F)$.

If we complete $K_G(X)$ with respect to the ideals $(x-1,x+1)^j K_G(X)$ (which is equivalent to 2-adic completion)[2] we obtain the isomorphism

$$K(F) \otimes \widehat{\mathbb{Z}}_2[x]/(x^2-1) \cong K(X_T)\hat{_2}\,.$$

We will use this relation in section 5 to give an 'algebraic description' of the K-theory[3] of the real points on a real algebraic variety.

Completions of Groups

Now we consider two kinds of completions for groups. First there are the profinite completions.

Let G be any group and ℓ a *non-void* set of primes in $\mathbb{Z}$. Denote the collection of those normal subgroups of G with index a product of primes in ℓ by $\{H\}_\ell$.

Now $\{H\}_\ell$ can be partially ordered by

$$H_1 \leqslant H_2 \text{ iff } H_1 \subseteq H_2\,.$$

Definition 1.4 *The ℓ-profinite completion of G is the inverse limit of the canonical finite ℓ-quotients of G –*

$$\widehat{G}_1 = \lim_{\substack{\longleftarrow \\ \{H\}_\ell}} (G/H)\,.$$

The ℓ-profinite completion $\widehat{G}_\ell$ is topologized by the inverse limit of the discrete topologies on the G/H's. Thus $\widehat{G}_\ell$ becomes a totally disconnected compact topological group.

The natural map

$$G \to \widehat{G}_\ell$$

is clearly universal[4] for maps of G into finite ℓ-groups.

This construction is functorial because the diagram

$$\begin{tikzcd} H = f^{-1}H' \arrow[r] \arrow[d] & H' \arrow[d] \\ G \arrow[r, "f"'] \arrow[d] & G' \arrow[d] \\ G/H \arrow[r, "\bar{f}"'] & G'/H' \end{tikzcd}$$

shows f induces a map of inverse systems –

$$\{H'\}_\ell \to \{f^{-1}H'\}_\ell \subseteq \{H\}_\ell$$

$$G/H \xrightarrow{\bar{f}} G'/H',\ H = f^{-1}H'$$

and thus we have $\widehat{f}$

$$\widehat{G}_\ell \equiv \varprojlim G/H \xrightarrow{\widehat{f}} \varprojlim G'/H' \equiv \widehat{G}'_\ell\,.$$

Examples

1) Let $G = \mathbb{Z}$, $\ell = \{p\}$. Then the only p-quotients are $\mathbb{Z}/p^n$. Thus

$$\begin{matrix} p\text{-profinite completion} \\ \text{of the group } G \end{matrix} \quad = \quad \varprojlim_n \mathbb{Z}/p^n$$

which agrees (additively) with the ring theoretic p-adic completion of $\mathbb{Z}$, the "p-adic integers".

2) Again $G = \mathbb{Z}$, $\ell = \{p_1, p_2, \dots\}$. Then

$$\mathbb{Z}_\ell = \varprojlim_\alpha \mathbb{Z}/p_1^{\alpha_1} \dots p_i^{\alpha_i}$$

where

$$\alpha = \{(\alpha_1, \alpha_2, \ldots, \alpha_i, 0, 0, 0, \ldots)\}$$

is the set of all non-negative exponents (eventually zero) partially ordered by

$$\alpha \leqslant \alpha' \text{ if } \alpha_i \leqslant \alpha_i' \text{ for all } i\,.$$

The cofinality of the sequence

$$\alpha_k = (\underbrace{k, k, \ldots, k}_{k \text{ places}}, 0, 0, 0, \ldots)$$

shows that

$$\begin{aligned} \widehat{\mathbb{Z}}_\ell &= \varprojlim_k \prod_{i=0}^{k} \mathbb{Z}/p_i^k \\ &= \prod_{p\in\ell} \varprojlim_k \mathbb{Z}/p^k \\ &= \prod_{p\in\ell} \widehat{\mathbb{Z}}_p \end{aligned}$$

3) For any Abelian group G

$$\widehat{G}_\ell \cong \prod_{p\in\ell} \widehat{G}_p\,.$$

4) The ℓ-profinite completion of a finitely generated Abelian group of rank n and torsion subgroup T is just

$$G \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_\ell \cong \underbrace{\widehat{\mathbb{Z}}_\ell \oplus \cdots \oplus \widehat{\mathbb{Z}}_\ell}_{n \text{ summands}} \oplus\, \ell\text{-torsion } T\,.$$

5) If G is ℓ-divisible, then ℓ-profinite completion reduces G to the trivial group. For example,

$$\widehat{\mathbb{Q}}_\ell = 0,\ (\mathbb{Q}/\mathbb{Z})_\ell = 0\,.$$

6) The p-profinite completion of the infinite direct sum $\bigoplus^{\infty} \mathbb{Z}/p$ is the infinite direct product $\prod^{\infty} \mathbb{Z}/p$.

From the examples we see that profinite completion is exact for finitely generated Abelian groups but is not exact in general, e.g.

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

becomes

$$0 \to \widehat{\mathbb{Z}}_\ell \to 0 \to 0 \to 0\,.$$

If we wanted a construction related to profinite completion which preserved exactness for non-finitely generated groups, we could simply make the

Definition 1.5 *The formal ℓ-completion of an Abelian group G, $\bar{G}_\ell$ is given by*

$$\bar{G}_\ell = G \otimes \widehat{\mathbb{Z}}_\ell\,.$$

Proposition 1.15 *The functor $G \to \bar{G}_\ell$ is exact. It is the unique functor which agrees with the profinite completion for finitely generated groups and commutes with direct limits.*

Proof: The first part follows since $\widehat{\mathbb{Z}}_\ell$ is torsion free. The second part follows from

i) any group is the direct limit of its finitely generated subgroups

ii) tensoring commutes with direct limits.

If ℓ is {all primes} then $\widehat{G}_\ell$ is called "the profinite completion" of G and denoted $\widehat{G}$. $\bar{G}_\ell$ is the "formal completion" of G and denoted $\bar{G}$. Thus $\bar{G} = G \otimes \bar{\mathbb{Z}} = G \otimes \widehat{\mathbb{Z}}$.

We note here that the profinite completion of G, $\widehat{G}$ is complete if we remember its topology. Namely, let $\{\widehat{H}\}$ denote the partially ordered set of *open* subgroups of $\widehat{G}$ of finite index. Then

$$\widehat{G} \overset{*}{\cong} \varprojlim_{\{\widehat{H}\}} = \text{“continuous completion of } \widehat{G}\text{”}\,.$$

It sometimes happens however that every subgroup of finite index in $\widehat{G}$ is open. This is true if $\widehat{G} = \widehat{\mathbb{Z}}$, in fact for the profinite completion of any finitely generated Abelian group. Thus in these cases the topology of $\widehat{G}$ can be recovered from the algebra using the isomorphism $*$.

The topology is essential for example in

$$\prod^{\infty} \mathbb{Z}/p = \text{profinite completion } \Big(\bigoplus^{\infty} \mathbb{Z}/p\Big)\,.$$

Examples from Topology and Algebra

Now we consider some interesting examples of "profinite groups".

1) Let X be an infinite complex and consider some extraordinary cohomology theory $h^*(X)$. Suppose that $\pi_i(X)$ is finite and $h^i(\mathrm{pt})$ is finitely generated for each i (or vice versa.)

   Then for each i, the reduced group $\widetilde{h}^i(X)$ is a profinite group.

   For example, the reduced K-theory of $\mathbb{R}P^\infty$ is the 2-adic integers.

   The profiniteness of $\widetilde{h}^i(X)$ follows from the formula

   $$\widetilde{h}^i(X) \;\cong \lim_{\substack{\longleftarrow \\ \text{skeleta } X}} \widetilde{h}^i(\text{skeleton } X)$$

   and the essential finiteness of $\widetilde{h}^i(\text{skeleton } X)$.

2) Let K containing k be an infinite Galois field extension. K is a union of finite Galois extensions of k,

   $$K = \bigcup_{\infty} L\,.$$

   Then the Galois group of K over k is the profinite group

   $$\mathrm{Gal}\,(K/k) = \lim_{\substack{\longleftarrow \\ \{L\}}} \mathrm{Gal}\,(L/k)\,.$$

Examples

i) Let k be the prime field F_p and $K = \widetilde{F}_p$, an algebraic closure of F_p. Then K is a union of fields with p^n elements, F_{p^n}, n ordered by divisibility, and

   $$\begin{aligned} \mathrm{Gal}\,(\widetilde{F}_p, F_p) &= \lim_{\substack{\longleftarrow \\ n}} \mathrm{Gal}\,(F_{p^n}, F_p) \\ &= \lim_{\substack{\longleftarrow \\ n}} \mathbb{Z}/n \\ &= \widehat{\mathbb{Z}}\,. \end{aligned}$$

   Moreover each Galois group has a natural generator, the Frobenius automorphism

   $$F_{p^n} \xrightarrow[x \mapsto x^p]{\mathscr{F}} F_{p^n}\,.$$

$\mathscr{F}$ is the identity on F_p because of Fermat's congruence

$$a^p \equiv a\,(\mathrm{mod}\ p)\,.$$

Thus the powers of $\mathscr{F}$ generate $\widehat{\mathbb{Z}}$ topologically (they are dense.)

The fixed fields of the powers of the Frobenius are just the various finite fields which filter $\widetilde{F}_p$.

ii) If $k = \mathbb{Q}$, and $K = A_{\mathbb{Q}}$ is obtained by adjoining all roots of unity to $\mathbb{Q}$, then

$$\mathrm{Gal}\,(A_{\mathbb{Q}}/\,\mathbb{Q}) = \widehat{\mathbb{Z}}^*\,,$$

the group of units in the ring $\widehat{\mathbb{Z}}$.

$A_{\mathbb{Q}}$ can be described intrinsically as a maximal Abelian extension of $\mathbb{Q}$, i.e. a maximal element in the partially ordered "set" of Abelian extensions of $\mathbb{Q}$ (Abelian Galois groups.)

The decomposition

$$\widehat{\mathbb{Z}}^* = \prod_p \widehat{\mathbb{Z}}_p^*$$

tells one how $A_{\mathbb{Q}}$ is related to the fields

$$A_{\mathbb{Q}}^p \;=\; \{\mathbb{Q} \text{ with all } p^\alpha \text{ roots of unity adjoined}\}\,,$$

for $\mathrm{Gal}\,(A_{\mathbb{Q}}^p, \mathbb{Q}) = \widehat{\mathbb{Z}}_p^*$, the group of units in the ring of p-adic integers.

iii) If $k = \mathbb{Q}$ and $K = \widetilde{\mathbb{Q}}$, an algebraic closure of $\mathbb{Q}$, then

$$\begin{aligned} G &= \text{“the Galois group of } \mathbb{Q}\text{”} \\ &= \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q}) \\ &= \varprojlim_{\text{Galois number fields } K} \mathrm{Gal}\,(K/\,\mathbb{Q}) \end{aligned}$$

is a profinite group of great importance.

G has very little torsion, only "complex conjugations", elements of order 2. These are all conjugate, and each one commutes with no element besides itself and the identity. This non-commuting fact means that our (conjectured) etale 2-adic homotopy type for *real* algebraic varieties (section 5) does not have Galois symmetry in general.

Notice also that in a certain sense G is only defined up to inner automorphisms (like the fundamental group of a space) but its profinite Abelianization

$$G/[G,G]$$

is canonically defined (like the first homology group of a space). Perhaps this is one reason why there is such a beautiful theory for determining $G/[G,G]$. This "class field theory for $\mathbb{Q}$" gives a canonical isomorphism

$$G/[G,G] \cong \widehat{\mathbb{Z}}^* .$$

We will see in the later sections how the $\widehat{\mathbb{Z}}^*$-Galois symmetry of the maximal Abelian extension of $\mathbb{Q}$, "the field generated by the roots of unity" seems to permeate geometric topology – in linear theory, C^∞-theory, and even topological theory.

iv) We will see below that the group of p-adic units is naturally isomorphic to a (finite group) direct sum (the additive group), e.g.

$$\widehat{\mathbb{Z}}_p^* \cong (\mathbb{Z}\,/p-1) \oplus \widehat{\mathbb{Z}}_p \quad (p>2)\,.$$

Thus there are non-trivial group homomorphisms

$$\widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}^*$$

e.g. $\widehat{\mathbb{Z}}_p$ maps non-trivially into $\widehat{\mathbb{Z}}_q^*$ for $q = p$, and $q \equiv 1 \pmod p$.

Such non-trivial maps

$$\begin{array}{ccc} \widehat{\mathbb{Z}} & \xrightarrow{\qquad\qquad\qquad} & \widehat{\mathbb{Z}}^* \\ \| & & \| \\ \text{“Galois group of } F_p\text{”} & & \begin{array}{c}\text{Abelianization of}\\ \text{“Galois group of } \mathbb{Q}\text{”}\end{array} \end{array}$$

allow us to connect "characteristic p" with "characteristic zero".

The p-adic units

Besides these interesting "algebraic occurrences" of profinite groups, in the p-adic case analytical considerations play a considerable role. For example, the p-adic analytic functions log and exp can be employed to prove

Proposition 1.16 *There is a "canonical" splitting of the (profinite) group of units in the p-adic integers*

$$\widehat{\mathbb{Z}}_p^* \cong (\mathbb{Z}/p-1) \oplus \widehat{\mathbb{Z}}_p \quad (p>2)$$
$$\widehat{\mathbb{Z}}_2^* \cong (\mathbb{Z}/2) \oplus \widehat{\mathbb{Z}}_2 \quad (p=2)$$

Proof: Consider the case $p > 2$. Reduction mod p and the equivalence

$$F_p^* \;=\; \text{multiplicative group of } F_p \;=\; \mathbb{Z}/p-1$$

yields an exact sequence

$$1 \to U \xrightarrow[\text{inclusion}]{} \widehat{\mathbb{Z}}_p^* \xrightarrow[\text{reduction mod } p]{} \mathbb{Z}/p-1 \to 1\,,$$

where U is the subgroup of units $U = \{1+u \text{ where } u = 0 \mod p\}$.

Step 1. There is a canonical splitting T of

$$\begin{tikzcd} 1 \arrow[r] & U \arrow[r] & \widehat{\mathbb{Z}}_p^* \arrow[r] & \mathbb{Z}/p-1 \arrow[r] \arrow[l, bend right, "T"'] & 1\,. \end{tikzcd}$$

Consider the endomorphism $x \mapsto x^p$ in $\widehat{\mathbb{Z}}_p^*$ and the effect of iterating it indefinitely (the Frobenius dynamical system on $\widehat{\mathbb{Z}}_p^*$). The Fermat congruences for $x \in \widehat{\mathbb{Z}}_p^*$

$$\begin{aligned} x^{p-1} &\equiv 1 \pmod{p} \\ x^{p(p-1)} &\equiv 1 \pmod{p^2} \\ &\vdots \\ x^{p^{k-1}(p-1)} &\equiv 1 \pmod{p^k} \\ &\vdots \end{aligned}$$

follow from counting the order of the group of units in the ring $\mathbb{Z}/p^k\,\mathbb{Z}$ (which "approximates" $\widehat{\mathbb{Z}}_p$.)

These show that the "Teichmüller representative"

$$\bar{x} = x + (x^p - x) + (x^{p^2} - x^p) + \dots$$

is a well defined p-adic integer for any x in $\widehat{\mathbb{Z}}_p^*$.

However, $\bar{x}$ is just
$$\bar{x} = \lim_{n\to\infty} x^{p^n} .$$
Thus, every point in $\widehat{\mathbb{Z}}_p^*$ flows to a definite point upon iteration of the Frobenius p^{th} power mapping, $x \mapsto x^p$.

The binomial expansion

$$\begin{aligned}(a+pb)^{p^n} &= \sum_{l+k=p^n} \binom{l+k}{l} a^l (pb)^k \\ &= a^{p^n} + p^n \Big(\sum_{\substack{l+k=p^n \\ k>0}} \frac{(p^n-1)(p^n-2)\dots(p^n-(k-1))}{1\cdot 2\dots(k-1)} \big(\frac{p^k}{k}\big) a^l b^k \Big)\end{aligned}$$

shows $(a+pb)^{p^n} \equiv a^{p^n} \pmod{p^n}$.

Thus $\bar{x}$ only depends on the residue class of x modulo p.

So each coset of U flows down over itself to a canonical point. The $(p-1)$ points constructed this way form a subgroup (comprising as they do the image of the infinite iteration of the Frobenius endomorphism), and this subgroup maps onto $\mathbb{Z}/p-1$.

This gives the required splitting T.

Step 2. We construct a canonical isomorphism
$$U \xrightarrow[\cong]{} \widehat{\mathbb{Z}}_p .$$
Actually we construct an isomorphism pair
$$U \underset{e}{\overset{l}{\rightleftarrows}} p\widehat{\mathbb{Z}}_p \subseteq \widehat{\mathbb{Z}}_p .$$
First,
$$(1+u) \xrightarrow{l} \log(1+u) \;=\; u - u^2/2 + u^3/3 - \dots$$
This series clearly makes sense and converges. If $u = pv$, the n^{th} terms u^n/n make sense for all n and approach zero as n goes to ∞ (which is sufficient for convergence in this non-Archimedian situation.)

The inverse for ℓ is constructed by the exponential function
$$x \overset{e}{\mapsto} e^x \;=\; 1 + x + x^2/2 + \dots + x^n/n! + \dots$$

which is defined for all x in the maximal ideal

$$p\widehat{\mathbb{Z}}_p \subseteq \widehat{\mathbb{Z}}_p\,.$$

Again one uses the relation $x = py$ to see that $x^n/n!$ makes sense for all n and goes to zero as n approaches infinity. The calculational point here is that

$$\nu_p(n!) \;=\; \lfloor \frac{n - \varphi_p(n)}{p-1} \rfloor$$

where $\nu_p(n)$ is defined by

$$n \;=\; \prod_p p^{\nu_p(n)}$$

and $\varphi_p(n) =$ the sum of the coefficients in the expansion

$$n \;=\; a_0 + a_1 p + a_2 p^2 + \dots$$

Since $e^{\log x} = x$, and $\log e^x = x$ are identities in the formal power series ring, we obtain an isomorphism $U \cong p\widehat{\mathbb{Z}}_p$. Since $\widehat{\mathbb{Z}}_p$ is torsion free $p\widehat{\mathbb{Z}}_p$ is canonically isomorphic to $\widehat{\mathbb{Z}}_p$. This completes the proof for $p > 2$. If $p = 2$ certain modifications are required.

In step 1, the exact sequence

$$1 \to U \to \widehat{\mathbb{Z}}_2^* \xrightarrow{T} \widehat{\mathbb{Z}}/2 \to 1$$

comes from "reduction modulo 4" and the equivalence

$$(\mathbb{Z}/4)^* \cong \mathbb{Z}/2\,.$$

The natural splitting is obtained by lifting $\mathbb{Z}/2 = \{0,1\}$ to $\{\pm 1\} \subseteq \widehat{\mathbb{Z}}_2$. But the exponential map is only defined on the square of the maximal ideal,

$$4\widehat{\mathbb{Z}}_2 \xrightarrow{e} U\,.$$

Namely,

$$\nu_2(n!) \;=\; n - \varphi_2(n)$$

means $\frac{(2y)^n}{n!}$ is defined (and even even) for all n but only approaches zero as required if y is also even.

From these functions we deduce that U is torsion free from the fact that $\widehat{\mathbb{Z}}_2$ is torsion free. ($x^n = 1$ implies $\log x^n = n \log x = 0$ implies $\log x = 0$ implies $x = e^{\log x} = 1$.)

Then we have

$$U \xrightarrow[\text{squaring}]{\cong} U^2 \xrightarrow[\cong]{\log} 4\widehat{\mathbb{Z}}_2 \xleftarrow[\text{multiplication by four}]{} \widehat{\mathbb{Z}}_2$$

where log is an isomorphism because the diagrams

$$\begin{tikzcd}
4\widehat{\mathbb{Z}}_2 \arrow[dd, "\text{inclusion}"'] \arrow[dr, "\exp"] & \\
& U \arrow[dl, "\log"] \\
2\widehat{\mathbb{Z}}_2 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
U^2 \arrow[dd, "\text{inclusion}"'] \arrow[dr, "\log"] & \\
& 4\widehat{\mathbb{Z}}_2 \arrow[dl, "\exp"] \\
U &
\end{tikzcd}$$

commute. (The first shows that $4\,\mathbb{Z}_2 \xrightarrow{\exp} U$ is injective. The second shows that $\exp\cdot\log$ is an isomorphism onto its image.)

Note: There is some reason for comparing the splittings

$$\widehat{\mathbb{Z}}_p^* \cong (\mathbb{Z}\,/p-1) \oplus \widehat{\mathbb{Z}}_p, \ \ p \text{ odd}$$
$$\widehat{\mathbb{Z}}_2^* \cong \mathbb{Z}\,/2 \oplus \widehat{\mathbb{Z}}_2$$

with

$$\mathbb{C}^* \xrightarrow[\cong]{\frac{1}{2\pi}\arg z \oplus \log|z|} S^1 \oplus \mathbb{R}^+$$
$$\mathbb{R}^* \xrightarrow[\cong]{\text{sign } x \oplus \log|x|} \mathbb{Z}\,/2 \oplus \mathbb{R}^+$$

where $\mathbb{C}$ is the complex numbers, $\mathbb{R}$ is the real numbers, $\mathbb{R}^+$ denotes the additive group of $\mathbb{R}$, and S^1 denotes $\mathbb{R}^+$ modulo the lattice of integers in $\mathbb{R}^+$.

Localization and Completion

Now let us compare localization and completion. Recall

localization: $G_\ell = \xrightarrow[(\ell',\ell)=1]{} \{G \xrightarrow{\cdot\ell'} G \xrightarrow{\cdot\ell'} G \to \dots\} = G \otimes \mathbb{Z}_\ell$.

profinite ℓ-completion: $\widehat{G}_\ell = \xleftarrow[\substack{\text{“subgroups} \\ \text{of index } \ell\text{”}}]{} \{\text{finite } \ell\text{-quotients of } G\}$

formal ℓ-completion: $\bar{G}_\ell = G \otimes \widehat{\mathbb{Z}}_\ell = \xrightarrow[\substack{\text{finitely generated} \\ \text{subgroups } H \subset G}]{} \{\widehat{H}_\ell\}$.

Let G be an Abelian group and let ℓ be a non-void set of primes.

Proposition *There is a natural commutative diagram*

$$\begin{tikzcd}
 & G_\ell \equiv G \otimes \mathbb{Z}_\ell \arrow[d] \\
G \arrow[ur] \arrow[r] \arrow[dr] & \bar{G}_\ell \equiv G \otimes \widehat{\mathbb{Z}}_\ell \arrow[d] \\
 & \widehat{G}_\ell \equiv \varprojlim \left\{ \begin{matrix} \textit{finite } \ell\textit{-quotients} \\ \textit{of } G \end{matrix} \right\}
\end{tikzcd}$$

Proof: First observe there is a natural diagram

$$\begin{tikzcd}
G \arrow[r, "\text{localization}"] \arrow[dr, "\text{profinite completion}"'] & G_\ell \arrow[d, "c"] \\
 & \widehat{G}_\ell
\end{tikzcd}$$

For taking direct limits over ℓ' prime to ℓ using

$$\begin{tikzcd}
G \arrow[r, "\cdot \ell'"] \arrow[d] & G \arrow[d] \\
G/H_\alpha \arrow[r, "\cdot \ell'", "\cong"'] & G/H_\alpha
\end{tikzcd}, \quad H_\alpha \text{ of finite } \ell\text{-index in } G$$

implies G_ℓ maps canonically to each finite ℓ-quotient of G. Thus G_ℓ maps to the inverse limit of all these, $\widehat{G}_\ell$. The diagram clearly commutes.

Using the map c and the expression

$$G \;=\; \varinjlim_{\alpha} H^\alpha\,, \quad H^\alpha \text{ finitely generated subgroup of } G$$

yields

$$G_\ell \xleftarrow{\substack{\text{natural}\\ \text{map}}} \varinjlim_{\alpha} H_\ell^\alpha \xrightarrow{c} \varinjlim_{\alpha} \widehat{H}_\ell^\alpha \xrightarrow{\substack{\text{natural}\\ \text{map}}} \widehat{G}_\ell\,.$$

But the first map is an isomorphism and the third group is just the formal completion. Thus we obtain the natural sequence

$$G_\ell \to \bar{G}_\ell \to \widehat{G}_\ell\,.$$

We can then form the desired diagrams. The upper triangle is just a direct limit of triangles considered above (for finitely generated groups). So it commutes. The lower triangle commutes by naturality,

$$\begin{tikzcd} \lim H^\alpha \arrow[r] \arrow[dr] & \varinjlim \widehat{H}^\alpha_\ell \arrow[d] \\ & (H^\alpha)\widehat{\ }_\ell\,. \end{tikzcd}$$

Corollary *In case G is finitely generated we have*

$$\begin{tikzcd} & & & 0 \arrow[d] \\ 0 \arrow[r] & \ell'\text{-}\mathit{torsion}\ G \arrow[r] & G \arrow[r] \arrow[d] & G_\ell \arrow[d] \\ & & \widehat{G}_\ell & \bar{G}_\ell\,. \arrow[l, "\cong"'] \end{tikzcd}$$

For $G = \mathbb{Z}$ we get the sequence of rings

$$\mathbb{Z} \xrightarrow[\mathit{localization}]{} \mathbb{Z}_\ell \xrightarrow[\mathit{completion}]{c} \widehat{\mathbb{Z}}_\ell\,,$$

and

$$G_\ell \xrightarrow{\substack{\mathit{natural} \\ \mathit{map}}} \bar{G}_\ell$$

is isomorphic to (identity G) $\otimes c$.

Regarding limits, it is clear that localization and formal completion commute with direct limits. The following examples show the other possible statements are false –

a) localization: $(\varprojlim_\alpha \mathbb{Z}/p^\alpha) \otimes \mathbb{Q} = \mathbb{Q}_p$, while $\varprojlim_\alpha (\mathbb{Z}/p^\alpha \otimes \mathbb{Q}) = 0$.

b) formal completion: $(\varprojlim_\alpha \mathbb{Z}/p^\alpha) \otimes \widehat{\mathbb{Z}}_p \neq \widehat{\mathbb{Z}}_p = \varprojlim (\mathbb{Z}/p^\alpha \otimes \widehat{\mathbb{Z}}_p)$.

c) profinite completion:

i) write $\mathbb{Q} = \varinjlim \mathbb{Z}$, then $\varinjlim \widehat{\mathbb{Z}} = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$, but $\widehat{\mathbb{Q}} = 0$.

ii) $\varprojlim(\dots \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \mathbb{Z}) = 0$,
but $\varprojlim(\dots \xleftarrow{2} \widehat{\mathbb{Z}} \xleftarrow{2} \widehat{\mathbb{Z}} \xleftarrow{2} \widehat{\mathbb{Z}}) = \prod_{p \neq 2} \widehat{\mathbb{Z}}_p$.

If we consider mixing these operations in groups, the following remarks are appropriate.

i) localizing and then profinitely completing is simple and often gives zero,

$$(G_\ell)\widehat{\ }_{\ell'} \;=\; \begin{cases} \prod\limits_{p \in \ell \cap \ell'} \widehat{G}_p & \text{if } \ell \cap \ell' \neq \emptyset \\ 0 & \text{if } \ell \cap \ell' = \emptyset\,. \end{cases}$$

e.g. $(G, l, \ell') = (\mathbb{Z}\,, \emptyset, p)$ gives $\widehat{\mathbb{Q}}_p = 0$.

ii) localizing and then formally completing leads to new objects, e.g.

a) $(\mathbb{Z}_0)\bar{\ }_p = \bar{\mathbb{Q}}_p = \mathbb{Q} \otimes \widehat{\mathbb{Z}}_p$, the "field of p-adic numbers", usually denoted by $\mathbb{Q}_p$. $\mathbb{Q}_p$ is the field of quotients of $\widehat{\mathbb{Z}}_p$ (although it is not much larger because only $1/p$ has to be added to $\widehat{\mathbb{Z}}_p$ to make it a field). $\mathbb{Q}_p$ is a locally compact metric field *whose unit disk is made up of the integers* $\widehat{\mathbb{Z}}_p$. The power series for $\log(1+x)$ discussed above converges for x in the interior of this disk, the maximal ideal $p\widehat{\mathbb{Z}}_p$ of $\widehat{\mathbb{Z}}_p$.
$\mathbb{Q}_p$ is usually thought of as being constructed from the field of rational numbers by completing with respect to the p-adic metric. It thus plays a role analogous to the real numbers, $\mathbb{R}$.

b) $(\mathbb{Z}_0)\bar{\ } = \bar{\mathbb{Q}} = \mathbb{Q} \otimes \widehat{\mathbb{Z}}$ is the *restricted* product over all p of the p-adic numbers. Namely,

$$\mathbb{Q} \otimes \widehat{\mathbb{Z}} \;=\; \widehat{\prod_p} \mathbb{Q}_p \subsetneq \prod_p \mathbb{Q}_p$$

is the subring of the infinite product consisting of infinite sequences

$$(r_2, r_3, r_5, \dots, r_p, \dots)$$

of p-adic numbers where all but finitely many of the r_p are actually p-adic integers.
Note that $\mathbb{Q}$ is contained in $\bar{\mathbb{Q}}$ as the diagonal sequences

$$n/m \to (n/m, n/m, \dots, n/m, \dots)\,.$$

If we combine this embedding with the embedding of $\mathbb{Q}$ in the reals we obtain an embedding

$$\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}} \times \{\text{real completion of } \mathbb{Q}\}$$

as a *discrete* subgroup with a *compact* quotient. $\bar{\mathbb{Q}} \times \{\text{real completion of } \mathbb{Q}\}$ is called the ring of Adeles (for $\mathbb{Q}$.)

Adeles can be constructed similarly for general number fields and even algebraic groups $\big($e.g. $\mathbb{Q}(\xi) = \mathbb{Q}(x)/(x^p - 1)$ and $\mathrm{GL}(n, \mathbb{Z})$.$\big)$

These Adele groups have natural measures, and the volumes of the corresponding compact quotients have interesting number theoretical significance. (See “Adeles and Algebraic Groups” Lectures by Andre Weil, the Institute for Advanced Study, 1961 (*Progress in Mathematics 23, Birkhäuser (1982).*)

In the number field case the Adeles form a ring. The units in this ring are called *ideles*. The ideles are used to construct Abelian extensions of the number field. (Global and local class field theory.)

The Arithmetic Square

Now we point out some “fibre square” relations between localizations and completions. The motive is to see how an object can be recovered from its localizations and completions.

Proposition 1.17 *The square of groups (rings) and natural maps*

$$\begin{tikzcd}
\mathbb{Z}_{(p)} \arrow[r, "\hat{}"] \arrow[d, "\otimes\mathbb{Q}"'] & \widehat{\mathbb{Z}}_p \arrow[d, "\otimes\mathbb{Q}"'] \\
\mathbb{Q} \arrow[r, "\otimes\widehat{\mathbb{Z}}_p"'] & \mathbb{Q}_p
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
\left\{\begin{matrix}\textit{integers localized}\\ \textit{at } p\end{matrix}\right\} \arrow[r, "\substack{p\text{-}adic\\ completion}"] \arrow[d, "\substack{localization\\ at\ zero}"] & \left\{\begin{matrix}p\text{-}\textit{adic}\\ \textit{integers}\end{matrix}\right\} \arrow[d, "\substack{localization\\ at\ zero}"'] \\
\left\{\begin{matrix}\textit{rational}\\ \textit{numbers}\end{matrix}\right\} \arrow[r, "\substack{formal\\ p\text{-}adic\ completion}"'] & \left\{\begin{matrix}p\text{-}\textit{adic}\\ \textit{numbers}\end{matrix}\right\}
\end{tikzcd}$$

is a fibre square of groups (rings).

Proof: We have to check exactness for

$$0 \to \mathbb{Z}_{(p)} \xrightarrow{\widehat{(\ )}\oplus(0)} \widehat{\mathbb{Z}}_p \oplus \mathbb{Q} \xrightarrow{i-j} \mathbb{Q}_p \to 0$$

where i and j are the natural inclusions

$$\widehat{\mathbb{Z}}_p \xrightarrow[\otimes\mathbb{Q}]{} \mathbb{Q}_p \ , \quad \mathbb{Q} \xrightarrow[\otimes\widehat{\mathbb{Z}}_p]{} \mathbb{Q}_p \ .$$

Take $n \in \mathbb{Z}$ and $q \in \widehat{\mathbb{Z}}_p$ then

$$(n/p^a) + q \ = \ (n + p^a q)/p^a$$

can be an arbitrary p-adic number. Thus $i - j$ is onto.

It is clear that $(\widehat{\ }) \oplus (0)$ has no kernel.

To complete the proof only note that a rational number n/m is also a p-adic integer when m is not divisible by p. Thus n/m is in $\mathbb{Z}$ localized at p.

Corollary *The ring of integers localized at p is the fibre product of the rational numbers and the ring of p-adic integers over the p-adic numbers.*

Proposition 1.18 *The square*

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "\otimes\mathbb{Q}"'] & \widehat{\mathbb{Z}} \arrow[d, "\otimes\mathbb{Q}"'] \\ \mathbb{Q} \arrow[r, "\otimes\widehat{\mathbb{Z}}"'] & \bar{\mathbb{Q}} \end{tikzcd} \quad \equiv \quad \begin{tikzcd} \{\textit{integers}\} \arrow[r, "\substack{\textit{profinite}\\ \textit{completion}}"] \arrow[d, "\substack{\textit{localization}\\ \textit{at zero}}"] & \left\{\substack{\textit{product over all } p\\ \textit{of the } p\textit{-adic}\\ \textit{integers}}\right\} \arrow[d, "\substack{\textit{localization}\\ \textit{at zero}}"'] \\ \left\{\substack{\textit{rational}\\ \textit{numbers}}\right\} \arrow[r, "\substack{\textit{formal}\\ \textit{completion}}"'] & \left\{\substack{\textit{restricted product}\\ \textit{over all } p \textit{ of the}\\ p\textit{-adic numbers}}\right\} \arrow[d, equal] \\ & \text{“\textit{finite Adeles}”} \end{tikzcd}$$

is a fibre square of rings.

Proof: Again we must verify exactness:

$$0 \to \mathbb{Z} \to \mathbb{Q} \oplus \prod_p \widehat{\mathbb{Z}}_p \xrightarrow{i-j} \widehat{\prod} \mathbb{Q}_p \to 0 \,,$$

where we have used the relations

$$\widehat{\mathbb{Z}} \ = \ \prod_p \widehat{\mathbb{Z}}_p \,,$$

$$\bar{\mathbb{Q}} \ = \ \text{the restricted product } \widehat{\prod_p} \mathbb{Q}_p \ .$$

An element in $\widehat{\prod_p} \mathbb{Q}_p$ is an infinite tuple

$$a \;=\; (r_2, r_3, r_5, \dots)$$

of p-adic numbers in which all but finitely many of the components are integers O_p.

If the non-integral component r_p equals O_p/p^α take N to be an integer in the residue class (mod p^α) of O_p. Then

$$a + (n/p^\alpha, n/p^\alpha, \dots, n/p^\alpha, \dots)$$

has one fewer non-integral component than a. This shows the finite Adeles $\widehat{\prod} \mathbb{Q}_p$ are generated by the diagonal $\mathbb{Q}$ and $\prod_p \widehat{\mathbb{Z}}_p$. Thus $i - j$ is onto.

As before the proof is completed by observing that a rational number which is also a p-adic integer for every p must actually be an integer.

Corollary *The ring of integers is the fibre product of the rational numbers and the infinite product of all the various rings of p-adic integers over the ring of finite Adeles.*

More generally, for a finitely generated Abelian group G and a non-void set of primes ℓ there is a fibre square

$$\begin{tikzcd}
G \otimes \mathbb{Z}_\ell \equiv G_\ell \arrow[r, "\ell\text{-adic completion}"] \arrow[d, "\text{localization at zero}"] & \widehat{G}_\ell \equiv G \otimes \widehat{\mathbb{Z}}_\ell \arrow[d, "\text{localization at zero}"] \\
G \otimes \mathbb{Q} \equiv G_0 \arrow[r, "\text{formal completion}"'] & (\widehat{G}_\ell)_0 \equiv (G_0)^-_\ell \equiv G \otimes \mathbb{Q} \otimes \widehat{Z}_\ell
\end{tikzcd}$$

Taking ℓ to be "all primes" we see that the group G can be recovered from appropriate maps of its localization at zero $G \otimes \mathbb{Q}$ and its profinite completion, $\prod_p \widehat{G}_p$ into $G\otimes$ "finite Adeles". Taking $\ell = \{p\}$ we see that G localized at p can be similarly recovered from its localization at zero and its p-adic completion.

We will be doing the same thing to spaces in the next two sections. Thus to understand a space X it is possible to break the problem up into pieces – a profinitely completed space $\widehat{X}$ and a rational space X_0. These each map to a common Adele space X_A, and any information about X may be recovered from this picture

$$\begin{array}{ccc} & & \widehat{X} \\ & & \downarrow \\ X_0 & \longrightarrow & X_A \,. \end{array}$$

This is the main idea of the first three sections.

Notes

1 In this context it is sufficient to assume that $d(x_n, x_{n+1}) \to 0$ to have a Cauchy sequence.

2 $(x-1, x+1)^2 \subset (2) \subset (x-1, x+1)$.

3 Tensored with the group ring of $\mathbb{Z}/2$ over the 2-adic integers.

4 There is a unique continuous map of the completion extending over a given map of G into a finite group.

Section 2. Homotopy Theoretical Localization

In this section we define a localization functor in homotopy theory. There is a cellular construction for simply connected spaces and a Postnikov construction for "simple spaces".

At the end of the section we give some (hopefully) enlightening examples.

We work in the category of "simple spaces" and homotopy classes of maps. A "simple space" is a connected space having the homotopy type of a CW complex and an Abelian fundamental group which acts trivially on the homotopy and homology of the universal covering space.

Let ℓ be a set of primes in $\mathbb{Z}$ which may be empty. ℓ will be fixed throughout the discussion and all localization will be made with respect to it.

Definition 2.1 *We say that X_ℓ is a* local space *iff $\pi_* X_\ell$ is local, i.e., $\pi_* X_\ell$ is a $\mathbb{Z}_\ell$-module. We say that a map of some space X into a local space X_ℓ*

$$X \xrightarrow{\ell} X_\ell$$

is a localization of X *if it is universal for maps of* X *into local spaces, i.e., given* f *there is a* unique f_ℓ *making the diagram*

$$\begin{tikzcd} X \arrow[rr, "\ell"] \arrow[dr, "f"'] & & X_\ell \arrow[dl, dashed, "f_\ell"] \\ & \begin{array}{c}\textit{local}\\ \textit{space}\end{array} & \end{tikzcd}$$

commutative

Local spaces and localization are characterized by

Theorem 2.1 *For a map*

$$X \xrightarrow{\ell} X'$$

of arbitrary "simple spaces" the following are equivalent

i) ℓ *is a localization*

ii) ℓ *localizes integral homology*

$$\begin{tikzcd} \widetilde{H}_* X \arrow[rr, "\ell"] \arrow[dr, "\otimes \mathbb{Z}_\ell"'] & & \widetilde{H}_* X' \arrow[dl, "\cong"] \\ & \widetilde{H}_* X \otimes \mathbb{Z}_\ell & \end{tikzcd}$$

iii) ℓ *localizes homotopy* $(* > 0)$

$$\begin{tikzcd} \pi_* X \arrow[rr, "\ell"] \arrow[dr, "\otimes \mathbb{Z}_\ell"'] & & \pi_* X' \arrow[dl, "\cong"] \\ & \pi_* X \otimes \mathbb{Z}_\ell & \end{tikzcd}$$

Taking $\ell =$ identity we get the

Corollary *For a "simple space" the following are equivalent*

i) X *is its own localization*

ii) X *has local homology*

iii) X *has local homotopy.*

Corollary *If $X \xrightarrow{\ell} X'$ is a map of local "simple spaces" then*

i) ℓ is a homotopy equivalence

ii) ℓ induces an isomorphism on local homotopy

iii) ℓ induces an isomorphism of local homology

are equivalent.

Note the case $\ell =$ all primes. We also note here that a map induces an isomorphism on local homology iff it does on rational homology and on mod p homology for $p \in \ell$.

The proof of the theorem is not uninteresting but long so we defer it to the end of this section.

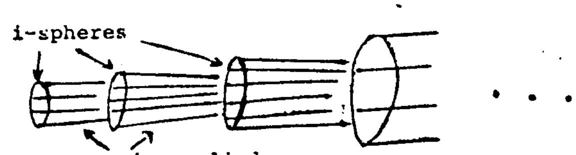
We go on to our construction of the localization which makes use of Theorem 2.1.

We begin the cellular construction by localizing the sphere.

Choose a cofinal sequence from the multiplicative set of integers generated by primes *not* in ℓ, e.g. if $\ell' = \{p_1, p_2, \dots\}$ let $\{\ell'_1, \ell'_2, \dots\}$ be $\{p_1, p_1^2 p_2^2, \dots, p_1^n \cdots p_n^n, \dots\}$.

Choose a map $S^i \xrightarrow{\ell'_n} S^i$ of degree ℓ'_n and define the "local sphere" S^i_ℓ to be the "infinite telescope" constructed from the sequence

$$S^i \xrightarrow{\ell'_1} S^i \xrightarrow{\ell'_2} S^i \to \dots .$$



picture of "the local sphere"

figure 1

The inclusion of $S^i \xrightarrow{\ell} S^i_\ell$ as the first sphere in the telescope clearly localizes homology, for $\widetilde{H}^j$ we have

$$\begin{array}{ll} 0 \xrightarrow{\ell} 0 & j \neq i \\ \mathbb{Z} \xrightarrow{\ell} \varinjlim_{l'} \mathbb{Z} = \mathbb{Z}_\ell & j = i\,. \end{array}$$

Thus by Theorem 2.1 ℓ also localizes homotopy,

$$\begin{tikzcd} \pi_* S^i \arrow[r, "\ell"] \arrow[dr, "\otimes \mathbb{Z}_\ell"'] & \pi_* S^i_\ell \arrow[d, phantom, "\wr\|"] \\ & \pi_* S^i \otimes \mathbb{Z}_\ell \end{tikzcd}$$

and ℓ is a localization. This homotopy situation is interesting because the map induced on homotopy by a degree d map of spheres is not the obvious one, e.g.

i) $S^2 \xrightarrow{d} S^2$ induces multiplication by d^2 on $\pi_3 S^2 = \mathbb{Z}$. (H. Hopf)

ii) $S^4 \xrightarrow{2} S^4$ induces a map represented by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\pi_8 S^4 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (David Frank).

Corollary *The map on d torsion of $\pi_j(S^i)$ induced by a map of degree d is nilpotent.*

Definition 2.2 *A* local *CW* complex *is built inductively from a point or a local 1-sphere by attaching cones over the local sphere using maps of the local sphere S_ℓ into the lower "local skeletons".*

Note: Since we have no local 0-sphere we have no local 1-cell.

Theorem 2.2 *If X is a CW complex with one zero cell and no one cells, there is a local CW complex X_ℓ and a "cellular" map*

$$X \xrightarrow{\ell} X_\ell$$

such that

i) ℓ induces an isomorphism between the cells of X and the local cells of X_ℓ.

ii) ℓ localizes homology.

Corollary *Any simply connected space has a localization.*

Proof: Choose a CW decomposition with one zero cell and zero one cells and consider
$$X \xrightarrow{\ell} X_\ell$$
constructed in Theorem 2.2. By Theorem 2.1 ℓ localizes homotopy and is a localization.

Proof of 2.2: The proof is by induction on the dimension. If X is a 2 complex with $X^{(1)}$ = point, then X is a wedge of 2-spheres $X = \bigvee S^2$. $\bigvee S^2 \to \bigvee S^2_{(\ell)}$ satisfies i) and ii), and is a localization. Assume the theorem true for all complexes of dimension less than or equal to $n-1$. Let X have dimension n. If $f : A \to A_\ell$ satisfies i), ii) and is a localization then $\sum f : \sum A \to \sum A_\ell$ does also. We have the Puppe sequence

$$\begin{tikzcd}
\bigvee S^{n-1} \arrow[r, "f"] \arrow[d, hook, "i"] & X^{(n-1)} \arrow[r, "c"] \arrow[d, hook, "\ell_{n-1}"] & X \arrow[r, "d"] & \bigvee S^n \arrow[r, "\sum f"] & \sum X^{(n-1)} \arrow[r] & \cdots \\
S^{n-1}_{(\ell)} \arrow[r, "f_\ell"] & X^{(n-1)}_{(\ell)} & & & &
\end{tikzcd}$$

f_ℓ exists and is unique since by Theorem 2.1 $X^{(n-1)}_{(\ell)}$ is a local space. let $X_{(\ell)}$ be the cofiber of f_ℓ. Then define $\ell : X \to X_{(\ell)}$ by
$$\ell \ = \ \ell_{n-1} \cup c(i) : X^{n-1} \cup_f c\big(\vee S^{n-1}\big) \to X^{n-1}_{(\ell)} \cup_{f_\ell} c\big(\vee S^{n-1}_{(\ell)}\big)\,.$$
ℓ clearly sets up a one to one correspondence between cells and local cells, since ℓ_{n-1} does. We may now form the ladder

$$\begin{tikzcd}
\bigvee S^{n-1} \arrow[r, "f"] \arrow[d, hook, "i"] & X^{(n-1)} \arrow[r, "c"] \arrow[d, "\ell_{n-1}"] & X \arrow[r, "d"] \arrow[d, "\ell"] & \bigvee S^n \arrow[r, "\sum f"] \arrow[d, hook, "i"] & \sum X^{(n-1)} \arrow[d, "\sum \ell_{n-1}"] \\
\bigvee S^{n-1}_{(\ell)} \arrow[r, "f_\ell"] & X^{(n-1)}_{(\ell)} \arrow[r, "c_\ell"] & X_{(\ell)} \arrow[r] & \bigvee S^n_{(\ell)} \arrow[r, "\sum f"] & \sum X^{(n-1)}_{(\ell)}\,.
\end{tikzcd}$$

All the spaces on the bottom line except possibly $X_{(\ell)}$ have local homology thus by exactness it does also. Since all the vertical maps localize homology except possibly ℓ, it does also. This completes the proof for finite dimensional complexes. If X is infinite take
$$X_\ell = \bigcup_{n=0}^{\infty} X_\ell^{(n)}\,.$$

This clearly satisfies i) and ii).

There is a construction dual to the cellular localization using a Postnikov tower.

Let X be a Postnikov tower

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \\
 & X^n \arrow[r] \arrow[d, "\text{fibre of } n\text{th } k\text{-invariant}"] & K(\pi_{n+1}, n+2) \\
X \arrow[ur] \arrow[r] \arrow[ddr] & X^{n-1} \arrow[r, swap, "\substack{n\text{th} \\ k\text{-invariant}}"] \arrow[d] & K(\pi_n, n+1) \\
 & \vdots \arrow[d] & \\
 & X^0 = * &
\end{tikzcd}
$$

We say that X is a "*local Postnikov tower*" if X^n is constructed inductively from a point using fibrations with "local $K(\pi, n)$'s". (Namely $K(\pi, n)$'s with π local.)

Theorem 2.1 *If X is any Postnikov tower*[1] *there is a local Postnikov tower X_ℓ and a Postnikov map*

$$X \to X_\ell$$

which localizes homotopy groups and k-invariants.

Proof: We induct over the number of stages in X. Induction starts easily since the first stage is a point. Assume we have a localization of partial systems

$$X^{(n-1)} \underset{l}{\to} X_\ell^{(n-1)}$$

localizing homotopy. Then the k-invariant

$$k \in H^{n+1}\big(X^{(n-1)}; \pi_n\big)$$

may be formally localized to obtain

$$\bar{k}_\ell \in H^{n+1}\big(X^{(n-1)}; \pi_n \otimes \mathbb{Z}_\ell\big)\,.$$

$\bar{k}_\ell$ determines a unique

$$k_\ell \in H^{n+1}\big(X_\ell^{(n-1)}; \pi_n \otimes \mathbb{Z}_\ell\big)$$

satisfying $\ell^* k_\ell = \bar{k}_\ell$. This follows from remarks in the proof of Theorem 2.1 and the universal coefficient theorem.

We can use the pair of compatible k-invariants (k, k_ℓ) to construct a diagram

$$\begin{tikzcd}
X^n \arrow[rr, "\ell_n"'] \arrow[d] & & X_\ell^n \arrow[d] & \\
X^{n-1} \arrow[rr, "\ell_{n-1}"'] \arrow[dr, "k"'] & & X_\ell^{n-1} \arrow[dr, "k_\ell"] & \\
& K(\pi_n, n) \arrow[rr, "\otimes \mathbb{Z}_\ell"'] & & K(\pi_n \otimes \mathbb{Z}_\ell\,, n)
\end{tikzcd}$$

where X^n is the fibre of k (by definition of k), X_ℓ^n is defined to be the fibre of k_ℓ, and ℓ_n is constructed by naturality.

Proceeding in this way we localize the entire X-tower.

Corollary *Any "simple space" has a localization.*

Proof: Choose a Postnikov tower decomposition for the "simple space". Localize the tower by 2.3 to obtain a "simple space" localizing the original.

We remark that any two localizations are canonically isomorphic by universality. Thus we speak of the localization functor.

Proposition 2.4 *In the category of "simple spaces" localization preserves fibrations and cofibrations.*

Proof: We will use the homology and homotopy properties of localization. Let

$$F \underset{i}{\to} E \underset{j}{\to} B$$

be a fibration of "simple spaces".

$$\begin{tikzcd}
{} \arrow[r] & \pi_*(F) \arrow[r] & \pi_*(E) \arrow[r] \arrow[dr] & \pi_*(B) \arrow[r] & \pi_{*-1}(F) \arrow[r] & \cdots \\
& & & \pi_*(E,F) \arrow[ur] & &
\end{tikzcd}$$

is an exact sequence. Now form

$$\begin{tikzcd} F_\ell \arrow[r, "i_\ell"] \arrow[d, "g_\ell"] & E_\ell \arrow[r, "f_\ell"] \arrow[d, equal] & B_\ell \arrow[d, equal] \\ \text{fibre } (f_\ell) \arrow[r] & E_\ell \arrow[r] & B_\ell\,. \end{tikzcd}$$

g_ℓ exists since $f_\ell \circ i_\ell = 0$. This gives rise to a commutative ladder:

$$(*) \qquad \begin{tikzcd} {} \arrow[r] & \pi_*(F_\ell) \arrow[r] \arrow[d] & \pi_*(E_\ell) \arrow[r, "f_{\ell *}"] \arrow[d, "||"] & \pi_*(B_\ell) \arrow[r] \arrow[d, "||"] & \pi_{*-1}(F_\ell) \arrow[r] \arrow[d] & \cdots \\ {} \arrow[r] & \pi_*(\text{fibre}) \arrow[r] & \pi_*(E_\ell) \arrow[r, "f_{l*}"] & \pi_*(B_\ell) \arrow[r] & \pi_{*-1}(\text{fibre}) \arrow[r] & \cdots \end{tikzcd}$$

To see that $(*)$ commutes recall that the following commutes:

$$\begin{tikzcd} \pi_*(B_\ell) \arrow[d, equal] & \pi_*(E_\ell, F_\ell) \arrow[l, "f_{l*}"', "\cong"] \arrow[r, "\delta"] \arrow[d, "||"] & \pi_*(F_{(\ell)}) \arrow[d, "g_{(\ell)}"] \\ \pi_*(B_\ell) & \pi_*(E_\ell, \text{fibre}) \arrow[l, "f_{l*}"', "\cong"] \arrow[r, "\delta"] & \pi_*(\text{fibre}) \end{tikzcd}$$

Thus $(*)$ is a commutative ladder. By the five lemma $g_\ell : F_\ell \to$ fiber is a homotopy equivalence. So that $F_\ell \xrightarrow{i_\ell} E_\ell \xrightarrow{f_\ell} B_\ell$ is a fibration. Let $A \xrightarrow{f} X \xrightarrow{i} X \cup_f CA$ be a cofibration of simple spaces. Then the following commutes up to homotopy:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & X \arrow[r] \arrow[d, "b"] & X \cup_f CA \arrow[r] \arrow[d, "b \cup C(a)"] & \sum A \arrow[r, "\sum f"] \arrow[d, "\sum a"] & \sum X \arrow[d, "\sum b"] \\ A_\ell \arrow[r, "f_\ell"] & X_\ell \arrow[r] & X_\ell \cup_{f_\ell} C(A_\ell) \arrow[r] & \sum A_\ell \arrow[r, "2 f_\ell"] & \sum X_\ell\,. \end{tikzcd}$$

There $b \cup C(a) : X \cup_f C(A) \to X_\ell \cup_{f_\ell} C(A_\ell)$ is an isomorphism of $\mathbb{Z}_\ell$ homology.

To complete the proof we need to show that

$$X_\ell \cup_{f_\ell} C(A_\ell) \;=\; Y_\ell$$

is a "simple space". Then it will follow from the second corollary to Theorem 2.1 that the natural map

$$X_\ell \cup_{f_\ell} C(A_\ell) \to (X \cup_f CA)_\ell$$

is an equivalence.

We leave to the reader the task of deciding whether this is so in case Y_ℓ is not simply connected.

We note here that no extension of the localization functor to the entire homotopy category can preserve fibrations and cofibrations. For this consider the diagram

$$\begin{tikzcd}
 & & S^2 \arrow[d, "\text{double cover}"] \\
S^1 \arrow[r, "\text{double cover}"'] & S^1 \arrow[r] & \mathbb{R}P^2 \arrow[d, "\text{natural inclusion}"] \\
 & & \mathbb{R}P^\infty = K(\mathbb{Z}/2,1)
\end{tikzcd}$$

The vertical sequence is a fibration and the horizontal sequence is a cofibration. If we localize "away from 2" (i.e. ℓ does not contain 2) we obtain

$$\begin{tikzcd}
 & & S^2_\ell \arrow[d] \\
S^1_\ell \arrow[r, "\cong"'] & S^1_\ell \arrow[r] & \mathbb{R}P^2_\ell \arrow[d] \\
 & & \mathbb{R}P^\infty_\ell \cong *
\end{tikzcd}$$

If cofibrations were preserved $\mathbb{R}P^2_\ell$ should be a point. If fibrations were preserved $\mathbb{R}P^2_\ell$ should be S^2_ℓ (which is not a point).

It would be interesting to understand what localizations are possible for more general spaces.[2]

We collect here some additional remarks and examples pertaining to localization before giving the proof of Theorem 2.1.

(1) We have used the isomorphism for local spaces X

$$[S^i_\ell, X]_{\text{based}} \cong \pi_i X\,.$$

This is a group isomorphism for $i > 1$ (since $S^i_\ell = \sum S^{i-1}_\ell$ the left hand side has a natural group structure) and imposes one on the left for $i = 1$.

(2) For a local space X,

$$\Omega^i X \cong \mathrm{Map}_{\mathrm{based}}(S^i_\ell, X)$$

generalizing (1).

(3) The natural map

$$(\Omega^i X)_\ell \to \Omega^i(X_\ell)$$

defined by universality from

$$\Omega^i \xrightarrow{\Omega^i \ell} \Omega^i X_\ell$$

is an equivalence between the components of the constant map. (Note $\Omega^i S^i$ has "$\mathbb{Z}$-components" but $\Omega^i(S^i_\ell)$ has "$\mathbb{Z}_\ell$-components"). Thus

$$\big(\mathrm{Map}_{\mathrm{based}}(S^i, S^i)_{+1}\big)_\ell \cong \mathrm{Map}_{\mathrm{based}}(S^i_\ell, S^i_\ell)_{+1}\,.$$

(4) If ℓ and ℓ' are two disjoint sets of primes such that $\ell \cup \ell' =$ all primes, then

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_\ell \arrow[d] \\ X_{\ell'} \arrow[r] & X_0 \end{tikzcd}$$

is a fibre square. (It is easy to check the exactness of

$$0 \to \pi_i X \to \pi_i X \otimes \mathbb{Z}_\ell + \pi_i X \otimes \mathbb{Z}_{\ell'} \to \pi_i X \otimes \mathbb{Q} \to 0$$

see Proposition 1.11.)

(5) More generally, X is the infinite product of its localizations at individual primes X_p

$$\begin{tikzcd} \cdots\ X_p \arrow[dr] & X_q \arrow[d] & X_r\ \cdots \arrow[dl] \\ & X_0 & \end{tikzcd}$$

over its localization at zero X_0.

$$\begin{aligned} X_{(2,3)} &\cong X_2 \times_{X_0} X_3 \\ X_{(2,3,5)} &\cong X_{(2,3)} \times_{X_0} X_5 \\ &\vdots \\ X &\cong \big((X_2 \times_{X_0} X_3) \times_{X_0} X_5\big) \times_{X_0} X_7 \dots \end{aligned}$$

(See Proposition 1.12.)

(7) X_0 is an H-space iff it is equivalent to a product of Eilenberg MacLane spaces. (See Milnor Moore – Hopf Algebras – Annals of Math 196-.)

(8) X is an H-space if and only if X_p is an H-space for each prime p with $H_*(X_p;\mathbb{Q})$ isomorphic as rings to $H_*(X_q;\mathbb{Q})$ for all p and q. (They are always isomorphic as groups.)

Proof: If X is an H-space, let $X \times X \xrightarrow{\mu} X$ be the multiplication. μ induces $\mu_p : (X \times X)_p \xrightarrow{\mu_p} X_p$, and thus $X_p \times X_p \xrightarrow{\mu_p} X_p$. Thus each X_p inherits an H-space structure from X. $(X_p)_0$ inherits an H-space structure from X_p. $(X_p)_0 = X_0$, and if we give X_0 the H-space structure from X, this homotopy equivalence is an H-space equivalence. Thus we have the ring isomorphism

$$\begin{array}{l} H_*\big((X_p)_0;\mathbb{Q}\big) \cong H_*(X_0;\mathbb{Q}) \\ \wr\| \\ H_*(X_p;\mathbb{Q})\,. \end{array}$$

Conversely if X_p is an H-space for each p and $H_*(X_p;\mathbb{Q}) \cong H_*(X_q;\mathbb{Q})$ as rings then $(X_p)_0 \simeq (X_q)_0$ as H-spaces, since the H-space structure on a rational space is determined by its Pontrjagin ring. Thus we have compatible multiplications in the diagram

$$\begin{tikzcd} X_2 \arrow[dr] & X_3 \arrow[d] & X_5 \ldots \arrow[dl] \\ & X_0 & \end{tikzcd}$$

This induces an H-space multiplication on the fibered product, X.

Note: If $H_*(X;\mathbb{Q})$ is 0 for $* \geqslant n$, some n, then the condition on the Pontrjagin rings is redundant. This follows since $H_*(X_p;\mathbb{Q}) = H_*(X_q;\mathbb{Q})$ as groups, and both must be exterior algebras on a finite set of generators. This implies that they are isomorphic as rings.

(9) If H is a homotopy commutative H-space then the functor $F = [\quad, H] \otimes \mathbb{Z}_\ell$ is represented by H_ℓ.

(10) If X has classifying space BX, then X_ℓ has one BX_ℓ.

(11) $(BU_n)_0 \overset{c}{\cong} \prod_{i=1}^{n} K(\mathbb{Q}, 2i)$, the isomorphism is defined by the rational Chern classes

$$c_i \in H^{2i}(BU_n, \mathbb{Q}), \;\; 1 \leqslant i \leqslant n\,.$$

To see this recall

$$H^*(BU_n; \mathbb{Q}) \cong \mathbb{Q}\,[c_1, \ldots, c_n]\,,$$
$$H^*\big(K(\mathbb{Q}, 2i); \mathbb{Q}\big) \cong \mathbb{Q}\,[x_{2i}]\,.$$

(12) Since $H^*(BSO_{2n}; \mathbb{Q}) \cong \mathbb{Q}\,[p_1, \ldots, p_n; \chi]/(\chi^2 = p_n)$

$$BSO_{2n} \xrightarrow{p_1, p_2, \ldots, p_{n-1}; \chi} \Big(\prod_{i=1}^{n-1} K(\mathbb{Q}, 4i)\Big) \times K(\mathbb{Q}, 2n)$$

defines the localization at zero of BSO_{2n}.

(13) $H^*(BSO_{2n-1}; \mathbb{Q}) \cong \mathbb{Q}\,[p_1, \ldots, p_{n-1}]$. In fact this is true if $\mathbb{Q}$ is replaced by $\mathbb{Z}\,[1/2]$. Thus

a) $(BSO_{2n-1})_0 \cong \prod_{i=1}^{n-1} K(\mathbb{Q}, 4i)$

b) If we localize at *odd primes*, the natural projection

$$BSO_{2n-1} \to BSO$$

has a canonical splitting over the $4n-1$ skeleton.

(14) The Thom space MU_n is the cofibre in the sequence

$$BU_{n-1} \to BU_n \to MU_n\,,$$

thus $(MU_n)_0$ is the cofibre of

$$\sum_{i=1}^{n-1} K(\mathbb{Q}, 2i) \to \sum_{i=1}^{n} K(\mathbb{Q}, 2i) \to (MU_n)_0$$

e.g. $(MU_n)_0$ has $K(\mathbb{Q}, 2n)$ as a canonical retract.

GEOMETRIC COROLLARY: *Every line in* H^{2i}(finite polyhedron; $\mathbb{Q}$) *contains a point which is "naturally" represented as the Thom class of a subcomplex*

$$V \subset \textit{polyhedron}$$

with a complex normal bundle.

(15) $(BU)_0$ is naturally represented by the direct limit over all n, k

$$BU_n \xrightarrow{\otimes k} BU_{kn}\,.$$

(16) We consider S^n for $n > 0$.

a) S_0^n is an H-space iff n is odd. In fact S_0^{2n-1} is the loop space of $K(\mathbb{Q}\,, 2n)$.

b) S_2^{2n-1} is an H-space iff $n = 1, 2$, or 4 and a loop space iff $n = 1$ or 2. (J. F. Adams)

c) S_p^{2n-1} is an H-space for all odd primes p and for all n. S_p^{2n-1} is a loop space iff p is congruent to 1 modulo n. (The necessity of the congruence is due to Adem, Steenrod, Adams, and Liulevicius. For the sufficiency see section 4, "principal spherical bundles".)

Thus each sphere S^{2n-1} is a loop space at infinitely many primes, for example S^7 at primes of the form $4k + 1$.

But at each prime p only finitely many spheres are loop spaces – one for each divisor of $p-1$ if $p > 2$ (or in general one for each finite subgroup of the group of units in the p-adic integers.)

(17) a) $PL \;=\; \left\{\begin{matrix}\text{piecewise linear}\\ \text{homeomorphisms}\\ \text{of } \mathbb{R}^n\end{matrix}\right\} \;\to\; \left\{\begin{matrix}\text{homeomorphisms}\\ \text{of } \mathbb{R}^n\end{matrix}\right\} \;=\; Top$

becomes an equivalence as $n \to \infty$. (Kirby-Siebenmann)

b) If G is the limit as $n \to \infty$ of proper homotopy equivalences of $\mathbb{R}^n$, then away from 2:

$$G/PL \;=\; G/Top \;=\; BO \qquad \text{(see section 6)}\,.$$

at 2:

$G/Top =$ product of Eilenberg Maclane spaces

$$\prod_{i=1}^{\infty} K(\mathbb{Z}/2, 4i+2) \prod_{i=1}^{\infty} K(\mathbb{Z}\,, 4i)$$

$G/PL =$ almost product of Eilenberg MacLane spaces

$$K(\mathbb{Z}/2, 2) \times_{\delta Sq^2} K(\mathbb{Z}\,, 4) \prod_{i=1}^{\infty} K(\mathbb{Z}/2, 4i+2) \prod_{i=1}^{\infty} K(\mathbb{Z}\,, 4i)\,.$$

c) Away from 2, the K-theories satisfy

$$\widetilde{K}_{PL} \cong \widetilde{K}_{Top} \cong K_0^* \oplus \text{Finite Theory}\,. \quad \text{(section 6)}$$

where K_0^* denotes the special units in K-theory $(1+\widetilde{K}O)$ and the natural map

$$\widetilde{K}O \to \widetilde{K}_{PL} \cong \widetilde{K}_{Top}$$

is given by a certain exponential operation θ in K-theory

$$\widetilde{K}O \xrightarrow{\theta\oplus\text{zero}} KO^* \oplus \text{Finite Theory}\,. \quad \text{(section 6)}$$

We will expand on these remarks in part II of this work.

Proof of Theorem 2.1: First we show i) and iii) are equivalent. For this we need three general remarks.

Remark a) For studying the map

$$X \xrightarrow{\ell} X'$$

we have its Postnikov decomposition

I

"right side up Postnikov system"

$$\begin{tikzcd}
X^n \arrow[r, "\ell_n"] \arrow[d] & (X')^n \arrow[d] & n = 1,2,3,\ldots \\
X^{n-1} \arrow[r, "\ell_{n-1}"] \arrow[d] & (X')^{n-1} \arrow[d, "n\text{th } k\text{-invariant of } X'"] & \\
K(\pi_n, n+1) \arrow[r, "k^n(\ell)"] & K(\pi'_n, n+1) &
\end{tikzcd}$$

where $X^0 = X^0_\ell = *$, the vertical sequences are fibrations, and

$$X \xrightarrow{\ell} X' \;=\; \varprojlim\{X^n \xrightarrow{\ell_n} (X')^n\}\,.$$

(The use of $\varprojlim$ here is innocuous because of the skeletal convergence of Postnikov systems. In section 3 we consider a more non-trivial

$\varprojlim$ situation and illustrate one of the pitfalls of $\varprojlim$.)

II

"upside down
Postnikov system"

$$\begin{tikzcd}
X_{n+1} \arrow[r, "\ell^{n+1}"] \arrow[d] & X'_{n+1} \arrow[d, "{n=1,2,3,\ldots}"] \\
X_n \arrow[r, "\ell^n"] \arrow[d] & X'_n \arrow[d] \\
K(\pi_n, n) \arrow[r, "k_n(\ell)"] & K(\pi'_n, n)
\end{tikzcd}$$

where $(X_1 \xrightarrow{\ell^1} X'_1) = (X \xrightarrow{\ell} X')$, the vertical maps are fibrations, and $X_n \xrightarrow{\ell^n} X'_n$ is the $(n-1)$ connected covering of $X \xrightarrow{\ell} X'$.

Remark b) For studying the maps

$$K(\pi_n, n+1) \underset{k_{n+1}(\ell)}{\overset{k^n(\ell)}{\rightleftarrows}} K(\pi'_n, n+1)$$

which are induced by homomorphisms $\pi \xrightarrow{k} \pi'$ we have the diagram

III

$$\begin{tikzcd}
K(\pi, n) \arrow[r, "k_n"] \arrow[d] & K(\pi', n) \arrow[d] \\
P \arrow[r] \arrow[d] & P \arrow[d] \\
K(\pi, n+1) \arrow[r, "k_{n+1}"] & K(\pi', n+1)\,.
\end{tikzcd}$$

Here P "the space of paths" is contractible and the vertical sequences are fibrations.

Remark c) Propositions 1.7 and 1.8 generalize easily to the following: if we have a map of fibrations

$$\begin{tikzcd}
F \arrow[r, "f"] \arrow[d] & F' \arrow[d] \\
E \arrow[r, "g"] \arrow[d] & E' \arrow[d] \\
B \arrow[r, "h"] & B'
\end{tikzcd}$$

then

i) If all spaces are connected, have π_1 Abelian and two of the maps f, g, and h localize homotopy then the third does also.

ii) If the fundamental groups act trivially on the homology of the fibres and two of the maps f, g, and h localize homology the third does also.

The proof of i) follows immediately from the exact ladder of homotopy as in Proposition 1.7.

The proof of ii) has two points. First, by Proposition 1.8 if two of the homologies

$$\widetilde{H}_* F' , \quad \widetilde{H}_* E' , \quad \widetilde{H}_* B'$$

are local the third is also. Second, if we know the homologies on the right are local then to complete the proof of ii) it is equivalent to check that f, g, h induce isomorphisms on

$$\widetilde{H}_*(\quad ; \mathbb{Z}_\ell)$$

since e.g.

$$\widetilde{H}_*(F') \cong \widetilde{H}_*(F') \otimes \mathbb{Z}_\ell \cong H_*(F'; \mathbb{Z}_\ell) \, .$$

But this last point is clear since if two of f, g, h induce isomorphisms on $H^*(\quad ; \mathbb{Z}_\ell)$ the third does also by the spectral sequence comparison theorem. With these remarks in mind it is easy now to see that a map of simple spaces

$$X \xrightarrow{\ell} X'$$

localizes homotopy iff it localizes homology.

Step 1. The case

$$(X \xrightarrow{\ell} X') \;=\; \bigl(K(\pi,1) \xrightarrow{\ell} K(\pi',1)\bigr) \, .$$

If ℓ localizes homology, then it localizes homotopy since $\pi = H_1 X$, $\pi' = H_1 X'$. If ℓ localizes homotopy then

$$(\pi \to \pi') \;=\; (\pi \to \pi_\ell) \, .$$

So ℓ localizes homology if

i) $\pi = \mathbb{Z}$. ℓ is just the localization

$$S^1 \to S^1_\ell$$

studied above.

ii)

$$\pi = \mathbb{Z}/p^n \text{ for } \pi_\ell = 0 \text{ if } p \notin \ell$$
$$\pi_\ell = \mathbb{Z}/p^n \text{ if } p \in \ell\,.$$

For general π, take finite direct sums and then direct limits of the first two cases.

Step 2. The case

$$(X \xrightarrow{\ell} X') \;=\; \big(K(\pi,n) \xrightarrow{\ell} K(\pi',n)\big)\,.$$

If ℓ localizes homology, then it localizes homotopy as in Step 1 because $\pi = H_n X$, $\pi' = H_n X'$.

If ℓ localizes homotopy, then we use induction, Step 1, diagram III in remark b) and remark c) to see that ℓ localizes homology.

Step 3. The general case $X \xrightarrow{\ell} X'$.

If ℓ localizes homology, apply the Hurewicz theorem for $n = 1$ to see that ℓ localizes π_1. Then use Step 1, diagram II in remark a) for $n = 1$ and remark c) to see that

$$X_2 \xrightarrow{\ell_2} X_2'$$

localizes homology. We apply Hurewicz here to see that ℓ_2 and thus ℓ localizes π_2. Now use Step 2 for $n = 2$, diagram II for $n = 2$ and remark c) to proceed inductively and find that ℓ localizes homotopy in all dimensions.

If ℓ localizes homotopy then apply Step 2 and diagram I inductively to see that each

$$X^n \xrightarrow{\ell_n} (X')^n$$

localizes homology for all n. Then

$$\ell \;=\; \varprojlim \ell_n$$

localizes homology.

Now we show that i) and ii) are equivalent.

If $X \xrightarrow{\ell} X'$ is universal for maps into local spaces Y, then by taking Y to be various $K(\pi,n)$'s with π local we see that ℓ induces an isomorphism of $H^*(\quad;\mathbb{Q})$ and $H^*(\quad;\mathbb{Z}/p)$, $p \in \ell$. Thus ℓ induces homomorphisms of

$$H_*(\quad;\mathbb{Q}) \text{ and } H_*(\quad;\mathbb{Z}/p),\ \ p \in \ell$$

which must be isomorphisms because their dual morphisms are, using the Bockstein sequence

$$\cdots \to H_i(\quad;\mathbb{Z}/p^n) \to H_i(\quad;\mathbb{Z}/p^{n+1}) \to H_i(\quad;\mathbb{Z}/p^n) \to \dots$$

and induction we see that ℓ induces an isomorphism on $H_*(\quad;\mathbb{Z}/p^n)$ for all n. Thus ℓ induces an isomorphism on $H_*(\quad;\mathbb{Z}/p^\infty)$ since taking homology and tensoring commute with direct limits, and

$$\mathbb{Z}/p^\infty = \varinjlim_n \mathbb{Z}/p^n,\ \ p \in \ell\,.$$

Finally ℓ induces an isomorphism of $H_*(\quad;\mathbb{Z}_\ell)$ using the coefficient sequence

$$0 \to \mathbb{Z}_\ell \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_\ell \to 0$$

and the equivalence

$$\mathbb{Q}/\mathbb{Z}_\ell = \bigoplus_{p\in\ell} \mathbb{Z}/p^\infty\,.$$

Now X' is a local space by definition. Thus the homology of X' is local by what we proved above. This proves i) implies ii).

To see that ii) implies i) consider the obstruction to uniquely extending f to f_ℓ in the diagram

$$\begin{tikzcd} X \arrow[r, "\ell"] \arrow[dr, "f"'] & X' \arrow[d, dashed, "f_\ell"] \\ & Y \end{tikzcd}$$

These lie in

$$H^*(X', X; \pi_* Y)\,.$$

Now $\pi_* Y$ is a $\mathbb{Z}_\ell$-module and ℓ induces an isomorphism of $\mathbb{Z}_\ell$ homology. Using the natural sequence (over $\mathbb{Z}_\ell$)

$$\begin{aligned} 0 \to \operatorname{Ext}\big(H_i(\quad;\mathbb{Z}_\ell), \mathbb{Z}_\ell\big) &\to H^{i+1}(\quad;\mathbb{Z}_\ell) \\ &\to \operatorname{Hom}\big(H_{i+1}(\quad;\mathbb{Z}_\ell), \mathbb{Z}_\ell\big) \to 0 \end{aligned}$$

we see that ℓ induces an isomorphism of $\mathbb{Z}_\ell$-cohomology. By universal coefficients (over $\mathbb{Z}_\ell$) the obstruction groups all vanish. Thus there is a unique extension f_ℓ, and ℓ is a localization.

Notes

1 We need not assume π_1 acts trivially on the homology of the universal cover to make this construction.

2 See equivariant localization in the proof of Theorem 4.2.

Section 3. Completions in Homotopy Theory

In this section we extend the completion constructions for groups to homotopy theory.

In spirit we follow Artin and Mazur[1], who first conceived of the profinite completion of a homotopy type as an inverse system of homotopy types with finite homotopy groups.

We "complete" the Artin-Mazur object to obtain an actual homotopy type $\widehat{X}$ for each connected CW complex X. This profinite completion $\widehat{X}$ has the additional structure of a natural compact topology on the functor, homotopy classes of maps into $\widehat{X}$,

$$[\quad, \widehat{X}]\,.$$

The compact open topology on the functor $[\quad, \widehat{X}]$ allows us to make inverse limits constructions in homotopy theory which are normally impossible.

Also under finite type assumptions on X (or $\widehat{X}$) this topology is *intrinsic* to the homotopy type of $\widehat{X}$. Thus it may be suppressed or resurrected according to the whim of the moment.

A formal completion $\bar{X}$ is constructed for *countable* complexes. $\bar{X}$ is a CW complex with a *partial topology* on the functor

$$[\quad, \bar{X}]\,.$$

We apply the formal completion to rational homotopy types where profinite completion gives only contractible spaces.

In this case, an essential ingredient in the extra topological structure on the functor $[\quad, \bar{X}]$ is a $\widehat{\mathbb{Z}}$-module structure on the homotopy groups of $\bar{X}$. This $\widehat{\mathbb{Z}}$-structure allows one to treat these groups which are enormous $\mathbb{Q}$-vector spaces.

These completion constructions and the localization of section 2 are employed to fracture a classical homotopy type into one rational and infinitely many p-adic pieces.

We discuss the reassembly of the classical homotopy types from these pieces using an Adele type and a homotopy analogue of the "arithmetic square" of section 1.

Construction of the Profinite Completion $\widehat{X}$

We outline the construction.

We begin with the following observation. Let F denote a space with finite homotopy groups. Then the functor defined by F,

$$[\quad, F]$$

may be *topologized* in a natural way. This (compact) topology arises from the equivalence

$$[Y, F] \xrightarrow{\cong} \varprojlim_{\text{finite subcomplexes } Y_\alpha} [Y_\alpha, F]$$

and is characterized by the separation property – Hausdorff.

Now given X consider the category $\{f\}$ of all maps

$$X \xrightarrow{f} F\,, \quad \pi_1 F \text{ finite}\,.$$

This category is suitable for forming inverse limits and a functor $\widehat{X}$ is defined by

$$\widehat{X}(Y) = \varprojlim_{\{f\}} [Y, F]\,.$$ [2]

The compact open topology on the right implies that the Brown requirements for the representability of $\widehat{X}$ hold.

Thus we have a well defined underlying homotopy type for the profinite completion $\widehat{X}$ together with a topology on the functor

$[\ \ ,\widehat{X}]$.

NOTE: The essential nature of this compactness for forming inverse limits is easily illustrated by an example – let L denote the inverse limit of the representable functor

$$[\ \ ,S^2]$$

using the self map induced by

$$S^2 \xrightarrow{\text{degree } 3} S^2 .$$

It is easy to check

$$\begin{aligned} &L(S^1) \cong L(S^2) \cong *, \text{ but} \\ &L(\mathbb{R}\,P^2) \text{ has two elements}. \end{aligned}$$

Thus L is not equivalent to $[\ \ ,B]$ for any space B.

We, perhaps prematurely, make the

DEFINITION 3.1 *The profinite completion $\widehat{X}$ of a connected CW complex X consists of the triple*

i) The contravariant functor $\widehat{X}$,

$$\begin{Bmatrix} \textit{homotopy} \\ \textit{category} \end{Bmatrix} \xrightarrow[\varprojlim_{\{f\}}[\ \ ,F]]{} \begin{Bmatrix} \textit{category of} \\ \textit{compact Hausdorff} \\ \textit{totally disconnected} \\ \textit{spaces} \end{Bmatrix}$$

ii) a CW complex (also denoted $\widehat{X}$) representing the composite functor into set theory

$$\begin{Bmatrix} \textit{homotopy} \\ \textit{category} \end{Bmatrix} \xrightarrow{\widehat{X}} \begin{Bmatrix} \textit{topological} \\ \textit{category} \end{Bmatrix} \xrightarrow{\substack{\textit{natural} \\ \textit{map}}} \begin{Bmatrix} \textit{set} \\ \textit{theory} \end{Bmatrix}$$

iii) The natural homotopy class of maps (profinite completion)

$$X \xrightarrow{c} \widehat{X}$$

corresponding to

$$\prod_{\{f\}}(X \xrightarrow{f} F) \ \ \textit{in} \ \ \varprojlim_{\{f\}}[X,F] .$$

The following paragraphs discuss and justify this definition.

Proposition 3.1 *If F has finite homotopy groups, $[\quad, F]$ may be naturally regarded as a functor into the topological category of compact Hausdorff totally disconnected spaces.*

A homotopy class of maps $F \to F'$ induces a continuous natural transformation of functors

$$[\quad, F] \to [\quad, F']$$

Proof: The proof is based on two assertions:

i) for each finite complex Y_α,

$$[Y_\alpha, F] \text{ is a finite set}.$$

ii) for an arbitrary complex Y the natural map

$$[Y, F] \xrightarrow{\text{restriction}} \varprojlim_{\substack{Y_\alpha \text{ a finite} \\ \text{subcomplex of } Y}} [Y_\alpha, F]$$

is a bijection of sets.

i) and ii) will be proved in the note below. Together they imply that $[Y, F]$ is naturally isomorphic to an inverse limit of finite discrete topological spaces. But from general topology we know that such "profinite spaces" are characterized by the properties compact, Hausdorff, and totally disconnected.

A homotopy class of maps $Y' \xrightarrow{f} Y$ induces a continuous map of profinite spaces

$$[Y, F] \to [Y', F].$$

A cellular representative of f induces a map of directed sets

$$\{Y'_\beta\} \to \{fY'_\beta\} \subseteq \{Y_\alpha\}$$

and thus a map (the other way) of inverse systems.

Similarly $F \xrightarrow{f} F'$ induces a continuous natural transformation of functors

$$[Y, F] \xrightarrow{\varprojlim_\alpha ([Y_\alpha, F] \xrightarrow{*} [Y_\alpha, F'])} [Y, F'].$$

Corollary *The full subcategory of the homotopy category determined by homotopy types with finite homotopy groups is canonically isomorphic to a category of functors from the homotopy category to the category of compact Hausdorff spaces,*

$$F \leftrightarrow [\quad ,F]\,.$$

Each space X determines a subcategory of this category of "compact representable functors".

First consider the category $\{f\}$, where

i) an object of $\{f\}$ is a (based) homotopy class of maps

$$X \xrightarrow{f} F\,,$$

where F has finite homotopy groups.

ii) a morphism in $\{f\}$, $f' \to f$, is a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f'"'] & & F \arrow[dl, "g"] \\ & F' & \end{tikzcd}$$

Proposition 3.2 *The category $\{f\}$ satisfies*

a) $\{f\}$ is directed, namely any two objects f and g can be embedded in a diagram

$$\begin{tikzcd} f \arrow[dr] & \\ & h \\ g \arrow[ur] & \end{tikzcd}$$

b) morphisms in $\{f\}$ are eventually unique, any diagram $f \rightrightarrows g$ can be extended to a diagram

$$f \rightrightarrows g \to h$$

where the compositions are equal.

Proof:

a) Given $X \xrightarrow{f} F$ and $X \xrightarrow{g} F'$ consider $X \xrightarrow{f\times g} F \times F'$. Then

$$\begin{tikzcd}
& F \times F' \arrow[dr, "p_2"] \arrow[dd, "p_1"] & \\
X \arrow[ur, "f\times g"] \arrow[rr, "g"'] \arrow[dr, "f"'] & & F' \\
& F &
\end{tikzcd}$$

b) Given

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & F' \arrow[d, shift right, "h"'] \arrow[d, shift left, "h'"] \\
& F
\end{tikzcd}$$

consider the "coequalizer of h and h'",

$$C(h,h') \to F' \underset{h'}{\overset{h}{\rightrightarrows}} F'\,, \ \ h\circ g \sim h \circ g'\,.$$

If h' is the point map, $C(h,h')$ is just the fibre of h (after h is made into a fibration). In general $C(h,h')$ may be described as the space mapping into F' which classifies (equivalence classes of) a map g into F' together with a homotopy between $h\circ g$ and $h'\circ g$ in F. (This is easily seen to be a representable functor in the sense of Brown.)

Or, more explicitly $C(h,h')$ may be described as a certain subset of the product)paths in F) $\times\, F'$, namely,

$$C(h,h') \ = \ \{P \in F^I, x \in F' \mid P(0) = h(x), P(1) = h'(x)\}\,.$$

As in the fibre case there is an exact sequence of homotopy groups

$$\cdots \to \pi_i C(h,h') \to \pi_i F' \xrightarrow{h_* - h'_*} \pi_i F \to \dots\,.$$

From the construction of $C(h,h')$ we can (by choosing a homotopy) form the diagram

$$\begin{tikzcd}
& F''\!= C(H,h') \arrow[d] \\
X \arrow[ur, "f''"] \arrow[r, "f"] \arrow[dr, "f"'] & F' \arrow[d, shift right, "h"'] \arrow[d, shift left, "h'"] \\
& F'\,.
\end{tikzcd}$$

From the exact sequence we see that $X \to F''$ is in the category $\{f\}$ $\big(F'' = C(h, h')$ has finite homotopy groups.$\big)$

We find ourselves in the following situation –

i) for each space X we have by Proposition 3.2 a *good indexing category* $\{X \to F\} = \{f\}$. (We also assume the objects in $\{f\}$ form a set by choosing one representative from each homotopy type with finite homotopy groups.)

ii) we have a functor from the indexing category $\{f\}$ to the category of "*compact representable functors*"

$$f \to [\quad, F]\,.$$

This should motivate

Proposition 3.3 *We can form the inverse limit of compact representable functors F_α indexed by a good indexing category $\{\alpha\}$. The inverse limit*

$$\varprojlim_{\alpha} F_\alpha$$

is a compact representable functor.

Proof: The analysis of our limit is made easier by considering for each Y and α, the "infinite image",

$$I_\alpha(Y) \;=\; \underset{\substack{\text{over all}\\ \alpha \to \beta}}{\text{intersection}}\ \{\text{image } F_\beta(Y) \to F_\alpha(Y)\}\,.$$

One can use the directedness and eventual uniqueness in $\{\alpha\}$ to see that all the morphisms from α to β

$$\alpha \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} \beta$$

induce one and the same morphism

$$I_\beta(Y) \xrightarrow{\beta_\alpha} I_\alpha(Y)\,.$$

For example, equalize $\alpha \rightrightarrows \beta$ by $\alpha \rightrightarrows \beta \to \gamma$, then

$$F_\gamma(Y) \to F_\beta(Y) \rightrightarrows F_\alpha(Y)$$

coequalizes

$$F_\beta(Y) \rightrightarrows F_\alpha$$

but $I_\beta(Y)$ is contained in the image of $F_\gamma(Y)$. Thus all maps of $F_\beta(Y)$ into $F_\alpha(Y)$ agree on $I_\beta(Y)$.

To see that this unique map

$$I_\beta(Y) \to F_\alpha(Y)$$

has image contained in $I_\alpha(Y)$ use the strong form of directedness of $\{\alpha\}$ – if

$$F_\gamma(Y) \to F_\alpha(Y)$$

is an *arbitrary* map into $F_\alpha(Y)$, dominate β and γ by δ', then equalize the compositions by δ

$$\begin{tikzcd} & \beta \arrow[dr] & & \\ \alpha \arrow[ur] \arrow[dr] & & \delta' \arrow[r] & \delta \\ & \gamma \arrow[ur] & & \end{tikzcd}$$

to obtain a commutative diagram

$$\begin{tikzcd} & F_\beta(Y) \arrow[dr] & \\ F_\delta(Y) \arrow[ur] \arrow[dr] & & F_\alpha(Y)\,. \\ & F_\gamma(Y) \arrow[ur] & \end{tikzcd}$$

Any point in $I_\beta(Y)$ is contained in the image of $F_\delta(Y)$. By commutativity its image in $F_\alpha(Y)$ must also be contained in the image of $F_\gamma(Y)$. Since

$$F_\gamma(Y) \to F_\alpha(Y)$$

was chosen arbitrarily we obtain

$$I_\beta(Y) \xrightarrow{\alpha\beta} I_\alpha(Y)\,.$$

It is clear that the inverse limit $\varprojlim_\alpha F_\alpha$ can be taken to be the ordinary inverse limit $\varprojlim I_\alpha$ over $\{\alpha\}$ with morphisms

$$\alpha \overset{\rightrightarrows}{\rightarrow} \beta \text{ collapsed to } \alpha \xrightarrow{\alpha\beta} \beta\,.$$

Moreover, $I_\alpha(Y)$ is clearly compact Hausdorff and non-void if all the $F_\beta(Y)$ are. $\big($To see that $I_\alpha(Y)$ is non-void an argument like the above is used to check the finite intersection property for the images of the $F_\beta(Y)$'s.$\big)$

Of course, $F - \alpha(Y)$ always contains the constant map so

$$Y \xrightarrow{\varprojlim_\alpha F_\alpha} \varprojlim_\alpha F_\alpha(Y)$$

assigns to each space Y a non-void compact Hausdorff space. (Here we use the fundamental fact $(**)$, the inverse limit of non-void compact Hausdorff spaces is a non-void compact Hausdorff space.)

To see that

$$G = \varprojlim_\alpha F_\alpha$$

is representable we need to check the Brown axioms –

i) the exponential law

ii) the Mayer Vietoris property.

The first property requires the natural map

$$G\big(\bigvee_\beta Y_\beta\big) \to \prod_\beta G(Y_\beta)$$

to be an isomorphism. But this is clear since inverse limits commute with arbitrary products.

The second property is more subtle and usually the one that fails. If $Y = A \cup B$, $Z = A \cap B$ (all complexes are subcomplexes), and we are given elements in $G(A)$ and $G(B)$ which restrict to the same element in $G(Z)$, then there should be at least one element in $G(Y)$ restricting to these elements in $G(A)$ and $G(B)$.

Now this is true at each index α since F_α is a representable functor. Moreover, the set of solutions is clearly a compact subset of $F_\alpha(A \cup B)$. Fundamental fact $(**)$ then insures that the inverse limit of these compact solution spaces will be non-void. Thus $\varprojlim F = G$ has the Mayer Vietoris property.

It is now easy to construct for such a functor as $\varprojlim F$ (universally defined and satisfying i) and ii) universally) a representing CW complex X

$$\varprojlim_{\alpha} F_\alpha \cong [\quad, X]\,.$$

(See for example, Spanier, *Algebraic Topology.*)

NOTE: We left two points open in Proposition 3.1.

First if Y is finite and $\pi_i F$ is finite for each i, then

$$[Y, F] \text{ is finite}\,.$$

This is proved by an easy finite induction over the cells of Y. One only has to recall that the set of homotopy classes of extensions of a map into F over the domain of the map with an i-cell adjoined has cardinality no larger than that of $\pi_i F$.

The second point was the isomorphism

$$[Y, F] \xrightarrow[r]{\cong} \varprojlim_{\{\alpha\}} [Y_\alpha, F]\,,$$

$\{\alpha\}$ the directed set of finite subcomplexes of Y.

Step 1. r is onto for all Y and F. Let x be an element of the inverse limit. Let β denote any subdirected set of $\{\alpha\}$ for which there is a map

$$Y_\beta = \bigcup_{\alpha \in \beta} Y_\alpha \xrightarrow{x_\beta} F$$

representing x/β. The set of such β's is partially ordered by inclusion and the requirement of compatibility up to homotopy of x_β. Any linearly ordered subset of the β's is countable because this is true for $\{\alpha\}$. We can construct a map on the infinite mapping telescope of the Y_β's in any linear chain to see that the partially ordered set of β's has an upper bound (by Zorn's lemma). This upper bound must be all of $\{\alpha\}$ because we can always adjoin any finite subcomplex to the domain of any x_β by a simple homotopy adjunction argument.

Step 2. r is injective if $\pi_i F$ is finite. Let f and g be two maps which determine the same element of the inverse limit. Then f and g are homotopic on every finite subcomplex Y_α. Since $\pi_i F$ is finite there are only *finitely many homotopy classes* of homotopies between f and g restricted to Y_α. These homotopy classes of homotopies between f and g form an inverse system $\big($over $\{\alpha\}\big)$ of finite sets. The inverse limit is then non-void (by compactness, again). Now we repeat step 1 to see that such an inverse limit homotopy can be realized to give an actual homotopy between f and g.

Some properties of the profinite completion

We study the homotopy and cohomology of the profinite completion $\widehat{X}$.

Proposition 3.4 *If X is $(k-1)$ connected, then*

$$\pi_k \widehat{X} \cong (\pi_k X)\hat{}\ , \ \textit{as topological groups}.$$

Proof: By definition of $\widehat{X}$

$$\pi_k \widehat{X} = \lim_{\substack{\longleftarrow \\ \{f\}}} \pi_k F\,.$$

Every finite quotient

$$\pi_k X \xrightarrow{r} \pi$$

occurs in this inverse system, namely

$$X \xrightarrow[\text{1st } k\text{-invariant}]{} K(\pi_k X, k) \xrightarrow{r} K(\pi, k)\,.$$

A covering space argument for $k = 1$ and an obstruction theory argument for $k > 1$ shows the full subcategory of $\{f\}$ where F is $(k-1)$ connected and $\pi_k X \xrightarrow{f} \pi_k F$ is onto and cofinal. This proves the proposition.

Before considering the relation between the higher homotopy of X and $\widehat{X}$, we must first consider cohomology.

There is a natural diagram

$$\begin{tikzcd}
 & H^i(\widehat{X};\mathbb{Z}/n) \arrow[dl, "r"'] & \\
H^i(X;\mathbb{Z}/n) & & \varinjlim_{\{f\}} H^i(F;\mathbb{Z}/n) \arrow[ul, "c"'] \arrow[ll, "\cong", "l"']
\end{tikzcd}$$

Proposition 3.5 *ℓ is an isomorphism for all n and i.*

PROOF: To see that ℓ is onto consider the map

$$(X \xrightarrow{f} F) \;=\; \big(X \xrightarrow{x} K(\mathbb{Z}/n, i)\big)$$

for some cohomology class x.

To see that ℓ is injective, consider the diagram

$$\begin{tikzcd} & F' \arrow[d] \\ X \arrow[ur, "f'"] \arrow[r, "f"] \arrow[dr, "0"'] & F \arrow[d, "x"] \\ & K(\mathbb{Z}/n, i) \end{tikzcd}$$

where x is some class in $H^i(F; \mathbb{Z}/n)$ which goes to zero in X, the vertical sequence is a fibration, and f' is some lifting of f.

The canonical direct summand $\{\text{image } c\}$ in $H^i(\widehat{X}; \mathbb{Z}/n)$ is closely related to the *continuous cohomology* of $\widehat{X}$, those maps

$$\widehat{X} \to K(\mathbb{Z}/n, i)$$

which induce *continuous* transformations between the respective compact representable functors

$$[\quad, \widehat{X}] \quad \text{and} \quad [\quad, K(\mathbb{Z}/n, i)]\,.$$

PROPOSITION 3.6 *The two natural subgroups of $H^i(\widehat{X}; \mathbb{Z}/n)$,*

$$\begin{aligned} L &= \varprojlim_{f} H^i(F; \mathbb{Z}/n) \cong H^i(X; \mathbb{Z}/n) \\ C &= \text{“}\textit{continuous cohomology of } \widehat{X} \end{aligned}$$

satisfy

$$L \subseteq C \subseteq (L \;\; \textit{closure})\,.$$

PROOF: Let us unravel the definition of C.

$$\widehat{X} \xrightarrow{x} K(\mathbb{Z}/n, i)$$

is a continuous cohomology class iff for each Y the induced map

$$\begin{array}{ccc} [Y, X] & \xrightarrow{x_*} & [Y, K(\mathbb{Z}/n, i)] \\ \| & & \| \\ \varprojlim_{f} [Y, F] & & \varprojlim_{\alpha} [Y_\alpha, K(\mathbb{Z}/n, i)] \end{array}$$

is continuous. This means that for each finite subcomplex $Y_\alpha \subset Y$ there is a projection $X \xrightarrow{f_\alpha} F_\alpha$, and a (continuous) map

$$[Y, F_\alpha] \xrightarrow{g_\alpha} H^i(Y_\alpha; \mathbb{Z}/n)$$

so that

$$I \qquad \begin{array}{ccc} [Y, \widehat{X}] & \xrightarrow{x_*} & [Y, K(\mathbb{Z}/n, i)] \\ \Big\downarrow \text{projection} & & \Big\downarrow \text{restriction} \\ [Y, F_\alpha] & \xrightarrow[g_\alpha]{} & [Y_\alpha, K(\mathbb{Z}/n, i)] \end{array}$$

commutes. Moreover, the map $\{Y_\alpha \to f_\alpha\}$ should be order preserving and the g_α's should be compatible as α varies.

Thus it is true that an element in

$$L = \varinjlim_{f} H^i(F; \mathbb{Z}/n)$$

determines a continuous cohomology class. For if we take an index $X \to F$ and a class $u \in H^i(F; \mathbb{Z}/n)$ for $Y - \alpha \subseteq Y$ define

$$f_\alpha = (X \to F)$$

$$g_\alpha = \big([Y, F] \xrightarrow{u/} [Y_\alpha, K(\mathbb{Z}/n, i)]\big)$$

to see that $(u) \in L$ is continuous. On the other hand, if $x \in H^i(\widehat{X}; \mathbb{Z}/n)$ is a continuous cohomology class take $Y = \widehat{X}$ and apply commutativity in (I) to the identity map of $\widehat{X}$. We obtain that for each finite subcomplex X_α of X the restriction of x factors through F_α,

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{x} & K(\mathbb{Z}/n, i) \\ \Big\uparrow & & \Big\uparrow x' \\ X_\alpha & \longrightarrow & F_\alpha \,. \end{array}$$

The element in L determined by x' has the same restriction to X_α as x. So for each finite subcomplex of $\widehat{X}$, L and C restrict to the same subgroup of $H^i(X_\alpha; \mathbb{Z}/n)$.

Thus C cannot be larger than (L closure) $= L'$. Any point outside L' is separated from L' in one of the finite quotients $H^i(X_\alpha; \mathbb{Z}/n)$

Lemma 3.7 *Suppose X has "countable type",*

$$\operatorname{Hom}(\pi_1 X; \text{finite group}) \quad \textit{and} \quad H^i(X; \mathbb{Z}/n)$$

are countable. Then $\widehat{X}$ is a simple inverse limit

$$X \cong \lim_{\overleftarrow{n}} \{\cdots \to F_n \to F_{n-1} \to \cdots \to *\}$$

where each F_n has finite homotopy groups.

Proof: Let $F_{(n)}$ denote the "coskeleton of F" obtained by attaching cells to annihilate the homotopy above dimension n. For spaces with finite homotopy groups

$$F \cong \lim_{\overleftarrow{n}} F_{(n)}$$

as compact representable functors.

Thus

$$\widehat{X} = \lim_{\overleftarrow{\{f\}}} F = \lim_{\overleftarrow{\{f\}}} \lim_{\overleftarrow{n}} F_{(n)}$$

is an inverse limit of spaces with only finitely many non-zero, finite homotopy groups.

The collection of such homotopy types is countable. The homotopy set defined by any two is finite.

Under our assumption the homotopy set $[X, F_{(n)}]$ is countable.

Thus X has an inverse limit over an indexing category C which has countably many objects and finitely many maps between any two of them.

One can now choose a linearly ordered *cofinal* subcategory

$$1 \to 2 \to \cdots \to n \to \ldots .$$

That is, each object O in C maps to n and any two maps

$$O \rightrightarrows n$$

can be equalized using $n \to m$:

$$O \rightrightarrows n \to m\,.$$

(Order the objects in C, $O_1, O_2, \ldots$ and suppose $1 \to 2 \to \ldots r$ has been chosen so that $O_1, \ldots O_{n-1}$ all map to r. Then choose

$$r \xrightarrow{f} r+1$$

so that O_{n+1} maps to $r+1$ and f equalizes all the elements in

$$\operatorname{Hom}(O_i, r) \quad i \leqslant n\,.$$

And so on.)

We get a sequence of spaces

$$\cdots \to F_n \to \cdots \to F_2 \to F_1$$

and

$$\widehat{X} \cong \varprojlim_n F_i\,.$$

Proposition 3.8 *Assume that* $\operatorname{Hom}(\pi_1 X,\ \text{finite group})$ *is countable for all* i, $H^i(X; \mathbb{Z}/n)$ *is finite. Then for all finite coefficients* A *the natural map gives*

$$H^i(X; A) \cong H^i(\widehat{X}; A) \quad \textit{for all } i\,.$$

Proof: We construct an auxiliary space F_∞ using lemma 3.7. If $\widehat{X} \cong \varprojlim F_n$ make each map $F_{n+1} \to F_n$ (inductively) into a fibration $F'_{n+1} \to F'_n$. Then let

$$F_\infty \;=\; \begin{matrix}\text{geometric}\\ \text{realization}\end{matrix} \left(\varprojlim \begin{matrix}\text{singular}\\ \text{complex}\end{matrix}\, F'_n \right)$$

Claim: *X is a retract of F_∞. The maps $f_\infty F'_n$ imply a map $F_\infty \to \widehat{X}$.*

We have maps $\widehat{X} \to F'_n$ and by the covering homotopy lemma we can construct an exactly commuting diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & F'_{n+1} \arrow[d] \\
\widehat{X} \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & F'_n \arrow[d] \\
 & \vdots
\end{tikzcd}$$

Thus we obtain a map $\widehat{X} \to F_\infty$.

Now the definition of $\widehat{X}$ implies the composition

$$\widehat{X} \to F_\infty \to \widehat{X}$$

is the identity.

Thus $H^*(X;\mathbb{Z}/n)$ is a direct summand of $h^*(F_\infty;\mathbb{Z}/n)$. But $H^*(F_\infty;\mathbb{Z}/n)$ is dual to $H_*(F_\infty;\mathbb{Z}/n)$. This homology may be computed from the chain complex

$$(\varprojlim_i C_i) \otimes \mathbb{Z}/n$$

where C_i is the singular chain complex (over $\mathbb{Z}$) of F_i'.

By the fibration property the chain maps

$$C_{i+1} \to C_i$$

are onto. Thus

$$(\varprojlim C_i) \otimes \mathbb{Z}/n \cong \varprojlim (C_i \otimes \mathbb{Z}/n)$$

and

$$H_*(F_\infty;\mathbb{Z}/n) \cong \text{Homology}\,(\varprojlim C_i')$$

where $C_i' = C_i \otimes \mathbb{Z}/n$.

Now $C_{i+1}' \to C_i'$ is still onto so by lemma 3.9 below

$$\text{Homology}\,(\varprojlim C_i') \cong \varprojlim(\text{Homology}\, C_i')\,.$$

Thus

$$H_*(F_\infty;\mathbb{Z}/n) \cong \varprojlim_i H_*(F_i;\mathbb{Z}/n)\,.$$

We know by proposition 3.5 that

$$\varinjlim H^*(F_i;\mathbb{Z}/n) \cong H^*(X;\mathbb{Z}/n)$$

is finite. We can dualize back again then and see that

$$H^*(F_\infty'\,\mathbb{Z}/n) \cong \varinjlim H^*(F_i;\mathbb{Z}/n)\,.$$

So in the diagram

$$\begin{tikzcd} & H^*(\widehat{X};\mathbb{Z}/n) \arrow[dl, "c"'] & \\ H^*(X;\mathbb{Z}/n) & \cong & \varprojlim H^*(F_i;\mathbb{Z}/n) \arrow[ul, "j"'] \end{tikzcd}$$

j is onto. c must be an isomorphism.

This proves the proposition for $A = \mathbb{Z}/n$. The general case follows by adding

Lemma 3.9 *Let*

$$\cdots \to C_{i+1} \xrightarrow{\varphi_i} C_i \to \cdots \to C_0$$

be an inverse system of chain complexes.

Suppose the chain maps φ_i are onto and the homology groups of C_i are finite in each dimension. Then

$$\text{Homology}\,(\varprojlim C_i) \cong \varprojlim(\text{Homology } C_i)\,.$$

Proof: Say that an inverse system of groups

$$\cdots \to A_i \to A_{i-1} \to \cdots \to A_0$$

is ML (Mittag-Leffler) if for each i

$$\bigcap_{n>i} \text{image}\,(A_n \to A_i) \;=\; \text{image}\,(A_N \to A_i)$$

for some $N \geqslant i$.

The point of the ML condition is the following – if

$$0 \to A_i \to B_i \to C_i \to 0$$

is an inverse system of short exact sequences and $\{A_i\}$ is ML, then

$$G \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i \to 0$$

is exact.

Assuming this let C denote the chain complex $\varprojlim C_i$ or one of its groups. Let Z, B and H denote the associated groups of cycles, boundaries, and homology classes in the appropriate (fixed) dimension. Consider C_i, Z_i, B_i, and H_i similarly.

Then we have the steps

0) $C = \lim C_i$ (by definition)

1) $Z = \varprojlim Z_i$ (always true)

2) $\{B_i\}$ is ML (φ_i onto)

3) $\{Z_i\}$ is ML (finite homology)

4) $0 \to \varprojlim Z_i \to \varprojlim C_i \to \varprojlim B_i \to 0$ (by 3))

5) $\lim B_i = B$ ($0 \to Z \to C \to B \to 0,$ 1), and 4))

6) $0 \to \varprojlim B_i \to \varprojlim Z_i \to \varprojlim H_i \to 0$ (by 2))

7) $\varprojlim H_i = H$ ($0 \to B \to Z \to H \to 0$, 10, 5), and 6).)

Before drawing some corollaries to the proof and statements of proposition 3.8 we need one more algebraic lemma.

LEMMA 3.10 *Suppose $\pi \xrightarrow{c} \pi'$ is a homomorphism of Abelian groups such that*

i) $c \otimes \mathbb{Z}/n$ is an isomorphism of finite groups

ii) π' is an inverse limit of finite groups.

Then $\pi' \cong \widehat{\pi}$ and c is profinite completion.

PROOF: Note that $G \otimes \mathbb{Z}/n$ finite implies

$$\widehat{G} \cong \varprojlim (G \otimes \mathbb{Z}/n)\,.$$

Thus c induces an isomorphism of profinite completions

$$\widehat{\pi} \;=\; \lim(\pi \otimes \mathbb{Z}/n) \xrightarrow{\varprojlim c \otimes \mathbb{Z}/n} \lim(\pi' \otimes \mathbb{Z}/n) \;=\; \widehat{\pi}'\,.$$

It suffices to prove that $\widehat{\pi}' \cong \pi'$. Topologize π' using the hypothesis

$$\pi' \cong \varprojlim_{\alpha} F_\alpha\,.$$

Now $\pi' \xrightarrow{\cdot n} \pi'$ is continuous since $\varprojlim_{n} (F_\alpha \xrightarrow{\cdot n} F_\alpha)$ is.

Thus the image $n\pi'$ is compact and closed. The quotient

$$\pi'/n\pi' \;=\; \pi' \otimes \mathbb{Z}/n$$

is finite, so $n\pi'$ is also open.

Thus the natural map

$$\pi' \xrightarrow{l} \lim(\pi' \otimes \mathbb{Z}/n) \;=\; \widehat{\pi}'$$

is continuous.

ℓ is onto by the usual compactness argument. But

$$\pi' \to \widehat{\pi}'$$

is always a monomorphism for any profinite group π'. This proves the lemma.

Note that we have shown the inverse limit topology on π' is intrinsic to the algebraic structure of π' if $\pi' \otimes \mathbb{Z}/n$ is finite. Namely

$$\pi' \cong \widehat{\pi}'\,.$$

This phenomenon generalizes to homotopy types.

For a brief moment let $|\widehat{X}|$ denote the underlying CW complex of $\widehat{X}$.

Corollary 3.11 *If $\pi_1 X = 0$, and $H^i(X;\mathbb{Z}/n)$ is finite for each n and i, then*

$$\big(|\widehat{X}|\big)^{\hat{}} \cong \widehat{X}\,.$$

Proof: The natural map

$$X \to |\widehat{X}|$$

induces an isomorphism of mod n cohomology by proposition 3.8.

By obstruction theory any map of X into a simply connected space with finite homotopy groups

$$X \to F$$

factors uniquely through $|\widehat{X}|$

$$|\widehat{X}| \to F\,.$$

Thus $\widehat{X}$ and $\big(|\widehat{X}|\big)^{\hat{}}$ can be defined by the same inverse system.

Corollary *In the simply connected "finite type" case the topology in $[\quad, \widehat{X}]$ is implied by the homotopy type of the representing complex $|\widehat{X}|$.*

Remark: The simply connected hypothesis is probably unnecessary in this corollary. Proposition 3.8 can hopefully be proved for finite twisted coefficients. Then we would know that the topology on $\widehat{X}$ is "intrinsic" for any X of "finite type".

This is of course true for any non-simply connected space with *finite* homotopy groups. ($\widehat{F} \cong F$.)

Corollary 3.12 *Suppose H_iX is finitely generated and* Hom$\,(\pi_1 X$, finite group) *is countable. Then*

$$H_iX \to H_i\widehat{X} \quad \textit{(integer coefficients)}$$

is profinite completion.

Proof: We use the space F_∞ constructed in the proof of Proposition 3.8. Consider the diagram (integer coefficients)

$$\begin{tikzcd}
& H_*\widehat{X} \arrow[dr, shift left, "j"'] & \\
H_*X \arrow[ur, "c"] \arrow[dr, "\ell"'] & & H_*F_\infty \arrow[ul, shift left, "i"'] \arrow[dl, "u"] \\
& \varprojlim_i H_*F_i &
\end{tikzcd}$$

By Lemma 3.9 u is an isomorphism. $i \circ j$ is the identity by construction.

Also all maps induce isomorphisms of finite groups with $\mathbb{Z}/n$-coefficients.

We can then use the natural sequence

$$0 \to \operatorname{Tor}(H_{i-1}, \mathbb{Z}/n) \to H_i(Y, \mathbb{Z}/n) \to H_iY \otimes \mathbb{Z}/n \to 0$$

for comparing $\widehat{X}$ and F_∞ and then $\widehat{X}$ and X.

If j is an isomorphism in dimension $i-1$ we see that $j \otimes \mathbb{Z}/n$ is an isomorphism in dimension i.

The proof of Lemma 3.10 shows

$$H_i F_\infty \cong \varprojlim_n (H_i F_\infty) \otimes \mathbb{Z}/n \,.$$

It follows that the direct summand $H_i \widehat{X}$ must be all of $H_i F_\infty$. By induction over i,

$$H_* \widehat{X} \cong H_* F_\infty \,.$$

We use the same argument to make the inductive argument – if $H_{i-1}X \to H_{i-1}\widehat{X}$ is profinite completion, then $H_i X \otimes \mathbb{Z}/n \to H_i \widehat{X} \otimes \mathbb{Z}/n$ is an isomorphism. Thus $h_i X \to H_i \widehat{X}$ is profinite completion by Lemma 3.10.

This uses $\mathrm{Tor}\,(G, \mathbb{Z}/n) \cong \mathrm{Tor}\,(\widehat{G}, \mathbb{Z}/n)$ which is true for finitely generated groups.

Now we consider the higher homotopy groups of $\widehat{X}$. These are profinite groups in general. They are simply related to the homotopy groups of X only under strong fundamental group assumptions.

We will assume that *all* the homotopy groups of X are finitely generated Abelian groups. We further assume that $\pi_1 X$ and $\pi_1 \widehat{X}$ act trivially on the mod n cohomology of the universal covers of X and $\widehat{X}$ respectively.

PROPOSITION 3.13 *Under these π_1 and finite assumptions on X, the natural map*

$$\pi_i X \to \pi_1 \widehat{X}$$

is profinite completion.

PROOF: The case $i = 1$ was considered above. However we reprove this case in such a way that it works for $i > 1$.

By Proposition 3.8 $X \xrightarrow{c} \widehat{X}$ is an isomorphism of cohomology mod n (which is finite). Now

$$\pi_1 X \to \pi_1 \widehat{X}$$

is isomorphic to

$$H_1 X \to H_1 \widehat{X}$$

by Hurewicz. This map becomes an isomorphism of finite groups when we tensor with $\mathbb{Z}/n$, and $\pi_1\widehat{X}$ is a profinite group. By Lemma 3.10

$$\pi_1 X \to \pi_1 \widehat{X}$$

is the profinite completion.

Thus in the map of universal cover sequences

$$\begin{tikzcd}
X_1 \arrow[r, "d"] \arrow[d] & (\widehat{X})_\ell = \text{universal cover of } X \arrow[d] \\
X \arrow[r, "c"] \arrow[d] & \widehat{X} \arrow[d] \\
K(\pi_1 X, 1) \arrow[r, "c_1"] & K(\pi_1 \widehat{X}, 1)
\end{tikzcd}$$

c_1 induces an isomorphism of cohomology mod n – which is finite. (Since $\pi_1 X$ is a finitely generated Abelian group – see note below.)

An easy spectral sequence argument then shows that X_1 and $(\widehat{X})_1$ also have finite (mod n) cohomology in each dimension.

By the spectral sequence sequence comparison theorem d is an isomorphism of cohomology mod n.

Now we apply the same argument to the map

$$X_1 \xrightarrow{d} (\widehat{X})_1$$

using the diagram

$$\begin{tikzcd}
X_2 \arrow[r, "e"] \arrow[d] & (\widehat{X})_2 = 2 \text{ connective cover of } \widehat{X} \arrow[d] \\
X_1 \arrow[r, "d"] \arrow[d] & (\widehat{X})_1 \arrow[d] \\
K(\pi_2 X_1, 2) \arrow[r, "d_2"] & K\big(\pi_2 (\widehat{X})_1, 2\big)
\end{tikzcd}$$

to see that

$$\pi_2 X \to \pi_2 \widehat{X}$$

is profinite completion (and so on).

The proof has the

Corollary 3.14 *If $X \to Y$ is a map of simply connected spaces such that*

a) $\pi_i X$ is finitely generated

b) $\pi_i Y$ is profinite

c) f induces an isomorphism of cohomology mod n,

then f is equivalent to profinite completion,

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\substack{\textit{profinite}\\ \textit{completion}}"'] & Y \arrow[d, "\cong"] \\ & \widehat{X} \end{tikzcd}$$

Proof: The proof of 3.13 shows that f is profinite completion on homotopy. By obstruction theory any map of X into a simply connected space with finite homotopy groups factors uniquely through Y. These factorings create a map $Y \to \widehat{X}$ which is an isomorphism of homotopy groups.

As a final summary we state the

Corollary 3.15 *For a map of simply connected spaces*

$$X \xrightarrow{c} Y$$

with $\pi_i X$ finitely generated the following are equivalent:

i) c completes the homotopy

$$\begin{tikzcd} \pi_i X \arrow[r, "c"] \arrow[dr, "\substack{\textit{profinite}\\ \textit{completion}}"'] & \pi_i Y \arrow[d, "\cong"] \\ & (\pi_i X)\hat{} \end{tikzcd}$$

ii) c completes the reduced integral homology

$$\begin{tikzcd} \widetilde{H}_i(X) \arrow[r, "c"] \arrow[dr] & \widetilde{H}_i Y \arrow[d, "\cong"] \\ & (\widetilde{H}_i X)\hat{} \end{tikzcd}$$

iii) $\pi_i Y$ *is profinite and* c *is an isomorphism of cohomology (mod* n*).*

iv) c *is profinite completion.*

Moreover if we are in one of these cases the homotopy type of $Y \cong \widehat{X}$ *determines the topology on* $[\quad, \widehat{X}]$ *since* $\widehat{Y} \cong Y$.

PROOF: We have proved that iv) implies i) and ii) and iii) is equivalent to iv). The argument of Proposition 3.13 and the note below show ii) implies i). Thus by universal coefficients ii) implies iii).

An induction over the (right side up) Postnikov system together with the note below shows i) implies iii). (See the end of the localization section for analogous details.)

We have left the following point open.

NOTE: If π is a finitely generated Abelian group, then

$$K(\pi, n) \to K(\widehat{\pi}, n)$$

induces an isomorphism of mod n cohomology.

Using the diagram

$$\begin{array}{ccc}
K(\pi, n-1) & \longrightarrow & K(\widehat{\pi}, n-1) \\
\downarrow & & \downarrow \\
\text{Paths} & & \text{Paths} \\
\downarrow & & \downarrow \\
K(\pi, n) & \longrightarrow & K(\widehat{\pi}, n)
\end{array}$$

we can reduce to the case $n = 1$.

This case can be reduced to the case $\pi = \mathbb{Z}$ or $\pi = \mathbb{Z}/n$.

The second case is clear.

The case $\pi = \mathbb{Z}$ may be proved as follows:

$$K(\widehat{\mathbb{Z}}, 1) \text{ is } \varprojlim_n K(\mathbb{Z}/n, 1)$$

as compact representable functors.

The argument of 3.8 shows the reduced (mod n) homology of $K(\widehat{\mathbb{Z}},1)$ is a direct summand of

$$\varprojlim_i \widetilde{H}_*\big(K(\mathbb{Z}/i,1);\mathbb{Z}/n\big)$$

which is easily calculated to be

$$\begin{array}{ll} \mathbb{Z}/n & \text{for } i=1 \\ 0 & \text{for } i>1\,. \end{array}$$

On the other hand,

$$H_1\big(K(\widehat{\mathbb{Z}},1),\mathbb{Z}/n\big) \cong \widehat{\mathbb{Z}} \otimes \mathbb{Z}/n \cong \mathbb{Z}/n\,.$$

Thus

$$H_*\big(K(\widehat{\mathbb{Z}},1),\mathbb{Z}/n\big) \xrightarrow{\cong} H_*\big(K(\widehat{\mathbb{Z}},1),\mathbb{Z}/n\big)\,.$$

Example 1

i) for G finitely generated and Abelian

$$K(G,n)\hat{\ } \cong K(\widehat{G},n) \cong K(G\otimes\widehat{\mathbb{Z}},n)$$

ii) if π_*X is finite, $X \cong \widehat{X}$

iii) $\widehat{S}^n \cong$ Moore space $M(\widehat{\mathbb{Z}},n)$

iv) $K(\mathbb{Q}/\mathbb{Z},1)\hat{\ } \overset{\widehat{\delta}}{\cong} (\mathbb{C}P^\infty)\hat{\ }$

(the fibration $K(\mathbb{Q},1) \to K(\mathbb{Q}/\mathbb{Z},1) \xrightarrow{\delta} K(\mathbb{Z},2) \cong \mathbb{Z}P^\infty$ shows δ induces an isomorphism of (mod n) cohomology which is finite. $\widehat{\delta}$ is a map between simple connected spaces with profinite homotopy groups, and $\widehat{\delta}$ also induces an isomorphism of the mod n cohomology. The arguments above show that $\widehat{\delta}$ is an equivalence.)

v) A recent theorem of Quillen comparing calculations of Milgram and Nakaoka suggests

profinite completion $K(S_\infty,1) \cong \varinjlim_n(\Omega^n S^n) \equiv F_0$ component of constant map

($S_\infty = \varinjlim_n$ symmetric group of order n.)

At least there is a map

$$K(S_\infty, 1)\hat{} \to F_0$$

inducing an isomorphism on

$$\pi_1 F_0 = \mathbb{Z}/2 \cong (S_\infty)\hat{}$$

and an isomorphism of (mod n) cohomology.

Assuming this we see that profinite completion converts the infinite symmetric group into the stable homotopy groups of spheres.

v) Suppose X has the homotopy type of a complex algebraic variety V. Then under mild assumptions on X Čech-like nerves of algebraic (etale) coverings of X give simplicial complexes with finite homotopy groups approximating the cohomology of $\widehat{X}$. (Lubkin – "On Conjectures of Weil", American Journal of Math, 1968.)

The work of Artin-Mazur[3] related to Grothendieck's cohomology implies

$$\widehat{X} \cong \varprojlim_{\text{etale covers}} (\text{Čech-like nerve})\,.$$

A beautiful consequence is the Galois symmetry that $\widehat{X}$ inherits from this algebraic description. This is a preview of section 5.

ℓ-profinite completion

One can carry out the previous discussion replacing finite groups by ℓ-finite groups. (ℓ is a set of primes and ℓ-finite means the order is a product of primes in ℓ. $\ell = \{\text{all primes}\}$ is the case already treated.)

The construction an propositions go through without essential change. For example, the hypothesis of Proposition 3.8 becomes

$$H^i(X; \mathbb{Z}/p) \text{ is finite}, \;\; p \in l\,,$$
$$\text{Hom}\,(\pi_1 X, \ell\text{-finite group}) \;\text{ is countable}\,.$$

The conclusion becomes

$$H^*(X; \mathbb{Z}/p) \cong H^*(\widehat{X}_\ell; \mathbb{Z}/p)\,, \;\; p \in l\,.$$

The further π_* assumptions analogous to that in Proposition 3.13 imply

$$(\pi_i X)\hat{}_\ell \cong \pi_i(X\hat{}_\ell)\,.$$

Again the topology in the functor

$$[\quad, \widehat{X}_\ell]$$

may be recovered from the homotopy type of $\widehat{X}_\ell$ if $\pi_1 X = 0$ and X has $H^i(X;\mathbb{Z}/p)$ finite, $p \in l$.

A new point in the simply connected case is a canonical splitting into p-adic components.

PROPOSITION 3.16 *If $(\pi_1 X)\hat{}_\ell = 0$, there is a natural splitting*

$$\widehat{X}_\ell \cong \prod_{p \in l} \widehat{X}_p$$

in the sense of compact representable functors. Furthermore any map

$$\widehat{X}_\ell \xrightarrow{f} \widehat{Y}_\ell$$

factors

$$f = \prod_{p \in l} f\hat{}_p\,.$$

PROOF: Write any space F with finite homotopy groups as an inverse limit (in the sense of compact representable functors) of it coskeletons

$$F = \varprojlim_n F^n\,.$$

F^n has the first n-homotopy groups of F.

If $\pi_1 F = 0$, each F^n may be decomposed (using a Postnikov argument) into a finite product of p-primary components

$$F^n = \prod_{p \in S_n} F^n_p\,.$$

Then

$$\begin{aligned} F &= \varprojlim_n \Big(\prod_{p \in S_n} F^n_p \Big) \\ &= \prod_p (\varprojlim_n F^n_p) \\ &= \prod_p F_p\,. \end{aligned}$$

If $\pi_i F$ is ℓ-finite, we obtain

$$F \cong \prod_{p \in l} F_p \, .$$

More generally,

$$\widehat{X}_\ell \cong \varprojlim_f F \, , \quad X \xrightarrow{f} F \, , \;\; F \;\ell\text{-finite} \, .$$

So we obtain

$$\begin{aligned} \widehat{X}_\ell &\cong \varprojlim_f \prod_{p \in l} F_p \\ &\cong \prod_{p \in l} \varprojlim_f F_p \\ &\cong \prod_{p \in l} \widehat{X}_p \, . \end{aligned}$$

Recall that the topology in $[\quad , F]$ was canonical so that it may be used or discarded at will. This should clarify the earlier manipulations.

The last equation

$$\varprojlim_f F_p \cong \widehat{X}_p \, ,$$

uses the splitting on the map level

$$(F \to F') \;\; = \;\; \prod_{p \in l} (F_p \xrightarrow{f_p} F'_p) \, .$$

This follows from the obstruction theory fact that any map

$$F_p \to F'_q \quad p \neq q$$

is homotopic to a constant map.

This generalizes (using obstruction theory) to – any map

$$\widehat{X}_p \xrightarrow{f} \widehat{Y}_q \quad p \neq q$$

is null homotopic.

Example Let $X = BO_2$, the classifying space of the 2 dimensional orthogonal group. Let ℓ be the set of odd primes. Then $\widehat{X}_\ell$ satisfies

i) $\pi_1 \widehat{X}_\ell = 0$

ii) $H^*(X_\ell; \mathbb{Z}/p) = \mathbb{Z}/p[x_4]$, p odd.

These imply

iii) $\Omega \widehat{X}_\ell \cong \widehat{S}^3_\ell$.

So the 2 non-zero homotopy groups of O_2 have been converted into the infinitely many non-zero groups of S^3_ℓ.

More precisely,

$$(BO_2)\hat{}_\ell \cong \prod_{\text{odd primes}} (BO_2)\hat{}_p$$

and for $(BO_2)\hat{}_p$,

$$\pi_2, \pi_3, \pi_4, \ldots, \pi_{2p} \;\cong\; 0, 0, 0, \widehat{\mathbb{Z}}_p, 0, 0, \ldots, 0, \mathbb{Z}/p, \ldots$$

The calculation ii) is discussed in more detail in section 4 under “principal spherical fibrations”.

Formal Completion

We give a construction which generalizes the completion construction

$$\text{rational numbers} \xrightarrow{\otimes \widehat{\mathbb{Z}}_\ell} \ell\text{-adic numbers}\,.$$

Let X be a countable complex[4]. Then X may be written as an increasing union of finite subcomplexes

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset X\,.$$

Definition (Formal ℓ-completion) *The formal ℓ-completion is defined by*

$$\bar{X}_\ell = \bigcup_{n=1}^{\infty} (X_n)\hat{}_\ell\,,$$

the infinite mapping telescope of the ℓ-profinite completions of the finite subcomlexes X_n.

Note that $\bar{X}_n$ is a CW complex and the functor

$$[\quad, \bar{X}_\ell]$$

has a *partial topology*. If Y is a *finite* complex then

$$\begin{aligned} [Y, \bar{X}_\ell] &\cong \varinjlim_n [Y, (X_n)\hat{}_\ell]^{5} \\ &\cong \varinjlim \{\text{profinite spaces}\} \end{aligned}$$

has the direct topology.

PROPOSITION 3.17 *The homotopy type of X_ℓ and the partial topology on*

$$[\quad, X_\ell]$$

only depend on the homotopy type of X.

PROOF: If $\{X_j\}$ is another filtering of X by finite subcomplexes, we can find systems of maps

$$X_i \to X_{j(i)}$$
$$X_j \to X_{i(j)}$$

because the CW topology on X forces any map of a compact space into X to be inside X_i (X_j) for some i (for some j). The compositions

$$X_j \to X_{i(j)} \to X_{j(i(j))}$$
$$X_i \to X_{j(i)} \to X_{i(j(i))}$$

are the given inclusions. So the induced maps

$$\bigcup \widehat{X}_i \rightleftarrows \bigcup \widehat{X}_j$$

$$\varinjlim_i [Y, \widehat{X}_i] \rightleftarrows \varinjlim_j [Y, \widehat{X}_j]\,, \quad Y \text{ finite}$$

are inverse. In the second line maps are continuous, so this proves the proposition.

A similar argument using cellular maps shows that a filtering on a homotopy equivalent space X' leads to the same result.

In fact, the formal completion is a functorial construction on the homotopy category.

The homotopy groups of $\bar{X}_\ell$ are direct limits of profinite groups. The maps in the direct system are continuous and $\widehat{\mathbb{Z}}_\ell$-module homomorphisms. So $\pi_i X_\ell$ is a topological group and a $\widehat{\mathbb{Z}}_\ell$-module. Similarly $\pi_i X \otimes \widehat{\mathbb{Z}}_\ell$ is a $\widehat{\mathbb{Z}}_\ell$-module and a topological group. The topology is the direct limit topology

$$\pi_i X \otimes \widehat{\mathbb{Z}}_\ell \cong \varinjlim_{\substack{\text{finitely generated}\\ \text{subgroups } H}} \widehat{H}_\ell \,.$$

Proposition 3.18 *If X is simply connected, there is a natural isomorphism*

$$\pi_i \bar{X}_\ell \cong \pi_i X \otimes \widehat{\mathbb{Z}}_\ell$$

of topological groups and $\widehat{\mathbb{Z}}_\ell$-modules.

Proof: Write X as an increasing union of simply connected finite subcomplexes $\{X^i\}$. Then

$$\begin{aligned}
\pi_j \bar{X}_\ell &\cong \varinjlim_i \pi_j \widehat{X}^i_\ell \quad \text{since } S^j \text{ is compact,} \\
&\cong \varinjlim_i (\pi_j X^i)\hat{}_\ell \quad \begin{array}{l}\text{since } \pi_1 X^i = 0 \text{ and} \\ X^i \text{ is finite} \\ \text{(Proposition 3.14),}\end{array} \\
&\cong \varinjlim_i \big((\pi_j X^i) \otimes \widehat{\mathbb{Z}}_\ell\big) \quad \text{since } \pi_j X^i \text{ is finitely generated,} \\
&\cong (\varinjlim_i \pi_j X^i) \otimes \widehat{\mathbb{Z}}_\ell \quad \begin{array}{l}\text{since tensoring commutes} \\ \text{with taking direct limits,}\end{array} \\
&\cong (\pi_j X) \otimes \widehat{\mathbb{Z}}_\ell \quad \text{again using the compactness of } S^j,
\end{aligned}$$

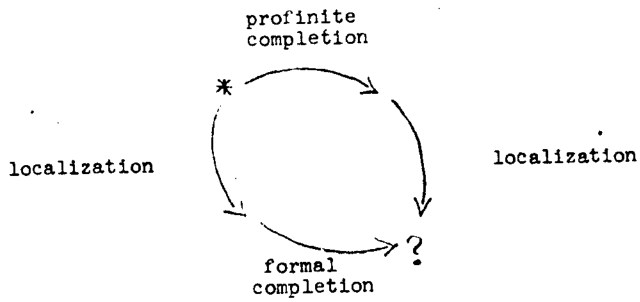
and all isomorphisms are $\widehat{\mathbb{Z}}_\ell$-homomorphisms and continuous.

The Arithmetic Square in Homotopy Theory

We consider the homotopy analogue of the arithmetic square,

$$\begin{tikzcd}
\text{rational integers } = \mathbb{Z} \arrow[r] \arrow[d] & \prod_p \widehat{\mathbb{Z}}_p = p\text{-adic integers} \arrow[d] \\
\text{rational numbers } = \mathbb{Q} \arrow[r] & \mathbb{Q} \otimes \widehat{\mathbb{Z}} = p\text{-adic numbers}\,.
\end{tikzcd}$$

Say that X is *geometric* if X is homotopy equivalent to a CW complex with only finitely many i-cells for each i. We consider the process



and assume for now that X is simply connected.

The *profinite completion* of X, $\widehat{X}$ is a CW complex together with a compact topology on the functor

$$[\quad, \widehat{X}] .$$

In this simply connected geometric case, we saw that this topology could be recovered from the homotopy type of $\widehat{X}$. The homotopy groups of X are the $\widehat{\mathbb{Z}}$-modules,

$$\pi_i \widehat{X} \cong (\pi_i X)\hat{} \ .$$

The *localization at zero* of X, X_0 is a countable complex whose homotopy groups are the finite dimensional $\mathbb{Q}$-vector spaces

$$\pi_i X_0 \cong \pi_i X \otimes \mathbb{Q} \ .$$

The *localization at zero* of $\widehat{X}$ gives a map of CW complexes

$$\widehat{X} \xrightarrow{\text{localization}} (\widehat{X})_0$$

which in homotopy satisfies

$$\begin{tikzcd}
\pi_i \widehat{X} \arrow[r, "\text{localization}"] \arrow[dr, "x \mapsto x \otimes 1"'] & \pi_i (\widehat{X})_0 \arrow[d, "\cong"] \\
& (\pi_i \widehat{X}) \otimes \mathbb{Q}
\end{tikzcd}$$

The isomorphism is uniquely determined by the requirement of commutativity. Thus $\pi_i(\widehat{X})_0$ has the natural $\widehat{\mathbb{Z}}$-module structure (or topology) of

$$\pi_i \widehat{X} \otimes \mathbb{Q} \cong \varinjlim_n (\pi_i \widehat{X} \xrightarrow{n} \pi_i \widehat{X}) \,.$$

The *formal completion* of X_0 may be defined since X_0 is countable. This gives a CW complex $(X_0)^-$ with a partial topology on

$$[\quad , (X_0)^-] \,.$$

In particular, the homotopy group are topological groups and $\widehat{\mathbb{Z}}$-modules; and these with structures

$$\pi_i (X_0)^- \cong (\pi_i X_0) \otimes \widehat{\mathbb{Z}} \,.$$

PROPOSITION 3.19 *Let X be a* geometric *simply connected complex. Then there is a natural homotopy equivalence between*

$$(X_0)^- \ \ \textit{and} \ \ (\widehat{X})_0 \,.$$

The induced isomorphism on homotopy groups preserves the module structure over the ring of "finite Adeles",

$$\mathbb{Q} \otimes \widehat{\mathbb{Z}} \,.$$

PROOF: Filter X by simply connected finite subcomplexes $\{X^i\}$. Then the natural map

$$\bar{X} \equiv \varinjlim_i (\widehat{X}^i) \to \widehat{X}$$

is a homotopy equivalence ($\varinjlim$ means infinite mapping telescope). This follows since our assumptions imply

$$\pi_i \bar{X} \cong \pi_i X \otimes \widehat{\mathbb{Z}} \cong (\pi_i X)\hat{} \cong \pi_i \widehat{X} \,.$$

So apply the formal completion functor to the map

$$X \xrightarrow{l} X_0 \,.$$

This gives a map

$$\bar{X} \xrightarrow{\bar{l}} (X_0)^-$$

which on homotopy is a map of $\widehat{\mathbb{Z}}$-modules

$$\begin{tikzcd}
\pi_i \bar{X} \arrow[r, "l^-"] \arrow[d, equal] & \pi_i (X_0)^- \arrow[d, equal, "\widehat{\mathbb{Z}}\text{-module isomorphism}"] \\
\pi_i X \otimes \widehat{\mathbb{Z}} \arrow[r, "l \otimes \text{identity } \widehat{\mathbb{Z}}"] & \pi_i X_0 \otimes \widehat{\mathbb{Z}} \, .
\end{tikzcd}$$

But ℓ is the localization

$$\begin{tikzcd}
\pi_i X \arrow[r, "l"] \arrow[dr, "x \mapsto x \otimes 1"'] & \pi_i X_0 \arrow[d, "\cong"] \\
& \pi_i X \otimes \mathbb{Q} \ .
\end{tikzcd}$$

So for the composed map ℓ'

$$\begin{tikzcd}
\widehat{X} \arrow[rr, "l'"] \arrow[dr, "\cong"'] & & (X_0)^- \arrow[dl, "l^-"] \\
& \bar{X} &
\end{tikzcd}$$

we can construct a diagram

$$\begin{tikzcd}
\pi_i \widehat{X} \arrow[r, "l'"] \arrow[dr, "x \mapsto x \otimes 1"'] & \pi_i (X_0)^- \arrow[d, equal, "\widehat{\mathbb{Z}}\text{-module isomorphism}"] \\
& \pi_i \widehat{X} \otimes \mathbb{Q} \ .
\end{tikzcd}$$

Thus $(X_0)^-$ is a localization at zero for $\widehat{X}$ with the correct $\widehat{\mathbb{Z}}$-module structure on its homotopy groups. It follows that it has the correct $(\widehat{\mathbb{Z}} \otimes \mathbb{Q})$-module structure.

Definition *Let $(\widehat{X})_0$ or $(X_0)^-$ be denoted by X_A, the "finite Adele type of X". The homotopy groups of X_A have the structure of modules over the ring of "finite Adeles",*

$$A = \mathbb{Q} \otimes \widehat{\mathbb{Z}} \, .$$

Using this equivalence we may form an arithmetic square in homotopy theory for X "geometric" and simply connected

$$\begin{tikzcd} X \arrow[r, "\substack{\text{profinite}\\ \text{completion}}"] \arrow[d, "\text{localization}"'] & \widehat{X} = \prod_p \begin{pmatrix} p\text{-adic completion} \\ \text{of } X \end{pmatrix} \arrow[d, "\text{localization}"] \\ \text{“rational type of } X\text{”} = X_0 \arrow[r] & X_A = \text{“Adele type of } X\text{”} \end{tikzcd}$$

In the homotopy level we have

$$\pi_* X \otimes \left\{ \begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \widehat{\mathbb{Z}} \arrow[d] \\ \mathbb{Q} \arrow[r] & \widehat{\mathbb{Z}} \otimes \mathbb{Q} \end{tikzcd} \right\} .$$

This proves the

PROPOSITION 3.20 *The arithmetic square is a "fibre square". That is, if*

$$\widehat{X} \to X_A , \quad X_0 \to X_A$$

are converted into fibrations, then

$$X \xrightarrow{\textit{localization}} X_0 \quad \textit{and}$$

$$X \xrightarrow{\substack{\textit{profinite}\\ \textit{completion}}} \widehat{X}$$

are equivalent to the induced fibration over X_0 and $\widehat{X}$, respectively.

COROLLARY 3.21 *X is determined by its rational type X_0, its profinite completion $\widehat{X}$, and the equivalence*

$$(X_0)^- \underset{e}{\cong} (\widehat{X})_0 \quad (\equiv \textit{“Adele type of } X) .$$

e is a $\widehat{\mathbb{Z}}$-module isomorphism of homotopy groups. The triple

$$(Y_0, \widehat{Y}, f)$$

also determines X iff there are equivalences

$$X_0 \xrightarrow{u} Y_0 , \quad \widehat{X} \xrightarrow{v} \widehat{Y}$$

so that

$$\begin{tikzcd} (X_0)^- \arrow[r, "e"] \arrow[d, "u^-"] & (\widehat{X})_0 \arrow[d, "v_0"] \\ (Y_0)^- \arrow[r, "f"] & (\widehat{Y})_0 \end{tikzcd}$$

commutes.

In this description of the homotopy type of X one should keep in mind the splitting of spaces

$$\widehat{X} = \prod_p \widehat{X}_p$$

(product of compact representable functors) and maps

$$v = \prod_p \widehat{v}_p \,.$$

Then one can see how geometric spaces X are built from infinitely many p-adic pieces and one rational piece. The fitting together problem is a purely rational homotopy question – with the extra algebraic ingredient of profinite module structures on certain homotopy groups.

In this pursuit we note the analogue of the construction of Adele spaces in Weil, "Adeles and Algebraic Groups".

Let S be any finite set of primes. Form

$$\widehat{X}_S \;=\; \big(\prod_{p\in S} (\widehat{X}_p)_0\big) \times \big(\prod_{p\notin S} \widehat{X}_p\big)\,.$$

The second factor is (as usual) constructed using the natural compact topology on the functors $[\quad, \widehat{X}_p]$.

The $\{X_S\}$ form a directed system and

Proposition 3.22 *The "Adele type" of X satisfies*

$$X_A \sim \varinjlim_S \widehat{X}_S \,.$$ [6]

The equivalence preserves the $\widehat{\mathbb{Z}}$-module structures on homotopy groups.

Proof: Using obstruction theory one sees there is a unique extension f in

$$\begin{tikzcd} Y \times Y' \arrow[d, "\text{localization} \times \text{identity}"'] \arrow[dr, "\text{localization}"] & \\ Y_0 \times Y' \arrow[r, dashed, "f"'] & (Y \times Y')_0 \, . \end{tikzcd}$$

Thus we have natural maps

$$\widehat{X}_S \to X_A$$

which imply a map

$$\varinjlim_S \widehat{X}_S \xrightarrow{A} X_A$$

It is easy to see that A induces a $\widehat{\mathbb{Z}}$-module isomorphism of homotopy. One only has to check the natural equivalence

$$\mathbb{Q} \otimes \widehat{\mathbb{Z}} \sim \varinjlim_S \Big(\prod_{p \in S} \mathbb{Q}_p \Big) \times \Big(\prod_{p \notin S} \widehat{\mathbb{Z}}_p \Big) \, ,$$

the ring of finite Adeles is the restricted product over all p of the p-adic numbers.

Note: This proposition could be regarded as a description of the localization map for a profinite homotopy type $\widehat{X}$,

$$\widehat{X} \to (\widehat{X})_0 \, .$$

Note: In summary, we apply this discussion to the problem of expressing a simply connected space y – whose homotopy groups are finitely generated $\widehat{\mathbb{Z}}$-modules – as the profinite completion of some geometric complex X,

$$Y \cong \widehat{X} \, .$$

We take Y and form its localization at zero. To do this we could make the direct limit construction using the individual p-adic components of Y as in Proposition 3.22 (using $Y \cong \widehat{Y} \cong \prod_p \widehat{Y}_p$). This gives the "Adele type", X_A, of X if it exists.

X_A is a rational space – its homotopy groups are $\mathbb{Q}$-vector spaces (of uncountable dimension). However, these homotopy groups are also $\widehat{\mathbb{Z}}$-modules. The problem of finding X then reduces to a problem in rational homotopy theory – *find an appropriate "embedding" of a rational space (with finitely dimensional homotopy groups) into* X_A. Namely, a map

$$X_0 \xrightarrow{c} X_A$$

so that

$$\pi_i X_0 \otimes \widehat{\mathbb{Z}} \underset{\cong}{\rightarrow} \pi_i X_A\,,$$

as $\widehat{\mathbb{Z}}$-modules.

Then the desired X is the induced "fibre product"

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "l"] \\ X_0 \arrow[r, "c"] & X_A \end{tikzcd}$$

For example, in the case of a complex algebraic variety X we see what is required to recover the homotopy type of X from the etale homotopy type of $X \sim \widehat{X}$.

We have to find an "appropriate embedding" of the rational type of X into the localized etale type

$$X_0 \to (\widehat{X})_0\,.$$

Note: It is easy to see that any simply connected space Y whose homotopy groups are finitely generated $\mathbb{Q}$-modules is the localization of some geometric space X,

$$Y \cong X_0\,.$$

One proceeds inductively over a local cell complex for $Y = \bigcup Y_n$ –

$$S_0^k \xrightarrow{a} Y_{n-1},\ \ Y_n = Y_{n-1}/S_0^k = \text{cofibre } a\,.$$

If X_{n-1} is constructed so that

$$(X_{n-1})_0 \cong Y_{n-1}\,,\ \ X_{n-1} \text{ geometric}$$

we can find a' so that

$$\begin{tikzcd} S^k \arrow[r, "a'"] \arrow[d, "\text{localization}"'] & X_{n-1} \arrow[d, "\text{localization}"] \\ S_0^k \arrow[r, "a"] & Y_{n-1} \end{tikzcd}$$

commutes, for some choice of the localization $S^k \to S_0^k$. In fact chose one localization of S^k, then the homotopy element

$$S^k \xrightarrow{l} S_0^k \xrightarrow{a} Y_{n-1}$$

may be written $(1/n) \cdot a'$, where a' is in the lattice

$$\text{image}\,(\pi_k X_{n-1} \to \pi_k Y_{n-1})\,.$$

Then a' works for the new localization

$$S^k \to S_0^k \underset{\cong}{\xrightarrow{n}} S_0^k$$

and $X_n = X_{n-1}/S^k \cong$ cofibre a' satisfies

$$(X_n)_0 \cong Y_n\,.$$

This argument also works for localization at some set of primes ℓ.

It would be interesting to analyze the obstructions for carrying out this argument in the profinite case.

The Local Arithmetic Square

There is an analogous discussion for constructing the "part of X at ℓ".

The square of groups

$$\begin{tikzcd} \mathbb{Z}_\ell \arrow[r] \arrow[d] & \widehat{\mathbb{Z}}_\ell \arrow[d] \\ \mathbb{Q} \arrow[r] & \mathbb{Q} \otimes \widehat{\mathbb{Z}}_\ell = l\text{-adic numbers} \end{tikzcd} \qquad \begin{array}{c} \ell \text{ a non-void} \\ \text{set of primes} \end{array}$$

has the homotopy analogue

$$\begin{tikzcd} X_\ell \arrow[r] \arrow[d] & \widehat{X}_\ell \arrow[d] \\ X_0 \arrow[r] & X_{A_\ell} = (\widehat{X}_\ell)_0 = (X_0)_\ell^- \,. \end{tikzcd}$$

The above discussion holds without essential change.

If $\ell = \{p\}$, the square becomes

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] & \widehat{X}_p \arrow[d] \\ X_0 \arrow[r] & X_{\mathbb{Q}_p}\,, \end{tikzcd}$$

where

$$X_{\mathbb{Q}_p} \;=\; (\widehat{X}_p)_0 \;=\; (X_0)_p^-$$

is the *form of* X *over the* p*-adic numbers* – the p-adic completion of $\mathbb{Q}$.

One is led to ask what $X_{\mathbb{R}}$ should be where $\mathbb{R}$ is the real numbers – the real completion of $\mathbb{Q}$, and how it fits into this scheme.

For example, the "finite Adele type" X_A should be augmented to *"the complete Adele type"*

$$X_A \times X_{\mathbb{R}}\,.$$

Then the fibre of the natural map

$$X_0 \to X_A \times X_{\mathbb{R}}$$

has *compact homotopy groups* and this map might be better understood.

If ℓ is the complement of $\{p\}$, then spaces of the form $\widehat{X}_\ell$ can be constructed algebraically from *algebraic varieties in characteristic* p.

The problem of filling in the diagram

$$\begin{tikzcd} & & \widehat{X}_\ell \arrow[d] \\ \text{"finite dimensional over } \mathbb{Q}\text{"} \arrow[r, hook] & X_0 \arrow[r, dashed] & (\widehat{X}_\ell)_0 \end{tikzcd}$$

appropriately $\big(\pi_*(\widehat{X}_\ell)_0 \cong \pi_* X_0 \otimes \widehat{\mathbb{Z}}_\ell$ as $\widehat{\mathbb{Z}}$-modules$\big)$ provides homotopy obstructions to *lifting the variety to characteristic zero.*

Notes

1 "Etale homotopy theory" Springer-Verlag Lecture Notes.

2 $\{f\}$ depends on X.

3 "Etale Homotopy", Springer Verlag Lecture Notes.

4 This condition is unnecessary – a more elaborate mapping cylinder construction may be used for higher cardinalities.

5 See proof of Proposition 3.17

6 As always direct limit here means infinite mapping cylinder over a cofinal set of S's.

Section 4. Spherical Fibrations

We will discuss the theory of fibrewise homotopy classes of fibrations where the fibres are l-local or l-adic spheres. These theories are interrelated according to the scheme of the arithmetic square and have interesting symmetry.

Definition. *A Hurewicz fibration*[1]

$$\xi : S \to E \to B$$

where the fibre is the local sphere S_l^{n-1}, $n > 1$, *is called a* local spherical fibration. *The local fibration is oriented if there is given a class in*

$$U_\xi \in H^n(E \to B; \mathbb{Z}_l)^2$$

which generates

$$H^n(S_l^{n-1} \to *; \mathbb{Z}_l) \cong \mathbb{Z}_l$$

upon restriction.

When l is the set of all primes the theory is more or less familiar,

i) the set of fibrewise homotopy equivalence classes of S^{n-1} fibrations over X is classified by a homotopy set

$$[X, BG_n] .$$

ii) BG_n is the classifying space of the associative H-space (composition)

$$G_n = \{S^{n-1} \xrightarrow{f} S^{n-1} \mid \deg f \in \{\pm 1\} = \mathbb{Z}^*\}$$

that is, $\Omega BG_n \cong G_n$ as infinitely homotopy associative H-spaces. (Stasheff, Topology 1963, A Classification Theorem for Fibre Spaces.)

iii) the oriented theory[2] is classified by the homotopy set

$$[X, BSG_n]$$

where BSG_n may be described in two equivalent ways

a) BSG_n is the classifying space for the component of the identity map of S^{n-1} in G_n, usually denoted SG_n.

b) BSG_n is the universal cover of BG_n, where $\pi_1 BG_n = \mathbb{Z}/2$.

iv) the involution on the oriented theory obtained by changing orientation

$$\xi \to -\xi$$

corresponds to the covering transformation of BSG_n.

v) there are natural inclusions $G_n \to G_{n+1}$, $BG_n \to BG_{n+1}$, corresponding to the operation of suspending each fibre. The union

$$BG = \bigcup_{n=1}^{\infty} BG_n$$

is the classifying space for the "stable theory".

The stable theory for finite dimensional complexes is just the direct limit of the finite dimensional theories under fibrewise suspension. This direct limit converges after a finite number of steps – so we can think of a map into BG as classifying a spherical fibration whose fibre dimension is much larger than that of the base.

For infinite complexes X we can say that a homotopy class of maps of X into BG is just the element in the inverse limit of the homotopy classes of the skeleta of X. This uses the finiteness of the homotopy groups of BG (see section 3). Such an element in the inverse limit can then be interpreted as an increasing union of spherical fibrations of increasing dimension over the skeletons of X.

The involution in the "stable theory" is trivial[3] and there is a canonical splitting

$$BG \cong K(\mathbb{Z}/2, 1) \times BSG\,.$$

Some particular examples can be calculated:

$$BG_1 = \mathbb{R}P^\infty, \quad BSG_1 = S^\infty \cong *$$
$$BG_2 = BO_2, \quad BSG_2 \cong \mathbb{C}P^\infty \cong BSO_2$$

All higher BG_n's are unknown although the (finite) homotopy groups of

$$BG = \bigcup_{n=1}^{\infty} BG_n$$

are much studied.

$$\pi_{i+1}BG \;\cong\; \begin{matrix}\text{stable}\\ i\text{-stem}\end{matrix} \;\equiv\; \varinjlim_k \pi_{i+k}(S^k)\,.$$

Stasheff's explicit procedure does not apply without (semi-simplicial) modification to S_l^{n-1}-fibrations for l a proper set of primes. In this case S_l^n is an infinite complex (though locally compact).

If we consider the *l-adic spherical fibrations*, namely Hurewicz fibrations with fibre $\widehat{S}_l^{n-1}$, the situation is even more infinite. $\widehat{S}_l^{n-1}$ is an uncountable complex and therefore not even locally compact.

However, Dold's theory of quasi-fibrations can be used[4] to obtain abstract representation theorems for theories of fibrations with arbitrary fibre.

THEOREM 4.1 (Dold). *There are connected CW complexes B_l^n and $\widehat{B}_l^n$ so that*

$$\left\{\begin{matrix}\textit{theory of}\\ S_l^{n-1}\ \textit{fibrations}\end{matrix}\right\} \;\cong\; [\quad, B_l^n]_{\text{free}}\,,$$
$$\left\{\begin{matrix}\textit{theory of}\\ \widehat{S}_l^{n-1}\ \textit{fibrations}\end{matrix}\right\} \;\cong\; [\quad, \widehat{B}_l^n]_{\text{free}}\,.$$

Actually, Dold must prove a based theorem first, namely

$$\left\{\begin{matrix}\text{based}\\ \text{fibrations}\end{matrix}\right\} \;\cong\; [\quad, B]_{\text{based}}\,.$$

then divide each set into the respective $\pi_1 B$ orbits to obtain the free homotopy statements of the theorem.

The Main Theorem

THEOREM 4.2

i) There is a canonical diagram

$$\begin{tikzcd} \left\{\begin{matrix} S^{n-1}\text{-fibration} \\ \text{theory} \end{matrix}\right\} \arrow[r, "\substack{\text{fibrewise}\\ \text{localization}}"] \arrow[dr] & \left\{\begin{matrix} S_l^{n-1}\text{-fibration} \\ \text{theory} \end{matrix}\right\} \arrow[d] \\ & \left\{\begin{matrix} \widehat{S}^{n-1}\text{-fibration} \\ \text{theory} \end{matrix}\right\} \end{tikzcd}$$

which corresponds to a diagram of classifying spaces

$$\begin{tikzcd} BG_n \arrow[r, "l"] \arrow[dr, "c"'] & B_l^n \arrow[d, "c"] \\ & \widehat{B}_l^n\,. \end{tikzcd}$$

ii) The diagram of fundamental groups is the diagram of units

$$\begin{tikzcd} \{\pm 1\} = \mathbb{Z}^* \arrow[r] \arrow[dr] & \mathbb{Z}_l^* \arrow[d] \\ & \widehat{\mathbb{Z}}_l^*\,. \end{tikzcd}$$

iii) The universal cover of the diagram of classifying spaces is canonically isomorphic to

$$\begin{tikzcd} GSG_n \arrow[r, "\text{localization at } l"] \arrow[dr] & (BSG_n)_l \arrow[d, "\text{completion}"] \\ & (BSG_n)\hat{_l}\,. \end{tikzcd}$$

This diagram classifies the diagram of oriented theories. The action of the fundamental groups on the covering spaces corresponds to the action of the units on the oriented theories –

$$(\xi, U_\xi) \to (\xi, \alpha U_\xi)$$

where ξ is the spherical fibration, $E \to B$, U_ξ is the orientation in $H^n(E \to B; R)$, α is a unit of $R = (\mathbb{Z}, \mathbb{Z}_l, \widehat{\mathbb{Z}}_l)$.

The proof of 4.2 is rather long so we defer it to the end of the section. However, as corollary to the proof we have

Corollary 1 *There are natural equivalences*

$$\begin{aligned} \pi_0 \operatorname{Aut} S_l^{n-1} &\cong \mathbb{Z}_l^*\,, \\ (\operatorname{Aut} S_l^{n-1})_l &\cong (SG_n)_l\,, \\ \pi_0 \operatorname{Aut} \widehat{S}_l^{n-1} &\cong \widehat{\mathbb{Z}}_l^*\,, \\ (\operatorname{Aut} \widehat{S}_l^{n-1})_1 &\cong (SG_n)\hat{}_l\,. \end{aligned}$$

Here $\operatorname{Aut} X$ is the singular complex of automorphisms of X – a simplex σ is a homotopy equivalence

$$\sigma \times X \to X\,.$$

The subscript "1" means the component of the identity.

The Galois Group

A second corollary of the main theorem which should be emphasized is the symmetry in

$$(BSG_n)_l \text{ and } (BSG_n)\hat{}_l\,.$$

Corollary 2 *Since the theorem shows these spaces classify the oriented theories they have compatible $\mathbb{Z}_l^*$ and $\widehat{\mathbb{Z}}_l^*$ symmetry.*

For $l = \{p\}$,

$$\mathbb{Z}_l^* \cong \mathbb{Z}/2 \oplus \begin{array}{c}\textit{Free Abelian group}\\ \textit{generated by the}\\ \textit{primes other}\\ \textit{than } p\end{array}$$

$$\widehat{\mathbb{Z}}_l^* \cong \begin{cases} \mathbb{Z}/p-1 \oplus \widehat{\mathbb{Z}}_p & (p > 2) \\ \mathbb{Z}/2 \oplus \widehat{\mathbb{Z}}_2 & (p = 2) \end{cases}$$

So in the complete theory we see how the rather independent symmetries of the local theory coalesce (topologically) into one compact (topologically) cyclic factor.

We will see below (Corollary 3) that the homotopy groups of BSG_n are finite except for one dimension –

$$\pi_n BSG_n = \mathbb{Z} \oplus \text{ torsion} \quad n \text{ even}$$
$$\pi_{2n-2} BSG_n = \mathbb{Z} \oplus \text{ torsion} \quad n \text{ odd}$$

The first $(n-1)$ finite homotopy groups correspond to the first $n-2$ stable homotopy groups of spheres.

Then $(BSG_n)\hat{}_l$ has for homotopy the l-torsion of these groups plus one $\widehat{\mathbb{Z}}_l$ (in dimension n or $2n-2$, respectively).

The units $\widehat{\mathbb{Z}}_l^*$ act trivially on the low dimensional, stable groups but non-trivially on the higher groups. For example, for n even we have the natural action of $\widehat{\mathbb{Z}}_l^*$ on

$$\pi_n(BSG_n)\hat{}_l/\text{mod torsion} \cong \widehat{\mathbb{Z}}_l\,.$$

On the higher groups the action measures the effect of the degree α map on the homotopy groups of a sphere. This action is computable up to extension in terms of Whitehead products and Hopf invariants. It seems especially interesting at the prime 2.

The Rational Theory

If l is vacuous, the local theory is the "rational theory". Using the fibration

$$[(\Omega^{n-1}S^{n-1})_1 \to SG_n \to S^{n-1}]_{\text{localized at } l\,=\,0}$$

it is easy to verify

Corollary 3 *Oriented $S_\emptyset^{n-1}$ fibrations are classified by*

i) an Euler class in $H^n(base;\mathbb{Q})$ for n even

ii) a "Hopf class" in $H^{2n-2}(base;\mathbb{Q})$ for n odd.

It is not too difficult[5] *to see an equivalence of fibration sequences*

$$\begin{tikzcd}
{[\dots} \arrow[r] & SG_{2n} \arrow[r] \arrow[d, "\cong"', "\substack{\textit{evalu-}\\ \textit{ation}}"] & SG_{2n+1} \arrow[r] \arrow[d, "\cong"] & SG_{2n+1}/SG_{2n} \arrow[r] \arrow[d, "\cong"] & BSG_{2n} \arrow[r] \arrow[d, "\substack{\textit{Euler}\\ \textit{class}}"] & BSG_{2n+1}]_{\emptyset} \arrow[d, "\substack{\textit{Hopf}\\ \textit{class}}"] \\
\cdots \arrow[r] & S_{\emptyset}^{2n-1} \arrow[r] & S_{\emptyset}^{4n-1} \arrow[r, "\substack{\textit{Whitehead}\\ \textit{product}}"'] & S_{\emptyset}^{2n} \arrow[r] & K(\mathbb{Q}\,, 2n) \arrow[r, "\substack{\textit{cup}\\ \textit{square}}"'] & K(\mathbb{Q}\,, 4n)
\end{tikzcd}$$

Corollary 3 has a "twisted analogue" for unoriented bundles.

Stably the oriented rational theory is trivial. The unoriented stable theory is just the theory of $\mathbb{Q}$ coefficient systems, $H^1(\quad;\mathbb{Q}^*)$.

Note that Corollary 3 (part i) twisted or untwisted checks with the equivalence

$$D_{\emptyset}^{2n-1} \cong K(\mathbb{Q}\,, 2n-1)\,.$$

The group of units in $\mathbb{Q}$

$$\mathbb{Q}^* \cong \mathbb{Z}/2 \oplus \begin{array}{c}\text{free Abelian group generated}\\ \text{by the primes}\end{array}$$

acts in the oriented rational theory by the natural action for n even and the square of the natural action for n odd.

The Stable Theory

Corollary 3

i) For the stable oriented theories, we have the isomorphisms

$$\begin{array}{c}\textit{oriented stable}\\ l\textit{-local}\\ \textit{theory}\end{array} \cong \begin{array}{c}\textit{oriented stable}\\ l\textit{-adic}\\ \textit{theory}\end{array} \cong \Big[\quad , \prod_{p \in l} (BSG_n)_p\Big]\,.$$

ii) The unoriented stable theory is canonically isomorphic to the direct product of the oriented stable theory and the theory of $\mathbb{Z}_l$

or $\widehat{\mathbb{Z}}_l$ coefficient systems.

$$\begin{matrix}\textit{stable}\\ \textit{l-local}\\ \textit{theory}\end{matrix} \cong \Big[\quad , K(\mathbb{Z}_l^*) \times \prod_{p \in l} (BSG_n)_p \Big],$$

$$\begin{matrix}\textit{stable}\\ \textit{l-adic}\\ \textit{theory}\end{matrix} \cong \Big[\quad , K(\widehat{\mathbb{Z}}_l^*) \times \prod_{p \in l} (BSG_n)_p \Big].$$

iii) The action of the Galois group is trivial in the stable oriented theory.

Proof:

i) Since BSG has finite homotopy groups there is a canonical splitting

$$BSG \;\cong\; \prod_p (BSG)_p$$

into its p-primary components.

Clearly, for finite dimensional spaces

$$\begin{aligned}\Big[\quad , \prod_{p \in l} (BSG)_p\Big] &\cong [\quad , (BSG)_l] \\ &\cong [\quad , \lim_{\to} (BSG_n)_l] \\ &\cong \lim_{\to} \text{ oriented } S_l^{n-1} \text{ theories} \\ &\cong \text{stable oriented local theory}.\end{aligned}$$

Also, because of the rational structure

$$\lim_{n \to \infty} \big((BSG_n)_l \to (BSG_n)_l\hat{\ }\big)$$

is an isomorphism. This completes the proof of i) since the stable oriented l-adic theory is classified by

$$\lim_{n \to \infty} (BSG_n)_l\hat{\ }.$$

ii) Consider the l-adic case. A coefficient system

$$\alpha \in H^1(\quad , \widehat{\mathbb{Z}}_l^*)$$

determines an $\widehat{S}^1_l$-bundle α by letting the units act on some representative of $\widehat{S}^1_l$ by homeomorphisms. (A functorial construction of $K(\widehat{\mathbb{Z}}_l, 1)$ will suffice.)

Represent a stable oriented l-adic fibration by an l-local fibration γ using the isomorphism of i). The fibrewise join $\alpha * \gamma$ determines a (cohomologically twisted) l-adic fibration since

$$\widehat{S}^1_l * S^{n-1}_l \cong \widehat{S}^{n-1}_l \,.$$

One easily checks (using the discussion in the proof of Theorem 4.2) that this construction induces a map

$$K(\widehat{\mathbb{Z}}^*_l, 1) \times (BSG)_l \to \big(\varinjlim_n \widehat{B}^n_l\big)$$

which is an isomorphism on homotopy groups.

The local case is similar. In fact an argument is unnecessary since one knows *a priori* (using Whitney join) that the stable local theory is additive.

iii) The action of the Galois group is clearly trivial using ii). Or, more directly in the local theory the action is trivial because there are automorphisms of

$$S^1_l \text{ fibre join}\,, \;\; \gamma \text{ an oriented local fibration}$$

which change the orientation by any unit of $\mathbb{Z}_l$. Then the action of the profinite group $\widehat{\mathbb{Z}}^*_l$ is trivial by continuity.

Note that part iii) of Corollary 3 is a "pure homotopy theoretical Adams Conjecture".

The Inertia of Intrinsic Stable Fibre Homotopy Types

We generalize the "purely homotopy theoretical Adams conjecture" of the previous Corollary 3, iii).

Let

$$B_0 \to \cdots \to B_n \xrightarrow{i} B_{n+1} \to \dots$$

be a sequence of spaces. Assume that each space is the base of a spherical fibration of increasing fibre dimension

$$S^{d_n} \to \gamma_n \to B_n \,,$$

and that these fit together in the natural way

$$i^*\gamma_{n+1} \overset{f_n}{\cong} \left(\gamma_n \text{ fibre join } S^{d_{n+1}-d_n}\right).$$

We will study the "stable bundle"

$$B_\infty \xrightarrow{\gamma} BG\,,$$

where B_∞ = mapping telescope $\{B_n\}$, and a representative of γ is constructed from the $\{\gamma_n\}$ and the equivalences $\{f_n\}$.[6]

The basic assumption about the "stable fibration" γ is that it is "intrinsic to the fibration" $\{B_n\}$ of B_∞ – there are arbitrarily large integers n so that the spherical fibration γ_{n+1} is strongly approximated by the map

$$B_n \to B_{n+1}\,,$$

i.e. the natural composition

$$B_n \xrightarrow{\text{cross section}} i^*\gamma_{n+1} \xrightarrow{i^*} \gamma_{n+1}$$

is an equivalence over the $d(n)$-skeleton where $\big(d(n)-d_n\big)$ approaches infinity as n approaches infinity.

The natural stable bundles

"orthogonal"	$BO \to BG\,,$
"unitary"	$BU \to BG\,,$
"symplectic"	$BSp \to BG\,,$
"piecewise linear"	$BPL \to BG\,,$
"topological"	$BTop \to BG\,,$
"homotopy"	$BG \xrightarrow[\text{identity}]{} BG$

are "intrinsic" to the natural fibrations

$$\{BO_n\}\,,\ \ \{BU_n\}\,,\ \ \{BSp_n\}\,,\ \ \{BPL_n\}\,,\ \ \{BTop_n\}\,,\ \ \{BG_n\}\,.$$

We make the analogous definition of intrinsic in the local or l-adic spherical fibration context. In the oriented case we are then studying certain maps,

$$B_\infty \to (BSG)_l \cong \prod_{p\in l} (BSG)_p\,.$$

It is clear that the localization or completion of an intrinsic stable fibration is intrinsic.

Theorem (Inertia of intrinsic fibre homotopy type)
Let γ be a stable spherical fibration over B_∞ (ordinary, local or complete) which is intrinsic to a filtration $\{B_n\}$ of B_∞. Let A_∞ be any filtered automorphism of B_∞,

$$B_\infty \xrightarrow{A_\infty} B_\infty \;=\; \lim_{n\to\infty} \left(B_n \underset{\cong}{\xrightarrow{A_n}} B_n\right).$$

Then A_∞ preserves the fibre homotopy type of γ, that is

$$\begin{tikzcd} B_\infty \arrow[rr] \arrow[dr] & & B_\infty \arrow[dl] \\ & BG & \end{tikzcd} \quad \textit{commutes.}$$

Proof: Assume for convenience $d_n = n$ and $d(n) = 2n$. The dimensions used below are easily modified to remove this intuitive simplification.

We have the spherical fibrations

$$\begin{tikzcd} S^{d_{n-1}} \arrow[r] & \gamma_n \arrow[d] & S^{d_n} \arrow[r] & \gamma_{n+1} \arrow[d] \\ & B_n \arrow[rr, "i"] & & B_{n+1} \end{tikzcd} \quad i^*\gamma_{n+1} \cong \gamma_n \text{ fibre join } S^{d_n - d_{n-1}}\,,$$

the filtered automorphism

$$\text{fibre } j = F_n \to \begin{tikzcd} B_n \arrow[r, "A_n"] \arrow[d, "j"] & B_n \arrow[d, "j"] \\ B_{n+1} \arrow[r, "A_{n+1}"] & B_{n+1} \end{tikzcd}$$

and the map e which is a $d(n)$-skeleton equivalence,

$$\begin{tikzcd} B_n \arrow[rr, "e"] \arrow[dr, "\text{cross section}"'] & & \gamma_{n+1} \\ & i^*\gamma_{n+1} \arrow[ur] & \end{tikzcd}$$

We can assume that A_{n+1} is a skeleton-preserving map, that A_n is fibre preserving in the sequence

$$F_n \to B_n \to B_{n+1}\,,$$

and that e is a fibre preserving map covering the identity.

We restrict e and (A_{n+1}, A_n) to the pertinent spaces lying over the n-skeleton of B_{n+1} –

$$\begin{tikzcd}
\gamma_{n+1}/ & B_n/ \arrow[l, "e/"'] \arrow[r, "A_n/"] \arrow[d] & B_n/ \arrow[d] \\
& (B_{n+1})_{n\text{-skeleton}} \arrow[r, "A_{n+1}/"] & (B_{n+1})_{n\text{-skeleton}}
\end{tikzcd}$$

Then we make $e/$ and $A_n/$ cellular and restrict these to $2n$-skeletons giving

$$(\gamma_{n+1}/)_{2n\text{-skeleton}} \xleftarrow{(e/)_{2n\text{-skeleton}}} (B_n/)_{2n\text{-skeleton}} \xrightarrow{(A_n/)_{2n\text{-skeleton}}} (B_n/)_{2n\text{-skeleton}}.$$

The second map is still an equivalence, while the first map becomes an equivalence by the intrinsic hypothesis. On the other hand,

$$\gamma_{n+1}/ \to (B_{n+1})_{n\text{-skeleton}}$$

is a S^n-fibration, so

$$\gamma_{n+1}/ \cong (\gamma_{n+1}/)_{2n\text{-skeleton}}\,.$$

We can then transform the automorphism of $(A_n/)_{2n\text{-skeleton}}$ using $(e/)_{2n\text{-skeleton}}$ to obtain an automorphism $\widetilde{A}$ of $\gamma_{n+1}/$ covering the automorphism $(A_{n+1})_{n\text{-skeleton}}$,

$$\begin{tikzcd}
\gamma_{n+1}/ \arrow[r, "\widetilde{A}", "\cong"'] \arrow[d] & \gamma_{n+1}/ \arrow[d] \\
(B_{n+1})_{n\text{-skeleton}} \arrow[r, "(A_{n+1})_{n\text{-skeleton}}"] & (B_{n+1})_{n\text{-skeleton}}
\end{tikzcd}$$

Then the composition

$$\gamma_{n+1}/ \xrightarrow{\widetilde{A}} \gamma_{n+1} \xleftarrow{A^*_{n+1}} A^*_{n+1}(\gamma_{n+1}/)$$

is a fibre homotopy equivalence

$$\gamma_{n+1}/ \sim A^*_{n+1}(\gamma_{n+1}/)$$

covering the identity map of $(B_{n+1})_{n\text{-skeleton}}$.

Letting n go to infinity gives the desired homotopy commutativity

$$\begin{tikzcd} B_\infty \arrow[rr, "A_\infty"] \arrow[dr] & & A_\infty \arrow[dl] \\ & BG & \end{tikzcd}$$

Corollary *Any automorphism of BG which keeps the filtration $\{BG_n\}$ invariant is homotopic to the identity map.*

Remark: There is at least one homotopy equivalence of BG which is not the identity – the homotopy inverse

$$x \mapsto x^{-1}$$

defined by the H-space structure in BG.

Remark: In the intrinsic examples above $BO, BPL, BG, \dots$, the filtered automorphisms are precisely those operations in bundles which give isomorphisms and preserve the geometric fibre dimension. That is they are isomorphisms which are geometric in character. The theorem can then be paraphrased "geometric automorphisms of bundle theories preserve the stable fibre homotopy type".

When is an l-adic fibration the completion of a local fibration?

According to sections 1 and 3 there are the fibre squares

$$\begin{tikzcd} \mathbb{Z}_l \arrow[r] \arrow[d] & \widehat{\mathbb{Z}}_l \arrow[d, "\otimes\mathbb{Q}"] \\ \mathbb{Q} \arrow[r, "\otimes\mathbb{Z}_l"'] & \bar{\mathbb{Q}}_l \end{tikzcd} , \qquad \begin{tikzcd} (BSG_n)_l \arrow[r] \arrow[d] & (BSG_n)\hat{}_l \arrow[d, "\text{formal } l\text{-completion}"] \\ (BSG_n)_0 \arrow[r] & (BSG_n)_0 \end{tikzcd}$$

This leads to the

Corollary 4 *An oriented $\widehat{S}_l^{n-1}$-fibration is the completion of a S_l^{n-1}-fibration iff*

a) for n even, the image of the Euler class under

$$H^n(base; \widehat{\mathbb{Z}}_l) \xrightarrow[\otimes\mathbb{Q}]{} H^n(base; \bar{\mathbb{Q}}_l)$$

is rational, namely in the image of

$$H^n(base;\mathbb{Q}) \xrightarrow[\otimes\widehat{\mathbb{Z}}_l]{} H^n(base;\bar{\mathbb{Q}}_l)\,.$$

b) For n odd, the Hopf class, which is only defined in

$$H^{2n-2}(base;\bar{\mathbb{Q}}_l)$$

is rational.

Proof: The fibre square above is equivalent to (n even)

$$\begin{tikzcd}
(BSG_n)_l \arrow[r] \arrow[d, "\substack{\text{rational}\\ \text{Euler class}}"'] & (BSG_n)\hat{}_l \arrow[d, "\substack{l\text{-adic}\\ \text{Euler class}}"] \\
K(\mathbb{Q}\,,n) \arrow[r] & K(\bar{\mathbb{Q}}_l,n)
\end{tikzcd}$$

The corollary is a restatement of one of the properties of a fibre square. Namely, in the fibre square of CW complexes

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

maps into C and B together with a class of homotopies between their images in D determine a class of maps into A.

Addendum. Another way to think of the connection is this – since

$$(BSG_n)\hat{}_l \;\cong\; \prod_{p\in l}(BSG_n)\hat{}_p\,,$$

a local S_l^{n-1}-fibration is a collection of $\widehat{S}_p^{n-1}$-fibrations one for each p in l together with the coherence condition that the characteristic classes (either Euler or Hopf, with coefficients in $\mathbb{Q}_p$) they determine are respectively in the image of a single rational class.

Principal Spherical Fibrations

Certain local (or l-adic) spheres are naturally homotopically equivalent to topological groups. Thus we can speak of principal spherical fibrations. The classifying space for these principal fibrations is easy to describe and maps into the classifying space for oriented spherical fibrations.

PROPOSITION (p odd) *$\widehat{S}_p^{n-1}$ is homotopy equivalent to a topological group (or loop space) iff n is even and n divides $2p-2$.*[7]

COROLLARY *S_l^{2n-1} is homotopy equivalent to a loop space iff*

$$l \subseteq \{p : \mathbb{Z}/n \subseteq p\text{-adic units}\}\,.$$

Let S_l^{2n-1} have classifying space $P^\infty(n,l)$, then

$$\Omega P^\infty(n,l) \cong S_l^{2n-1}\,.$$

The fibration

$$S_l^{2n-1} \to * \to P^\infty(n,l)$$

implies

i) $H^*\big(P^\infty(n,l);\mathbb{Z}_l\big)$ is isomorphic to a polynomial algebra on one generator in dimension $2n$.

ii) For each choice of an orientation of S_l^{2n-1} there is a natural map

$$P^\infty(n,l) \to (BSG_{2n})_l\,.$$

In cohomology the universal Euler class in $(BSG_{2n})_l$ restricts to the polynomial generator in $P^\infty(n,l)$.

PROOF OF THE PROPOSITION: A rational argument implies a spherical H-space has to be odd dimensional.

If $\widehat{S}_p^{n-1}$ is a loop space, ΩB_n, it is clear that the mod p cohomology of B_n is a polynomial algebra on one generator in dimension n. Looking at Steenrod operations implies λ divides $(p-1)p^k$ for some k, where $n = 2\lambda$. Looking at secondary operations – using Liulevicius' mod p analysis generalizing the famous mod 2 analysis of Adams shows $k = 0$, namely λ divides $p-1$.

On the other hand, if λ divides $p-1$ we can construct B_n directly –

i) embed $\mathbb{Z}/\lambda$ in $\mathbb{Z}/p-1 \subseteq \widehat{\mathbb{Z}}_p^*$,

ii) choose a functorial $K(\widehat{\mathbb{Z}}_p, 2)$ in which $\widehat{\mathbb{Z}}_p^*$ acts freely by cellular homeomorphisms,

iii) form

$$B_n = \big(K(\widehat{\mathbb{Z}}_p, 2)/(\mathbb{Z}/\lambda)\big)_p^{\hat{}}\,.$$

We obtain a p-adically complete space which is simply connected, has mod p cohomology a polynomial algebra on one generator in dimension n and whose loop space is $\widehat{S}_p^{n-1}$.

In more detail, the mod p cohomology of $K(\widehat{\mathbb{Z}}_p, 2)/(\mathbb{Z}/\lambda)$ is the invariant cohomology in $K(\widehat{\mathbb{Z}}_p, 2)$ –

$$(1, x, x^2, x^3, \dots) \xrightarrow{\alpha} (1, \alpha x, \alpha^2 x^2, \dots), \quad \alpha^\lambda = 1\,.$$

This follows since λ is prime to p and we have the spectral sequence of the fibration

$$K(\widehat{\mathbb{Z}}_p, 2) \to K(\widehat{\mathbb{Z}}_p, 2)/(\mathbb{Z}/\lambda) \to K(\mathbb{Z}/\lambda, 1)\,.$$

B_n is simply connected since

$$\pi_1\big(K(\widehat{\mathbb{Z}}_p, 2)/(\mathbb{Z}/\lambda)\big)_p^{\hat{}} \;=\; (\mathbb{Z}/\lambda)_p^{\hat{}} \;=\; 0\,.$$

B_n is then $(n-1)$-connected by the Hurewicz theorem.

The space of loops on B_n is an $(n-2)$-connected p-adically complete space whose mod p cohomology is one $\mathbb{Z}/p$ in dimension $n-1$. The integral homology of ΩB_n is then one $\widehat{\mathbb{Z}}_p$ in dimension $n-1$ and we have the p-adic sphere $\widehat{S}_p^{n-1}$.

Proof of the Corollary. If l is contained in $\{p : n \text{ divides } (p-1)\}$[8] construct the "fibre product" of

$$\begin{tikzcd} & \prod_{p\in l} B_{2n}^p \arrow[d] \\ K(\mathbb{Q}, 2n) \arrow[r] & K(\bar{\mathbb{Q}}_l, 2n) \end{tikzcd}$$

where B_{2n}^p is the de-loop space of $\widehat{S}_p^{2n-1}$ constructed above.

If we take loop spaces, we get the fibre square

$$\begin{tikzcd} S_l^{2n-1} \arrow[r] \arrow[d] & \prod_{p\in l} \widehat{S}_p^{2n-1} \arrow[d] \\ S_0^{2n-1} \arrow[r] & (S_0^{2n-1})_l^- . \end{tikzcd}$$

Thom Isomorphism

(Thom) A S_l^{n-1}-fibration with orientation

$$U_\xi \in H^n(E \to X; \mathbb{Z}_l)$$

determines a Thom isomorphism

$$H^i(X;\mathbb{Z}_l) \xrightarrow[\cong]{\cup U_\xi} H^{i+n}(E \to X;\mathbb{Z}_l)\,.$$

This is proved for example by induction over the cells of the base using Mayer-Vietoris sequences.

Conversely (Spivak), given a pair $A \xrightarrow{f} X$ and a class

$$U \in H^n(A \to X; \mathbb{Z}_l)$$

such that

$$H^i(X;\mathbb{Z}_l) \xrightarrow{\cup U} H^{i+n}(A \to X;\mathbb{Z}_l)$$

is an isomorphism, then under appropriate fundamental group assumptions $A \xrightarrow{f} X$ determines an oriented S_l^{n-1}-fibration.

For example if the fundamental group of X acts trivially on the fibre of f, then an easy spectral sequence argument shows that

$$H^*(\text{fibre } f;\mathbb{Z}_l) \cong H^*(S_l^{n-1};\mathbb{Z}_l)\,.$$

If further fibre f is a "simple space" a fibrewise localization[9] is possible, and this produces a S_l^{n-1}-fibration over X,

$$\begin{tikzcd} \text{fibre } f \arrow[r] \arrow[d, "\text{localization}"'] & A \arrow[r] \arrow[d, "\substack{\text{fibrewise}\\ \text{localization}}"] & X \arrow[d, equal] \\ S_l^{n-1} \arrow[r] & E \arrow[r] & X \end{tikzcd}$$

A similar situation exists for l-adic spherical fibrations.

Whitney Join

The Whitney join operation defines pairings between the S_l^{n-1}, S_l^{m-1} theories and S_l^{n+m-1} theories. We form the join of the fibres (S_l^{n-1} and S_l^{m-1}) over each point in the base and obtain a S_l^{n+m-1}-fibration.

This of course uses the relation

$$S_l^{n-1} * S_l^{m-1} \cong S_l^{n+m-1}\,.$$

The analogous relation is not true in the complete context. However, we can say that

$$(\widehat{S}_l^{n-1} * \widehat{S}_l^{m-1})\hat{}_l \cong \widehat{S}_l^{n+m-1}\,.$$

Thus fibre join followed by fibrewise completion defines a pairing in the l-adic context.

Proof of 4.2: (page 102)

i) The map l is constructed by *fibrewise localization*. Let ξ be a fibration over a simplex σ with fibre F, and let ∂l

$$\begin{tikzcd} \xi \mid \partial\sigma \arrow[rr, "\partial l"] \arrow[dr, "\text{fibre } F"'] & & \partial\xi' \arrow[dl, "\text{fibre } F_l"] \\ & \partial\sigma & \end{tikzcd} \qquad (F \text{ a “simple space”})$$

be a fibre preserving map which localizes each fibre. Then filling in the diagram

$$\begin{tikzcd} \xi \mid \partial\sigma \arrow[r] \arrow[d, "\partial l"'] & \xi \arrow[r, phantom, "\overset{\substack{\text{arbitrary}\\\text{trivialization}}}{\cong}"] & \sigma \times F \arrow[d, "\text{projection}"] \\ \partial\xi' \arrow[r, dashed, "t"] & F_l & F \arrow[l, "\text{localization}"'] \end{tikzcd}$$

gives an extension of the fibrewise localization ∂l to all of σ. Namely,

$$\begin{tikzcd} \xi \arrow[rr, "l"] \arrow[dr] & & \xi' \arrow[dl] \\ & \sigma & \end{tikzcd} \qquad \begin{array}{c} \xi' = \text{mapping cone of } t \\ \wr\| \\ \sigma \times F_l \end{array} .$$

But t exists by obstruction theory,

$$H^*(\partial\xi', \xi \mid \partial\sigma; \pi_* F_l) \cong H^*\big(\partial\sigma \times (F_l, F); \mathbb{Z}_l\text{-module}\big) = 0\,.$$

Thus we can fibrewise localize any fibration with "simple fibre" by proceeding inductively over the cells of the base. We obtain a "homotopically locally trivial" fibration which determines a unique Hurewicz fibration with fibre F_l.

The same argument works for fibrewise completion,

$$F \to \widehat{F}_l$$

whenever

$$H^*(\widehat{F}_l, F; \widehat{\mathbb{Z}}_l) \;=\; 0\,.$$

But this is true for example when $F = S^{n-1}$ or S_l^{n-1}.

This shows we have the diagram of i) for objects. The argument for maps and commutativity is similar.

To prove ii) and iii) we discuss the sequence of theories

$$U : \left\{\begin{matrix}\text{oriented} \\ S_R\text{-fibrations}\end{matrix}\right\} \xrightarrow{f} \{S_R\text{-fibrations}\} \xrightarrow{w} H^1(\quad, R^*)$$

where $S_R = S^{n-1}$, S_l^{n-1}, or $\widehat{S}_l^{n-1}$; and R is $\mathbb{Z}$, bbZ_l, or $\widehat{\mathbb{Z}}_l$.

The first map forgets the orientation.

The second map replaces each fibre by its reduced integral homology. This gives an R coefficient system classified by an element in $H^1(\quad; R^*)$.

Now the covering homotopy property implies that an S_R fibration over a sphere S^{i+1} can be built from a homotopy automorphism of $S^i \times S_R$ preserving the projection $S^i \times S_R \to S^i$. We can regard this as a map of S^i into the singular complex of automorphisms of S_R, $\operatorname{Aut} S_R$.

For $i = 0$, the fibration is determined by the component of the image of the other point on the equator. But in the sequence

$$\pi_0 \operatorname{Aut} S_R \to [S_R, S_R] \to \pi_{n-1} S_R \to H_{n-1} S_R$$

the first map is an injection and the second and third are isomorphisms. Thus

$$\pi_0 \operatorname{Aut} S_R \cong R^* \cong \operatorname{Aut}(H_{n-1} S_R)\,.$$

This proves ii) and the fact that oriented bundles over S^1 are all equivalent.

More generally, an orientation of an S_R fibration determines an embedding of the trivial fibration $S_R \to *$ into it. This embedding in turn determines the orientation over a connected base.

Thus if the orientation sequence U corresponds to the sequence of classifying spaces

$$\widetilde{B}R \xrightarrow{f} BR \xrightarrow{w} K(R^*, 1)$$

we see that for $i > 0$

$$\begin{aligned}
\pi_{i+1}\widetilde{B}R &\cong [S^{i+1}, \widetilde{B}R]_{\text{free}} \\
&\cong \text{oriented bundles over } S^{i+1} \\
&\cong \text{based bundles over } S^{i+1} \\
&\cong [S^{i+1}, BR]_{\text{based}} \\
&\cong \pi_{i+1} BR\,.
\end{aligned}$$

So on homotopy we have

$$* \xrightarrow{f} R^* \xrightarrow[\cong]{w} R^* \quad \text{for } \pi_1\,,$$

$$\pi_{i+1} \xrightarrow[\cong]{f} \pi_{i+1} \to * \quad \text{for } \pi_{i+1}\,.$$

Therefore, U is the universal covering space sequence.

Also the correspondence between based and oriented bundles shows the R^* actions correspond as stated in iii).

We are left to prove the first part of iii).

The cell by cell construction of part i) shows we can construct (cell by cell) a natural diagram

$$\begin{tikzcd}
G_n = \operatorname{Aut} S^{n-1} \arrow[r] \arrow[dr] & \operatorname{Aut} S_l^{n-1} \arrow[d] \\
& \operatorname{Aut} \widehat{S}_l^{n-1}
\end{tikzcd}$$

Also, a based or oriented S_R-fibration over S^{i+1} has a well defined characteristic element,

$$S^i \to \text{component of the identity of } \operatorname{Aut} S_R\,.$$

So we need to calculate the homotopy in the diagram

$$\begin{tikzcd}
SG_n \arrow[r] \arrow[dr, "c"'] & (\operatorname{Aut} S_l^{n-1})_1 \arrow[d] \\
& (\operatorname{Aut} \widehat{S}_l^{n-1})_1\,.
\end{tikzcd}$$

For example, to study c look at the diagram

$$\begin{tikzcd}
(S^{n-1}, S^{n-1})_1^{\text{based}} \arrow[r, "c_1", "\text{completion}"'] \arrow[d] & (\widehat{S}_l^{n-1}, \widehat{S}_l^{n-1})_1^{\text{based}} \arrow[d] \\
SG_n \arrow[r, "c", "\text{completion}"'] \arrow[d] & (\operatorname{Aut} \widehat{S}_l^{n-1})_1 \arrow[d] \\
S^{n-1} \arrow[r, "c_0", "\text{completion}"'] & \widehat{S}_l^{n-1}\,.
\end{tikzcd}$$

Now c_0 just tensors the homotopy with $\widehat{\mathbb{Z}}_l$.

An element in π_i of the upper right hand space is just a homotopy class of maps

$$S^i \times S_R \to S_R\,, \quad S_R = \widehat{S}_l^{n-1}$$

which is the identity on $* \times S_R$ and constant along $S^i \times *$. By translation to the component of the constant we get a map which is also constant along $* \times S_R$, thus a homotopy class of maps

$$S^i \wedge S_R \to S_R\,.$$

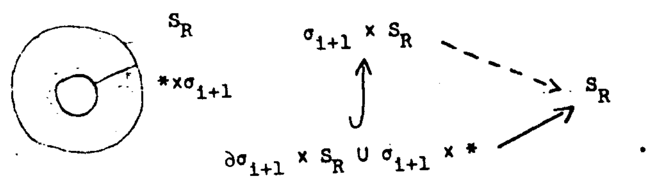
This homotopy set is isomorphic to

$$[\widehat{S}_l^{n+i-1}, \widehat{S}_l^{n-1}] \;\cong\; [S^{n+i-1}, \widehat{S}_l^{n-1}] \;\cong\; \pi_{n+i-1}S^{n-1} \otimes \widehat{\mathbb{Z}}_l\,.$$

The naturality of this calculation shows c_1 also l-adically completes the homotopy groups.

There is a long exact homotopy sequence for the vertical sequence on the left.

A proof of this which works on the right goes as follows. Look at the obstructions to completing the diagram



These lie in

$$H^k\big((\sigma_{i+1}, \partial\sigma_{i+1}) \times (S_R, *); \pi_{k-1}S_R\big)\,.$$

But these are all zero except $k = n + i$ and this group is just

$$\pi_{n+i-1}\widehat{S}_l^{n-1} \cong \pi_i(\text{“fibre”})\,.$$

So we can construct an exact homotopy sequence on the right.

This means that c also tensors the homotopy with $\widehat{\mathbb{Z}}_l$.

This proves that

$$BSG_n \to \text{universal cover } \widehat{B}_n^l$$

is just l-adic completion.

The localization statement is similar so this completes the proof of iii) and the theorem.

Notes

1 $E \to B$ has the homotopy lifting property for maps of spaces into B.

2 The fibre homotopy equivalences have to preserve the orientation.

3 This is the germ of the Adams phenomenon.
4 See Springer Verlag, Lecture Notes 12, "Halbexakte Homotopie Funktoren", p. 16.8.
5 It is convenient to compare the fibrations $SO_n \to SO_{n+1} \to S^n$, $(\Omega S^n)_1 \to SG_{n+1} \to S^n$.
6 Recall from section 3 that mapping into BG defines a compact representable functor. γ_n over B_n determines a unique map $B_n \xrightarrow{(\gamma_n)} BG$ and γ is the unique element in the inverse limit $\varprojlim [B_n, BG] \cong [B_\infty, BG]$ defined by these. In particular, γ is independent of the equivalences $\{f_n\}$.
7 For $p = 2$, it is well known that only S^1, S^3 and S^7 are H-spaces, and S^7 is not a loop space.
8 The case left out is taken care of by S^3_l.
9 See proof of Theorem 4.2.

Section 5. Algebraic Geometry (Etale[1] Homotopy Theory)

Introduction

We discuss a beautiful theory from algebraic geometry about the homotopy type of algebraic varieties.

The main point is this – there is a purely algebraic construction of the profinite completion of the homotopy type of a complex algebraic variety.

We will be concerned with *affine varieties* over the complex numbers, $V \subset \mathbb{C}^k$, such as

$$\mathrm{GL}\,(n,\mathbb{C}) \subseteq \mathbb{C}^{n^2+1} = \{(x_{11}, x_{12}, \ldots, x_{nn}, y) \mid \det(x_{ij}) \cdot y = 1\}\,.$$

We also want to consider *algebraic varieties of finite type* built from a finite collection of affine varieties by algebraic pasting such as

$$P^1(\mathbb{C}) \;=\; \{\text{space of lines in } \mathbb{C}^2\} \;=\; A_1 \cup A_2$$

where the pasting diagram

$$A_1 \leftarrow A_1 \cap A_2 \rightarrow A_2$$

is isomorphic to

$$\mathbb{C} \xleftarrow[\text{inclusion}]{} \mathbb{C} - \{0\} \xrightarrow[\text{inversion}]{} \mathbb{C}\,.$$

The role of $\mathbb{C}$ in the above definition is only that of "field of definition". Both of *these* definition schemes work over any commutative ring R giving varieties (defined over R)

$$\mathrm{GL}\,(n, R) \text{ and } P^1(R)\,.$$

For this last remark it is important to note that the coefficients in the defining relation of $\mathrm{GL}\,(n, \mathbb{C})$ and the pasting function for $P^1(\mathbb{C})$ naturally lie in the ring R (since they lie in $\mathbb{Z}$).

The notion of a *prescheme over* R has been formulated[2] to describe the object obtained by the general construction of glueing together affine varieties over R.

The notion of algebraic variety over R (or scheme) is derived from this by imposing a closure condition on the diagonal map.

One of the main points is –

To any prescheme S there is a naturally associated *etale homotopy type* $\varepsilon(S)$. The etale homotopy type is an inverse system of ordinary homotopy types. These homotopy types are constructed from "algebraic coverings" (= etale coverings) of the prescheme S.

One procedure is *analogous*[3] to the construction of the Čech nerve of a topological covering of a topological space.

As in the Čech theory the homotopy types are indexed by the set of coverings which is partially ordered by refinement.

In Artin-Mazur[4], a number of theorems are proved about the relationship between

i) the classical homotopy type of a variety over the complex numbers and its etale homotopy type.

ii) various etale homotopy types for one variety over different ground rings.

For example,

THEOREM 5.1 *If V is an algebraic variety of finite type over $\mathbb{C}$, then there is a natural homotopy class of maps*

$$\begin{array}{cccc} V_{\mathrm{cl}} & \xrightarrow{f} & V_{\mathrm{et}} & = \begin{array}{c} \textit{inverse system} \\ \textit{of "nerves" of} \\ \textit{etale covers} \\ \textit{of } V \end{array} \\ \begin{array}{c} \textit{classical} \\ \textit{homotopy type} \end{array} & & \begin{array}{c} \textit{etale} \\ \textit{homotopy type} \end{array} & \end{array}$$

inducing an isomorphism on cohomology with finite coefficients

$$H^*(V;A) \underset{\cong}{\longrightarrow} \lim_{\substack{\longrightarrow \\ \{etale\ cover\}}} H^*(nerve;A)$$

A finite Abelian (twisted or untwisted).

Actually, Artin and Mazur use a more elaborate (than Čech like) construction of the "nerves" – interlacing the actual nerves for systems of etale coverings.

This construction uses Verdier's concept of a hypercovering.

They develop some homotopy theory for inverse systems of homotopy types. The maps

$$\{X_i\} \to \{Y_i\}$$

are

$$\lim_{\substack{\longleftarrow \\ j}} \lim_{\substack{\longrightarrow \\ i}} [X_i, X_j]\,.$$

Theorem 5.1 then implies for X a complex algebraic variety of finite type

$$\begin{array}{rcl} \text{etale homotopy type of } X & \equiv & \text{inverse system of} \\ & & \text{homotopy types with} \\ & & \text{finite homotopy groups}^5 \\ & \underset{\substack{\text{in the sense} \\ \text{of maps} \\ \text{described above}}}{\cong} & \{F\}_{\{f\}}\,. \end{array}$$

$\{f\}$ is the category used in section 3, $\{X \xrightarrow{f} F\}$, to construct the profinite completion of the classical homotopy type.

Thus we can take the inverse limit of the "nerve" of etale coverings and obtain an algebraic construction for the profinite completion of the classical homotopy type of X.

We try to motivate the success of the etale method. To do this we discuss a slight modification of Lubkin's (Čech-like) construction in certain examples. Then we study the "complete etale homotopy type" and its Galois symmetry in the case of the finite Grassmannian.

For motivation one might keep in mind the "inertia lemma" of section 4.

Intuitive Discussion of the Etale Homotopy Type

Let V be an (irreducible) complex algebraic variety. Let us consider the problem of calculating the cohomology of V by *algebraic means*. It will turn out that we can succeed if we consider only finite coefficients. (Recall that the cohomology of a smooth manifold can be described *analytically* in terms of differential forms if we consider real coefficients.)

The singular method for calculating cohomology involves the notion of a continuous map

$$\text{simplex} \longrightarrow \text{underlying topological space of } V$$

so we abandon it.

The Čech method only involves the formal lattice of open sets for the topology of V. Some part of this lattice is algebraic in nature. In fact we have

$$\left\{\begin{array}{c}\text{lattice of}\\ \text{subvarieties of}\\ V\end{array}\right\} \xrightarrow{c} \left\{\begin{array}{c}\text{lattice}\\ \text{of open sets}\end{array}\right\}$$

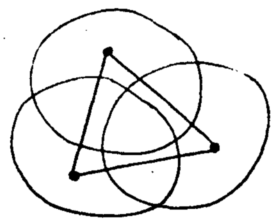
which assigns to each algebraic subvariety the complement – a "Zariski open set".

The open sets in $\{\text{image } c\}$ define the Zariski topology in V – which is algebraic in nature.

Consider the Čech scheme applied to this algebraic part of the topology of V. Let $\{U_\alpha\}$ be a finite covering of V by Zariski open sets.

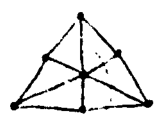
Proposition *If $\{U_\alpha\}$ has $(n+1)$ elements, then the Čech nerve of $\{U_\alpha\}$ is the n-simplex, Δ^n.*

The proof follows from the simple fact that any finite intersection of non-void Zariski open sets is a non-void Zariski open set – each is everywhere dense being the complement of a real codimension 2 subvariety.



n = 2

Notice also that if we close U under intersections (and regard all the intersections as being distinct), the simplices of the first barycentric subdivision of the n-simplex,



correspond to strings of inclusions

$$U_{i_1} \subseteq U_{i_2} \subseteq \cdots \subseteq U_{i_n} .$$

Since the homology of the n-simplex is trivial we gain nothing from a *direct* application of the Čech method to Zariski coverings.

At this point Grothendieck enters with a simple generalization with brilliant and far-reaching consequences.

i) First notice that the calculation scheme of Čech cohomology for a covering may be defined for any category. (We regard $\{U_\alpha\}$ as a category – whose objects are the U_α and whose morphisms are the inclusions $U_\alpha \subseteq U_\beta$.)

   We describe Lubkin's geometric scheme for doing this below.

ii) Consider categories constructed from coverings of X by "etale mappings",

$$\widetilde{U}_\alpha \xrightarrow{\pi} U_\alpha \subseteq X .$$

   U_α is a Zariski open set in X and π is a *finite*[6] covering map. (The images of π are assumed to cover X.)

   The objects in this category are the maps

$$\widetilde{U}_\alpha \xrightarrow{\pi} X$$

and the morphisms are the commutative diagrams

$$\begin{tikzcd} \widetilde{U}_\alpha \arrow[rr, shift left=1.5] \arrow[rr] \arrow[rr, shift right=1.5] \arrow[dr] & & \widetilde{U}_\beta \arrow[dl] \\ & X & \end{tikzcd} \qquad \text{(three morphisms)}$$

So the generalization consists in allowing several maps $\widetilde{U}_\alpha \rightrightarrows \widetilde{U}_\beta$, instead of only considering inclusions.

Actually Grothendieck generalizes the notion of a topology using such etale coverings and develops sheaf cohomology in this context.

It is clear from the definition that the lattice of all such categories (constructed from all the etale coverings of X) contains the following information:

i) the formal lattice of coverings by Zariski open sets

ii) that aspect of the "lattice of fundamental groups" of the Zariski open sets which is detected by looking at finite (algebraic) coverings of these open sets.

To convert this information into homotopy theory we make the

DEFINITION (The nerve of a category) *If C is a category define a semi-simplicial complex $S(C)$ as follows –*

the vertices are the objects of C
the l-simplices are the morphisms of C
$\vdots$
the n-simplices are the strings of n-morphisms in C,

$$O_0 \xrightarrow{f_1} O_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} O_n\,.$$

The face operations are obtained by composing maps. The degeneracies are obtained by inserting the identity map to expand the string.

The "nerve of the category" is the geometric realization of $S(C)$. The homology, cohomology and homotopy of the category are defined to be those of the nerve.

To work with the nerve of a category geometrically it is convenient to suppress the degenerate simplices – those strings

$$O_0 \xrightarrow{f_1} O_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} O_n$$

where some f_i is the identity.

The examples below begin to give one a feeling for manipulating the homotopy theory of categories directly without even going to the geometric realization of the category.

EXAMPLE 0

i) The non-degenerate part of the nerve of the category

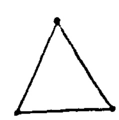
$$A \to B \leftleftarrows C$$

is the semi-simplicial complex

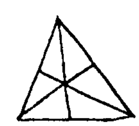
ii) If K is a simplicial complex, let $C(K)$ denote the category whose objects are the simplices of K and whose morphisms are the face relations

$$\sigma < \tau\,.$$

Then the non-degenerate part of the nerve of $C(K)$ is the first barycentric subdivision of K.



K



nerve of C(K)

iii) If there is at most one morphism between any two objects of C, then nerve $C(U)$ forms a simplicial complex.

iv) If C has a final object A, then the nerve is contractible. In fact

$$\text{nerve } C \;\cong\; \text{cone}\,\big(\text{nerve } (C - A)\big)\,.$$

v) C and its opposite category have isomorphic nerves.

The next example illustrates Lubkin's ingenious method of utilizing the nerves of categories.

EXAMPLE 1 Let $U = \{U_\alpha\}$ be a finite covering of a finite polyhedron K. Suppose

i) $K - U_\alpha$ is a subcomplex

ii) U_α is contractible

iii) $\{U_\alpha\}$ is "locally directed", i.e. given $x \in U_\alpha \cap U_\beta$ there is a U_γ such that $x \in U_\gamma \subseteq U_\alpha \cap U_\beta$.

Then following Lubkin we can construct the subcovering of "smallest neighborhoods", $C(U)$. Namely $U_\alpha \in C(U)$, iff there is an x such that U_α is the smallest element of U containing x. (Property iii) and the finiteness of U implies each x has a unique "smallest neighborhood".)

We regard $C(U)$ as a category with objects the U_α and morphisms the inclusions between them.

PROPOSITION 5.1

$$\text{nerve } C(U) \;\cong\; K\,.$$

PROOF: Suppose K is triangulated so that

i) each simplex of K is in some element U_α of $C(U)$.

ii) For each U_α of $C(U)$ the maximal subcomplex of the first barycentric subdivision of K, $K(U_\alpha)$ which is contained in U_α is contractible.

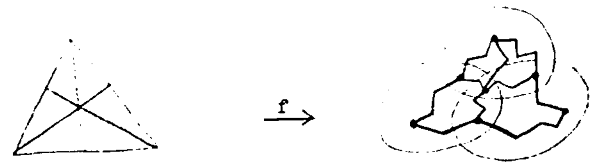
(If this is not so we can subdivide K so that

i) each simplex is smaller than the Lebesgue number of U

ii) each complement $K - U_\alpha$ is full – it contains a simplex iff it contains the vertices.)

Then the category $C(U)$ is equivalent to a category of contractible subcomplexes of K, $\{K(U_\alpha)\}$. It is easy now to define a canonical homotopy class of maps

$$\text{nerve } C(U) \xrightarrow{f} K$$



Map each vertex U_α of nerve $C(U)$ to a vertex of K_α. Map each 1-simplex $U_\alpha \to U_\beta$ of nerve $C(U)$ to a (piecewise linear) path in K_β. We proceed in this way extending the map piecewise-linearly over nerve $C(U)$ using the contractibility of the $K(U_\alpha)$'s.

We construct a map the other way.

To each simplex σ associate U_σ the smallest neighborhood of the barycenter of σ. One can check[7] that U_σ is the smallest neighborhood of x for each x in the interior of σ.

It follows that $\sigma < \tau$ implies $U_\sigma \subseteq U_\tau$.

So map each simplex in the first barycentric subdivision of K,

$$\sigma_1 < \sigma_2 < \cdots < \sigma_n\,, \quad \sigma_i \in K$$

to the simplex in nerve $C(U)$,

$$U_{\sigma_n} \subseteq U_{\sigma_{n-1}} \subseteq \cdots \subseteq U_{\sigma_1}\,.$$

This gives a simplicial map

$$K' \underset{g}{\to} \text{nerve } C(U)\,, \quad K' = \text{1st barycentric subdivision}\,.$$

Now consider the compositions

$$K' \underset{g}{\to} \text{nerve } C(U) \to K$$
$$\text{nerve } C(U) \xrightarrow{f} K = K' \xrightarrow{g} \text{nerve } C(U)\,.$$

One can check for the first composition that a simplex of K'

$$\sigma_1 < \cdots < \sigma_n$$

has image in $K(U_{\alpha_1})$. A homotopy to the identity is then easily constructed.

For the second composition consider a triangulation L of nerve $C(U)$ so that f is simplicial.[8] For each simplex σ of L consider

a) the smallest simplex of nerve $C(U)$ containing σ, say $U_1 \subset U_2 \subset \cdots \subset U_n$.

b) all the open sets in $C(U)$ obtained by taking "smallest neighborhoods" of barycenters of $f_*\tau$, τ a face of σ.

Let $C(\sigma)$ denote the subcategory of $C(U)$ generated by $U_1, U_2, \ldots, U_n$; the open sets of b); and all the inclusions between them.

The construction of f/σ takes place in U_n so all the objects of $C(\sigma)$ lie in U_n. This implies nerve $C(\sigma)$ is a contractible subcomplex of $C(U)$ $\big(C(\sigma)$ has a final object – $U_n.\big)$

Also by construction $\tau < \sigma$ implies $C(\tau) \subseteq C(\sigma)$. $g \circ f$ and the identity map are both carried by

$$\sigma \to C(\sigma)\,,$$

that is

$$I(\sigma) \subseteq C(\sigma)\,,$$
$$g \circ f(\sigma) \subseteq C(\sigma)\,.$$

The desired homotopy is now easily constructed by induction over the simplices of L.

NOTE: The canonical map

$$K \xrightarrow{g_U} \text{nerve } C(U) \qquad \begin{aligned} U &= \{U_\alpha\} \\ C(U) &= \text{category of "smallest neighborhoods"}\,. \end{aligned}$$

is defined under the assumptions

i) $K - U_\alpha$ is a subcomplex

ii) U is finite and locally directed.

We might prematurely say that maps like g_U (for slightly more complicated U's) comprise Lubkin's method of approximating the homotopy type of algebraic varieties.

The extra complication comes from allowing the categories to have many morphisms between two objects.

EXAMPLE 2 Let π be a group. Let $C(\pi)$ be the following category:

$C(\pi)$ has one object π 

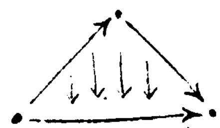
the morphisms of this object are the elements of π. These compose according to the group law in π.

Then

$$N \;=\; \text{nerve of } C(\pi) \;\cong\; K(\pi, 1)\,,$$

the space with one non-zero homotopy group π in discussion one.

Proof: First consider $\pi_1 N$. In the van Kampen description of π_1 of a complex we take the group generated by the edge paths beginning and ending at the base point with relations coming from the 2-cells,



(See Hilton and Wylie, "Homology Theory".)

Now N has only one vertex (the object π).

An element of π then determines a loop at this vertex. From the van Kampen description we see that this gives an isomorphism

$$\pi \xrightarrow[\cong]{} \pi_1 N\,.$$

Now consider the cells of N. The non-degenerate n-cells correspond precisely to the n-tuples of elements of π,

$$\{(g_1, \ldots, g_n) : g_i \neq 1\}\,.$$

An n-cell of the universal cover $\widetilde{N}$ is an n-cell of N with an equivalence class of edge paths connecting it to the base point.

We obtain a non-degenerate n-cell in $\widetilde{N}$ for each element in π and each non-degenerate n-cell in N.

It is not hard to identify the chain complex obtained from the cells of the universal cover $\widetilde{N}$ with the "bar resolution of π". (MacLane – "Homology" Springer Verlag, p. 114.)

Thus $\widetilde{N}$ is acyclic.

But $\widetilde{N}$ is also simply connected, thus

$$\pi \sim K(\pi, 1)\,.$$

The preceding two examples admit a common generalization which seems to be the essential *topological* fact behind the success of etale cohomology.

EXAMPLE 3 Let U be a locally directed covering of X by $K(\pi, 1)$'s,

$$\pi_i U_\alpha = 0 \text{ if } i > 1\,,\ \ U_\alpha \in U\,.$$

Consider the "generalized etale"[9] covering of X by the universal covers of the U_α

$$\widetilde{U}_\alpha \to X$$

and construct the category of "smallest neighborhoods" as before. (We assume for all $x \in U$ only finitely many U's contain x). The maps are now all commutative diagrams

$$\begin{tikzcd} \widetilde{U}_\alpha \arrow[rr, shift left=2] \arrow[rr] \arrow[rr, shift right=2] \arrow[dr] & & \widetilde{U}_\beta \arrow[dl] \\ & X & \end{tikzcd} \quad .$$

Then X is homotopy equivalent to

$$\text{nerve } C(U)\,.$$

A map $X \to$ nerve $C(U)$ is slightly harder to construct than in example 1. It is convenient to use contractible "Čech coverings" which refine U as in Theorem 5.12 below.

One homological proof can then be deduced from Lubkin "On a Conjecture of Weil" American Journal of Math. Theorem 2, p. 475.

Rather than pursue a precise discussion we consider some specific examples. They illustrate the geometric appeal of the Lubkin construction of etale cohomology.

i) (the circle, S^1) Consider the category determined by the single covering map,

$$R \xrightarrow[\pi]{x \mapsto e^{ix}} S^1 .$$

This category $\mathbb{Z}$ has one object, the map π, and infinitely many morphisms, the translations of R preserving π.

The nerve of Z has the homotopy type of a $K(\mathbb{Z}, 1) \cong S^1$.

ii) (the two sphere, S^2) Let p and σ denote two distinct points of S^2. Consider the category over S^2 determined by the maps

object	name
$S^2 - p \xrightarrow{\subseteq} S^2$	e
$S^2 - q \xrightarrow{\subseteq} S^2$	e'
(universal cover $S^2 - p - q$) $\to S^2$	$\mathbb{Z}$.

The category has three objects and might be denoted

$$C \equiv \{e \leftarrow \mathbb{Z} \to e'\} .$$

e and e' have only the identity self-morphism while $\mathbb{Z}$ has the covering transformations π^n, $-\infty < n < \infty$. There are unique maps

$$\mathbb{Z} \to e \text{ and } \mathbb{Z} \to e' .$$

The nerve of the subcategory $\{\mathbb{Z}\}$ is equivalent to the circle. The nerve of $\{e\}$ is contractible and the nerve of $\{\mathbb{Z} \to e\}$ looks like the cone over the circle. Thus the nerve of C looks like two cones over S^1 glued together along S^1, or

$$\text{nerve } C \cong S^2 .$$

iii) (the three sphere S^3) Consider $\mathbb{C}^2 - 0$ and the cover

$$U_1 = \{z_1 \neq 0\} \cong S^1 ,$$

$$U_2 = \{z_2 \neq 0\} \cong S^1 ,$$

$$U_1 \cap U_2 = \{z_1 \neq 0, z_2 \neq 0\} \cong S^1 \times S^1 .$$

Let C be the category determined by the universal covers of these open sets mapping to $\mathbb{C}^2 - 0$. Then C might be denoted

$$\{\mathbb{Z} \underset{p_1}{\longleftarrow} \mathbb{Z} + \mathbb{Z} \underset{p_2}{\longrightarrow} \mathbb{Z}\} ,$$

where p_1 and p_2 are the two projections. (We think of the elements in $\mathbb{Z}$ or $\mathbb{Z}+\mathbb{Z}$ and p_1 and p_2 as generating the morphisms of the category.)

The nerve of C is built up from the diagram of spaces

$$\{S^1 \underset{p_1}{\longleftarrow} S^1 \times S^1 \underset{p_2}{\longrightarrow} S^1\}\,,$$

corresponding to the decomposition of S^3 into two solid tori glued together along their common boundary.

the realization of the natural functor

$$\{\mathbb{Z} \leftarrow \mathbb{Z}+\mathbb{Z} \rightarrow \mathbb{Z}\} \rightarrow \{e \leftarrow \mathbb{Z} \rightarrow e'\}$$

corresponds to the hemispherical decomposition of the Hopf map

$$S^3 \xrightarrow{H} S^2\,.$$

iv) (Complex projective plane) Consider the three natural affine open sets in

$$\mathbb{C}\,P^2 \;=\; \{(z_0, z_1, z_2) \text{ “homogeneous triples”}\}\,,$$

defined by

$$U_i \;=\; \{(z_0, z_1, z_2) : z_i \neq 0\} \quad i = 0, 1, 2\,.$$

U_i is homeomorphic to $\mathbb{C} \times \mathbb{C}$,

$U_i \cap U_j$ is homeomorphic to $(\mathbb{C} - 0) \times \mathbb{C}$ $i \neq j$,

$U_0 \cap U_1 \cap U_2$ is homeomorphic to $(\mathbb{C} - 0) \times (\mathbb{C} - 0)$.

The category of universal covering spaces over $\mathbb{C}\,P^2$,

$$\begin{tikzcd}
 & & \widetilde{U}_0 & & \\
 & \widetilde{U}_0 \cap \widetilde{U}_1 \arrow[ur] \arrow[ddl] & & \widetilde{U}_0 \cap \widetilde{U}_2 \arrow[ul] \arrow[ddr] & \\
 & & \widetilde{U}_0 \cap \widetilde{U}_1 \cap \widetilde{U}_2 \arrow[ul, shift left] \arrow[ul, shift right] \arrow[ur, shift left] \arrow[ur, shift right] \arrow[d, shift left] \arrow[d, shift right] & & \\
\widetilde{U}_1 & & \widetilde{U}_1 \cap \widetilde{U}_2 \arrow[ll] \arrow[rr] & & \widetilde{U}_2
\end{tikzcd}$$

might be denoted

$$
\begin{tikzcd}
 & & e & & \\
 & Z \arrow[ur] \arrow[ddl] & & Z \arrow[ul] \arrow[ddr] & \\
 & & Z+Z \arrow[ul, "\ell_0"'] \arrow[ur, "\ell_1"] \arrow[d, "\ell_2"] & & \\
e & & Z \arrow[ll] \arrow[rr] & & e
\end{tikzcd}
$$

where

$$
\begin{aligned}
l_0(a \oplus b) &= a\,, \\
l_1(a \oplus b) &= b\,, \\
l_2(a \oplus b) &= a + b\,.
\end{aligned}
$$

Notice that C looks like the mapping cone of the Hopf map

$$
\{e\} \leftarrow \left\{ \begin{array}{c} \mathbb{Z} \\ \uparrow \\ \mathbb{Z}+\mathbb{Z} \\ \downarrow \\ \mathbb{Z} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} e \\ \uparrow \\ \mathbb{Z} \\ \downarrow \\ e \end{array} \right\},
$$

$e^4 \cup_H S^2 \cong \mathbb{C}\,P^2$.

Note that the category C has an involution obtained by reflecting objects about the l_2 axis an mapping morphisms by sending a into b.

The fixed subcategory is

$$
\{e \leftarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z}\}
$$

whose nerve is the mapping cone of

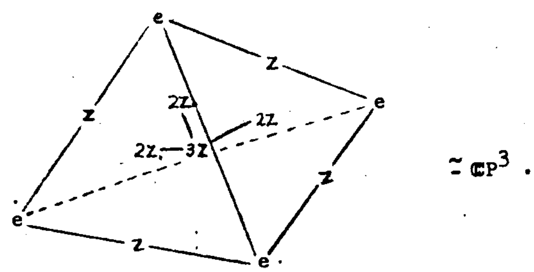
$$
S^1 \xrightarrow{\text{degree } 2} S^1
$$

or the real projective plane $\mathbb{R}\,P^2$.

v) (Complex projective n-space) Again consider the category determined by the universal covers of the natural affines in $\mathbb{C}\,P^n$

$$
U_i = \{(z_0, \ldots, z_n) : z_i \neq 0\}, \;\; 0 \leqslant i \leqslant n
$$

and their intersections. This category is modeled on the n-simplex, one imagines an object at the barycenter of each face whose self morphisms form a free Abelian group of rank equal to the dimension of the face,



The nerve of this category is equivalent to $\mathbb{C}\,P^n$.

This decomposition of $\mathbb{C}\,P^n$ occurs in other points of view:

i) the nth stage of the Milnor construction of the classifying space for S^1,

$$\underbrace{S^1 * S^1 * \cdots * S^1}_{n \text{ join factors}} / S^1$$

ii) the dynamical system for the Frobenius map

$$(z_0, z_1, \ldots, z_n) \xrightarrow{\mathscr{F}_q} (z_0^q, z_1^q, \ldots, z_n^q)\,.$$

If we iterate $\mathscr{F}_q$ indefinitely the points of $\mathbb{C}\,P^n$ flow according to the scheme of arrows in the n-simplex. Each point flows to a torus of the dimension of the face.

For example, if we think of $P^1(\mathbb{C})$ as the extended complex plane, then under iteration of the q^{th} power mapping

the points inside the unit disk flow to the origin,
the points outside the unit disk flow to infinity,
the points of the unit circle are invariant.

This corresponds to the scheme

$$S^2 \cong P^1(\mathbb{C}) \cong \{e \leftarrow \mathbb{Z} \rightarrow e\}\,.$$

(John Guckenheimer).

This (n-simplex) category for describing the homotopy type of $\mathbb{C}P^n$ has natural endomorphisms –

$$F^q \;=\; \begin{cases} \text{identity in objects}\,, \\ f \to f^q \text{ for self morphisms}\,. \end{cases}$$

(F^q is then determined on the rest of the morphisms.)

The realization of F^q is up to homotopy the geometric Frobenius discussed above, $\mathscr{F}^q$.

From these examples we see that there are simple categories which determine the homotopy type of these varieties

$$\begin{aligned} V :\ & \mathbb{C}^1 - 0, \mathbb{C}^2 - 0, \dots \\ & \mathbb{C}P^1, \mathbb{C}P^2, \dots, \mathbb{C}P^n, \dots\,. \end{aligned}$$

These categories are constructed from Zariski coverings of V by considering the universal covering spaces of the elements in the Zariski cover.

These categories cannot be constructed from etale coverings of V, collections of *finite to one* covering maps whose images are Zariski open coverings of V.

If we consider these algebraically defined categories we can however "profinitely approximate" the categories considered above.

For example, in the case $V = S^2 = \mathbb{C}P^1$, look at the etale covers

$$\begin{aligned} U_n :\ & S^2 - p \xrightarrow{\subseteq} S^2\,, \\ & S^2 - q \xrightarrow{\subseteq} S^2\,, \\ & S^2 - p - q \xrightarrow[\text{degree } n]{} S^2\,. \end{aligned}$$

The category of this cover,

$$C(U_n) \cong \{e \leftarrow \mathbb{Z}/n \to e\}$$

has nerve the suspension of the infinite dimensional lens space

$$\text{suspension } K(\mathbb{Z}/n, 1)\,.$$

These form an inverse system by refinement

$$\begin{tikzcd}[column sep=small, row sep=small]
 & {} \arrow[dr] & & \\
{} \arrow[dr] & & C(U_n) \arrow[dr] & \\
 & C(U_m) \arrow[ur] \arrow[dr] & & C(U_k) \\
{} \arrow[ur] & & C(U_l) \arrow[ur] &
\end{tikzcd}$$

(n ordered by divisibility).

The nerves give an inverse system of homotopy types with finite homotopy groups

$$\{\text{suspension } K(\mathbb{Z}/n,1)\}_n \,.$$

We can think of this inverse system as representing the etale homotopy type of S^2 (it is homotopically cofinal in the inverse system of all nerves of etale coverings).

We can form a homotopy theoretical limit if we wish[10],

$$\widehat{X} \cong \varprojlim_n \{\text{suspension } K(\mathbb{Z}/n,1)\}\,.$$

The techniques of section 3 show

$$\begin{aligned}
\widetilde{H}_i\widetilde{X} &\cong \varprojlim_n H_i\{\text{suspension } K(\mathbb{Z}/n,1)\} \\
&= \begin{cases} \varprojlim_n (\mathbb{Z}/n) & \text{if } i=2 \\ \quad 0 & \text{if } i\neq 2 \end{cases} \\
&= \begin{cases} \widehat{\mathbb{Z}} & \text{if } i=2 \\ 0 & \text{if } i\neq 2\,. \end{cases}
\end{aligned}$$

Since $\pi_1 X$ is zero we have the profinite completion of S^2, constructed algebraically from etale coverings of $\mathbb{C}P^1$.

In a similar manner we construct the profinite completions of the varieties above using the nerves of categories of etale coverings of these varieties.

For a more general complex variety V Lubkin considers all the (locally directed, punctually finite) etale covers of V. Then he takes the nerves of the category of smallest neighborhoods. This gives an inverse system of homotopy types from which we can form the profinite completion of the homotopy type of V.

The success of the construction is certainly motivated from the topological point of view by the topological proposition and examples above.

From the algebraic point of view one has to know

i) there are enough $K(\pi,1)$ neighborhoods in a variety

ii) the lattice of algebraic coverings of these open sets gives the profinite completion of the fundamental group of these open sets.

The following sketch (which goes back to Lefschetz[11]) provides credibility for i) which was proved in a more general context by Artin.

Consider the assertion

K_n: for each Zariski open set U containing $p \in V^n$, there is a $K(\pi,1)$ Zariski open U' so that

$$p \in U' \subseteq U\,,$$

where V^n is a non-singular subvariety of projective space $\mathbb{C}\,P^n$.

"Proof of K_n":

a) K_n is true for $n = 1$. V^1 is then a Riemann surface and

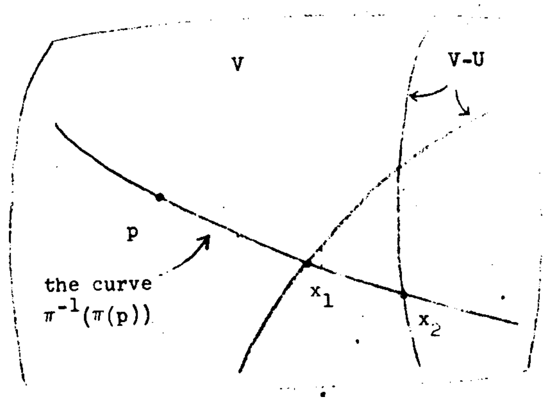
$$U \;=\; V - \text{finite set of points}\,.$$

But then U is a $K(\pi,1)$.

b) If $F \to E \to B$ is a fibration with F and B $K(\pi,1)$'s then E is a $K(\pi,1)$. This follows from the exact homotopy sequence.

c) K_n is true for all V if K_{n-1} is true for $V = P^{n-1}(\mathbb{C})$.

Consider $U \subseteq V$ a Zariski open set about p. Choose a generic projection π of $P^n(\mathbb{C})$ onto $P^{n-1}(\mathbb{C})$,



so that

i) $\pi(p)$ is a regular value of π/V,

ii) the non-singular Riemann surface $C = \pi^{-1}\big(\pi(p)\big)$ cuts $V - U$ transversally in a finite number of points $x_1, \ldots, x_n$.

Choose a $K(\pi, 1)$ neighborhood W of $\pi(p)$ in $P^{n-1}(\mathbb{C})$ contained in the set of images of points where i) and ii) (with constant n) hold.

Then by b) $\pi^{-1}W - (V - U)$ is a $K(\pi, 1)$ neighborhood of p contained in U. It is fibred by a punctured curve over a $K(\pi, 1)$ neighborhood in $P^{n-1}(\mathbb{C})$. Q. E. D.

For the fundamental group statement of ii) it is easy to see that an etale map determines a finite covering space of the image. (One may first have to apply a field automorphism of $\mathbb{C}$ to obtain a local expression of the map by complex polynomials.)

The converse is harder – the total space of a finite cover has an analytic structure. This analytic structure is equivalent in enough cases to an algebraic structure to compute $\widehat{\pi}_1$. For $n = 1$, this fact is due to Riemann.

The Complete Etale Homotopy Type

In order to apply etale homotopy we pass to the limit using the compactness of profinite sets (as in section 3). We obtain a single homotopy type from the inverse system of homotopy types comprising the etale type. This "complete etale homotopy type" is the profinite completion of the classical homotopy type for complex varieties.

We study the arithmetic square (in the simply connected case) and look at the examples of the Grassmannians – "real" and "complex".

Let X denote the homotopy type of an algebraic variety of finite type over $\mathbb{C}$ or a direct limit of these, for example

$$G_n(\mathbb{C}) \;=\; \lim_{k\to\infty} (\text{Grassmannian of } n\text{-plane in } k\text{-space})\,.$$

The etale homotopy type of X is an inverse system of CW complexes with finite homotopy groups.[12]

Each one of these complexes determines a compact representable functor (section 3) and the inverse limit of these is a compact representable functor. Let us denote this functor by $\widehat{X}$.

$$\begin{matrix}\text{Homotopy}\\ \text{Category}\end{matrix} \xrightarrow{\hat{X}} \begin{matrix}\text{category of}\\ \text{compact Hausdorff}\\ \text{spaces}\end{matrix}\;.$$

Recall that $\widehat{X}$ determined a single CW complex (also denoted $\widehat{X}$).

Definition 5.2. *The compact representable functor $\widehat{X}$,*

$$\begin{matrix}\textit{Homotopy}\\ \textit{Category}\end{matrix} \xrightarrow[\underset{\textit{etale covers}}{\varprojlim}\;[\quad,\text{nerve}]]{} \begin{matrix}\textit{compact}^{13}\\ \textit{Hausdorff}\\ \textit{spaces}\end{matrix}$$

together with its underlying CW complex $\widehat{X}$ is the complete etale homotopy type of the algebraic variety X.[14]

Theorem 5.2 *Let V be a complex algebraic variety of finite type. The "complete etale homotopy type", defined by*

$$[\quad,\widehat{V}] \equiv \varprojlim_{\text{etale covers}}\;[\quad,\text{nerve}]$$

is equivalent to the profinite completion of the classical homotopy type of V.

The integral homology of $\widehat{V}$ is the profinite completion of the integral homology of V,

$$\widetilde{H}_i \widehat{V} \cong (\widetilde{H}_i V)\hat{}\ .$$

If V is simply connected, the homotopy groups of $\widehat{V}$ are the profinite completions of those of V

$$\pi_i \widehat{V} \cong (\pi_i V)\hat{}\ .$$

In this simply connected case $\widehat{V}$ is the product over the primes of p-adic components

$$\widehat{V} \cong \prod_p \widehat{V}_p\,.$$

Moreover, the topology on the functor

$$[\quad , \widehat{V}]$$

is intrinsic to the homotopy type of the CW complex $\widehat{V}$.

PROOF: The first part is proved, for example, by completing both sides of the Artin-Mazur relation among inverse systems of homotopy types,

$$\text{etale type} \equiv \begin{Bmatrix} \text{inverse system} \\ \text{of “nerves”} \end{Bmatrix} \cong \{F\}_{\{f\}}\,. \quad \text{(see section 3)}$$

To do this we appeal to the lemma of section 3 – the arbitrary inverse limit of compact representable functors is a compact representable functor.

Since the inverse system $\{F\}_{\{f\}}$ was used to construct the profinite completion of V the result follows.

NOTE: Actually it is immaterial in the simply connected case whether we use the Artin-Mazur pro-object or Lubkin's system of nerves to construct the complete etale type. The characterization of section 3 shows that from a map of V into an inverse system of spaces $\{X_i\}$ with the correct cohomology property

$$H^*(V;A) \cong \lim_{\rightarrow} H^*(X_i;A) \quad A \text{ finite}$$

we can construct the profinite completion of V,

$$\widehat{V} \cong \lim_{\leftarrow} \widehat{X}_i\,.$$

(This is true for arbitrary π_1 if we know the cohomology isomorphism holds for A twisted.)

To understand how much information was carried by the profinite completion we developed the "arithmetic square"

$$\begin{tikzcd} X \arrow[r, "\text{profinite completion}"] \arrow[d, "\text{localization}"'] & \widehat{X} \arrow[d, "\text{localization}"] & = \text{"profinite type of } X\text{"} \\ \text{"rational type of } X\text{"} = X_0 \arrow[r, "\text{formal completion}"'] & X_A \arrow[d, equal] & \cong (X_0)^- \cong (\widehat{X})_0 \\ & \text{Adele type of } X & \end{tikzcd}$$

If for example $\widehat{X}$ is simply connected then

i) $\widehat{X} \cong \prod_p \widehat{X}_p$.

ii) The homotopy of the "arithmetic square" is

$$\pi_* X \otimes \left\{ \begin{tikzcd} \mathbf{Z} \arrow[r] \arrow[d] & \widehat{\mathbf{Z}} \arrow[d] \\ \mathbf{Q} \arrow[r] & \widehat{\mathbf{Z}} \otimes \mathbf{Q} \end{tikzcd} \right\} .$$

Under finite type assumptions,

$$H^i(X; \mathbb{Z}/n) \text{ finite for each } i$$

the topology on the homotopy functor is "intrinsic" to the homotopy type of $\widehat{X}$.

Thus we can think of $\widehat{X}$ as a homotopy type which happens to have the additional property that there is a natural topology in the homotopy sets $[\quad, \widehat{X}]$.

This is analogous to the topology on $\widehat{\mathbb{Z}}$ which is intrinsic to its algebraic structure,

$$\widehat{\mathbb{Z}} \cong \varprojlim(\widehat{\mathbb{Z}} \otimes \mathbb{Z}/n) .$$

iii) The problem of determining the classical homotopy type of X from the complete etale type $\widehat{X}$ was the *purely rational homotopy* *problem* of finding an "appropriate embedding" of the rational type of X into the Adele type of X,

$$X_0 \xrightarrow{i} X_A = (\widehat{X})_{\text{localized at zero}}\,.$$

The map i has to be equivalent to the formal completion considered in section 3.

Then X is the "fibre product" of the i and ℓ in the diagram

$$\begin{tikzcd}
 & & & \widehat{X} \arrow[d, "\ell"', "\text{localization}"] \\
X_0 \arrow[r, "\text{formal}", "\text{completion}"'] \arrow[rrr, dashed, bend left, "i"] & (X_0)^- \arrow[r, phantom, "\cong", "\substack{\text{natural}\\ \text{equivalence}}"'] & \; & (\widehat{X})_0
\end{tikzcd} \equiv (\text{Adele type of } X)\,.$$

Notice that we haven't characterized the map i, we've only managed to construct it as a function of X_0. This is enough in the examples below.

Examples:

1) $X = \lim\limits_{k\to\infty} G_{n,k}(\mathbb{C}) = G_n(\mathbb{C})$, the classifying space for the unitary group, BU_n.

   i) The profinite vertex of the "arithmetic square" is the "direct limit" of the complete etale homotopy types of the complex Grassmannians (thickened)

$$G_{n,k} \cong \mathrm{GL}\,(n+k,\mathbb{C})/\,\mathrm{GL}\,(n,\mathbb{C}) \times \mathrm{GL}\,(k,\mathbb{C})^{15}\,.$$

   More precisely, the underlying CW complex of $\widehat{X}$ is the infinite mapping telescope

$$\widehat{G}_{n,n}(\mathbb{C}) \to \widehat{G}_{n,n+1}(\mathbb{C}) \to \dots\,.$$

   The functor $[\quad,\widehat{X}]$ is determined on finite complexes by the direct limit

$$\varinjlim_{k}[\quad,\widehat{G}_{n,k}(\mathbb{C})]\,.$$

Almost all maps in the direct limit are isomorphisms so the compact topology is preserved.
The functor $[\quad, \widehat{X}]$ on general complexes is the unique extension via inverse limits

$$\lim_{\leftarrow}[\text{finite subcomplex}, \widehat{X}]\,.$$

ii) The rational vertex of the "arithmetic square" is the product

$$\prod_{i=1}^{n} K(\mathbb{Q}\,, 2i)$$

of Eilenberg MacLane spaces. The rational Chern classes give the "localization at zero"

$$BU_n \xrightarrow{(c_1, c_2, \ldots, c_n)} \prod_{i=1}^{n} K(\mathbb{Q}\,, 2i)\,.$$

This map induces an isomorphism of the rational cohomology algebras and is thus a localization by Theorem 2.1.

iii) For the Adele type of X we have

$$\begin{aligned} X_A &\cong \text{formal completion of localization at zero } BU_n \\ &\cong \text{localization at zero of } (BU_n)\hat{} \cong \prod_{i=1}^{n} K(\mathbb{Q} \otimes \widehat{\mathbb{Z}}, 2i)\,. \end{aligned}$$

(Since the latter is the formal completion of the rational type $\prod K(\mathbb{Q}\,, 2i)$.)
The $\widehat{\mathbb{Z}}$ module structure on the homotopy groups on X_A determined by the partial topology in

$$[\quad, X_A]$$

is the natural map in

$$\prod K(\mathbb{Q} \otimes \widehat{\mathbb{Z}}, 2i)$$

iv) We have the fibre square

$$\begin{array}{ccc} BU_n & \longrightarrow & \prod_p \widehat{X}_p = \{\text{complete etale type}\} \\ \big\downarrow & & \big\downarrow \\ \left\{\begin{matrix}\text{rational} \\ \text{type}\end{matrix}\right\} = \prod_{i=1}^{n} K(\mathbb{Q}\,, 2i) & \longrightarrow & \prod_{i=1}^{n} K(\mathbb{Q} \otimes \widehat{\mathbb{Z}}, 2i) \ = \{\text{Adele type}\} \end{array}$$

2) (The "real Grassmannian").

Consider the "complex orthogonal group"

$$O(n,\mathbb{C}) \subseteq \mathrm{GL}\,(n,\mathbb{C}) = \{A \in \mathrm{GL}\,(n,\mathbb{C}) \mid (Ax, Ax) = (x,x)\}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ and $(x,x) = \sum x_i^2$.

There is a diagram

$$\begin{tikzcd} O(n,\mathbb{C}) \arrow[r, "j"] & \mathrm{GL}\,(n,\mathbb{C}) \\ O(n) \arrow[u, "c"] \arrow[r, "i"] & \mathrm{GL}\,(n,\mathbb{R})\,. \arrow[u, "r"'] \end{tikzcd}$$

j is an inclusion of complex algebraic groups,

i is an inclusion of real algebraic groups,

c and r are the inclusions of the real points into the complex plane.

Studying the inclusion

$$\mathrm{GL}\,(n,\mathbb{R}) \to \mathrm{GL}\,(n,\mathbb{C})$$

homotopy theoretically is equivalent to studying the map of complex algebraic groups

$$O(n,\mathbb{C}) \to \mathrm{GL}\,(n,\mathbb{C})$$

because c and i are both homotopy equivalences.

i is an equivalence because of the well-known Gram-Schmidt deformation retraction. That c is an equivalence follows for example from Chevalley [16] – who showed that any compact Lie group[17] G has a *complex* algebraic form $\big($in our case $O(n)$ and $O(n,\mathbb{C})\big)$ which is topologically a product tubular neighborhood of the compact form.

If we adjoin the equation $\det A = 1$ we get the special complex orthogonal group

$$SO(n,\mathbb{C})\,,$$

the component of the identity of $O(n,\mathbb{C})$.

Now consider the "algebraic form" of the oriented real Grassmannians (thickened)

$$\widetilde{G}_{n,k}(\mathbb{R}) \cong SO(n+k,\mathbb{C})/SO(n,\mathbb{C}) \times SO(k,\mathbb{C})\,.$$

These are nice complex algebraic varieties which have the same homotopy type as the orientation covers of the real Grassmannians

$$\mathrm{GL}\,(n+k,\mathbb{R})/\,\mathrm{GL}\,(n,\mathbb{R})\times\mathrm{GL}\,(k,\mathbb{R})\,.$$

Let $X=\lim\limits_{k\to\infty}\widetilde{G}_{n,k}(\mathbb{R})\cong BSO_n$, the classifying space for the special orthogonal group.

Using the Euler class and Pontrjagin classes we can compute the "arithmetic square" as before:

$$\begin{tikzcd}
& BSO_n \arrow[r] \arrow[d] & \prod\limits_p \widehat{X}_p = \begin{Bmatrix}\text{complete etale}\\ \text{homotopy type}\end{Bmatrix} \arrow[d] \\
\begin{Bmatrix}\text{rational}\\ \text{type}\end{Bmatrix} = & \prod\limits_{i\in S} K(\mathbb{Q}\,,i) \arrow[r] & \prod\limits_{i\in S} K(\mathbb{Q}\otimes\widehat{\mathbb{Z}},i) = \{\text{Adele type}\}
\end{tikzcd}$$

where

$$\begin{aligned} S &= (4,8,12,\ldots,2n-4,n) \quad && n \text{ even}\\ S &= (4,8,12,\ldots,2n-2) && n \text{ odd}\end{aligned}$$

The Galois Symmetry in the Grassmannians BU_n and BSO_n

Now we come to the most interesting point about the "algebraic" aspect of etale homotopy type.

The varieties that we are considering – the "Grassmannians"

$$G_{n,k}(\mathbb{C}),\ \ \widetilde{G}_{n,k}(\mathbb{R})$$

are defined over the field of rational numbers. The coefficients in the defining equations for

$$\mathrm{GL}\,(n,\mathbb{C})\ \text{ and }\ O(n,\mathbb{C})$$

are actually integers.

Thus any field automorphism of the complex numbers fixes the coefficients of these equations and defines an *algebraic automorphism* of these Grassmannian varieties. Such an algebraic automorphism determines an automorphism of the system of algebraic coverings, an automorphism of the system of corresponding nerves, and finally a homotopy equivalence of the complete etale homotopy type.[18]

We can describe the action informally. Our variety V is built from a finite number of affine varieties

$$V \;=\; \bigcup A_i \,,$$

$$\mathbb{C}^n \supseteq A_i = \{(x_1, \ldots, x_n) \mid f_{ji}(x_1, \ldots, x_n) = 0, \;\; 0 \leqslant j \leqslant k\} \,.$$

The A_i are assembled with algebraic isomorphisms between complements of subvarieties of the A_i.

To say that V is *defined over the rationals* means the coefficients of the defining equation $f_{ji} = 0$ lie in $\mathbb{Q}$ and the polynomials defining the pasting isomorphisms have coefficients in $\mathbb{Q}$.

If σ is an automorphism of $\mathbb{C}$,

$$z \mapsto z^\sigma \,,$$

we have an *algebraic isomorphism* of $\mathbb{C}^n$

$$(z_1, \ldots, z_n) \mapsto (z_1^\sigma, \ldots, z_n^\sigma) \,.$$

σ has to be the identity on the subfield $\mathbb{Q}$ ($1^\sigma = 1$, $2^\sigma = 2$, $(\frac{1}{2})^\sigma = \frac{1}{2}\sigma = \frac{1}{2}$, etc.) [19]

Thus σ preserves the equations

$$f_{ji}(z_1, \ldots, z_n) = 0$$

defining the affines A_i and the pasting isomorphisms from which we build V. This gives

$$V \xrightarrow{\sigma} V \,,$$

an algebraic isomorphism of V.

We describe the automorphism induced by σ on the system of "etale nerves".

Let U be an etale covering of V,

$$\begin{tikzcd}
\ldots U_\alpha \arrow[dr, "\pi_\alpha"'] & U_\beta \arrow[d, "\pi_\beta"] & U_\gamma \ldots \arrow[dl, "\pi_\gamma"] \\
& V &
\end{tikzcd}$$

Each π is a finite algebraic covering of the complement of a subvariety of V. The images of the π cover V.

Given σ we proceed as in Čech theory and form the "inverse image" or pullback of the π

$$\begin{tikzcd} V \times U_\alpha \supseteq \sigma^* U_\alpha \arrow[r, "\sigma^*"] \arrow[d, "\sigma^* \pi_\alpha"] & U_\alpha \arrow[d, "\pi_\alpha"] \\ V \arrow[r, "\sigma"'] & V\,, \end{tikzcd}$$

$$\sigma^* U_\alpha = \{(v,u) : \sigma(v) = \pi_\alpha(u)\}\,.$$

Thus we have a "backwards map" of the indexing set of etale coverings used to construct the etale type,

$$\{U\} \mapsto \{\sigma^* U\}\,.$$

On the other hand there is a natural map of the categories determined by U and $\sigma^* U$ in the other direction

$$C(\sigma^* U) \xrightarrow{\sigma} C(U)\,.$$

This is defined by

$$(\sigma^* U_\alpha) \mapsto U_\alpha\,,$$

$$(\sigma^* U_\alpha \xrightarrow{f} \sigma^* U_\beta) \mapsto (U_\alpha \xrightarrow{\sigma_* f} U_\beta)\,,$$

where σ_* is defined so that

$$\begin{tikzcd} \sigma^* U_\alpha \arrow[r, "\sigma^*"] \arrow[d, "f"] & U_\alpha \arrow[d, "\sigma_* f"] \\ \sigma^* U_\beta \arrow[r, "\sigma^*"] & U_\beta \end{tikzcd}$$

commutes.

Thus σ induces a map of inverse systems of categories

indexing set : $\{U\} \to \{\sigma^* U\} \subseteq \{U\}\,,$

inverse system of categories : $\{C(\sigma^* U)\} \to \{C(U)\}\,.$

Similarly we have a map of the inverse system of nerves

$$\{U\} \to \{\sigma^* U\}\,,$$

$$\{\text{nerve } C(\sigma^* U)\} \to \{\text{nerve } C(U)\}\,.$$

Finally we pass to the limit and obtain an automorphism

$$\widehat{V} \xrightarrow{\sigma} \widehat{V}$$

of the profinite completion of the homotopy type of V.

Remark: (General Naturality). In this case when σ is an isomorphism σ^* is also an isomorphism so that it is clear that $\sigma_* f$ exists and is unique.

If σ were a more general morphism, say the inclusion of a subvariety

$$W \hookrightarrow V$$

then the "smallest neighborhood" concept plays a crucial role.

The elements in $C(U)$ were really "smallest neighborhoods" (see example 1 in the preceding subsection – "Intuitive Discussion of the Etale Homotopy Type") constructed from the locally finite locally directed etale covering U. $\sigma^* U$ is also locally finite and locally directed so it has "smallest neighborhoods", which are the objects of $C(\sigma^* U)$.

One can check that

$$\sigma^* U_\alpha \subseteq \alpha^* U_\beta \qquad \begin{array}{l}\sigma^* U_\alpha, \sigma^* U_\beta \text{ “smallest} \\ \text{neighborhoods” in } W \\ \text{of } \omega_\alpha \text{ and } \omega_\beta\end{array}$$

implies

$$U_{\omega_\alpha} \subseteq U_{\omega_\beta} \qquad \begin{array}{l}U_{\omega_\alpha}, U_{\omega_\beta} \text{ “smallest} \\ \text{neighborhoods” in } V \\ \text{of } \omega_\alpha \text{ and } \omega\beta\,.\end{array}$$

So we have a functor

$$C(\sigma^* U) \to C(U)$$

induced by

$$\sigma^* U_\alpha \mapsto U_{\omega_\alpha}\,.$$ [20]

The functor is not canonical, but the induced map on nerves is well-defined up to homotopy – as in the Čech theory.

Lubkin uses an interesting device which enlarges the calculation scheme restoring canonicity on the map level. We use this device in the proof about the "real variety conjecture".

Example $V = \mathbb{C} - 0$.

The etale coverings with one element

$$\{U_n\} \qquad U_n \xrightarrow[\substack{\text{covering of}\\ \text{degree } n}]{\pi} \mathbb{C} - 0$$

are sufficient to describe the etale homotopy.

The associated category is the one object category $\mathbb{Z}/n$ – the self-morphisms form a cyclic group of order n.

Now $U_n \xrightarrow{\pi} \mathbb{C} - 0$ is equivalent to $\mathbb{C} - 0 \xrightarrow{F_n} \mathbb{C} - 0$ where F_n is raising to the n^{th} power. So

$$\begin{tikzcd} \sigma^* U_n \arrow[r, "\sigma^*"] \arrow[d, "\sigma^*"'] & U_n \arrow[d, "\pi"] \\ (\mathbb{C} - 0) \arrow[r, "\sigma"] & (\mathbb{C} - 0) \end{tikzcd} \quad \cong \quad \begin{tikzcd} (\mathbb{C} - 0) \arrow[r, "\sigma"] \arrow[d, "F_n"] & (\mathbb{C} - 0) \arrow[d, "F_n"] \\ (\mathbb{C} - 0) \arrow[r, "\sigma"] & (\mathbb{C} - 0) \end{tikzcd}.$$

The covering transformations of π are rotations by n^{th} roots of unity

$$f_\xi : z \mapsto \xi z\,, \ \ \xi^n = 1\,.$$

Now $\sigma_* f_\xi$ is defined by

$$\begin{tikzcd} (\mathbb{C} - 0) \arrow[r, "\sigma"] \arrow[d, "f_\xi"] & (\mathbb{C} - 0) \arrow[d, "\sigma_* f_\xi"] \\ (\mathbb{C} - 0) \arrow[r, "\sigma"] & (\mathbb{C} - 0) \end{tikzcd}$$

or

$$(\sigma_* f_\xi)(z^\sigma) \ = \ \big(f_\xi(z)\big)^\sigma\,.$$

This reduces to

$$\begin{aligned}\sigma_* f_\xi(z) &= \left(f_\xi(z^{\sigma^{-1}})\right)^\sigma \\ &= \left(\xi \cdot (z^{\sigma^{-1}})\right)^\sigma \\ &= \xi^\sigma \cdot z\end{aligned}$$

or

$$\sigma_* f_\xi \;=\; f_\xi \sigma\,.$$

So if we identify the one object "group" categories $C(U_n)$ with the group of n^{th} roots of unity, the automorphism of $\mathbb{C}$ acts on the category via its action on the roots of unity

$$\xi \mapsto \xi^\sigma\,, \;\; \xi^n = 1\,.$$

Notice that the automorphism induced by σ on the etale homotopy of $\mathbb{C}-0$ only depends on how σ moves the *roots of unity* around.

This is not true in general[21] however it is true for a general variety V that the automorphism induced by σ on the etale homotopy of V only depends on how σ moves the *algebraic numbers* around.

Roughly speaking $\widetilde{\mathbb{Q}}$, the field of algebraic numbers and $\mathbb{C}$ are two *algebraically closed* fields containing $\mathbb{Q}$

$$\mathbb{Q} \subseteq \widetilde{\mathbb{Q}} \subseteq \mathbb{C}\,,$$

and nothing new happens from the point of view of etale homotopy in the passage from $\widetilde{\mathbb{Q}}$ to $\mathbb{C}$.

So we have the "Galois group of $\mathbb{Q}$",

$$\operatorname{Gal}(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$$

acting naturally on the profinite completion of the homotopy types of the Grassmannians

$$G_{n,k}(\mathbb{C}) \;\text{ and }\; \text{“}G_{n,k}(\mathbb{R})\text{”}\,.$$

$\big(\text{“}G_{n,k}(\mathbb{R})\text{”} \equiv O(n+k,\mathbb{C})/O(n,\mathbb{C}) \times O(k,\mathbb{C}).\big)$

The existence of the action is a highly non-trivial fact. The automorphisms of the ground field $\widetilde{\mathbb{Q}}$ are very discontinuous when extended to the complex plane. Thus it is even quite surprising that this group of automorphisms acts on the (mod n) cohomology of algebraic varieties (defined over the rationals).[22]

We have some additional remarks about the Galois action.

i) Many varieties if not defined over $\mathbb{Q}$ are defined over some number field – a finite extension of $\mathbb{Q}$. For such a variety V defined over K, a subgroup of finite index in the "Galois group of $\mathbb{Q}$",

$$\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/K) \subseteq \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$$

acts on the profinite completion of the homotopy type of V.

ii) The actions of these profinite Galois groups are continuous with respect to the natural topology on the homotopy sets

$$[\quad, \widehat{V}]\,.$$

For example, in the case V is defined over $\mathbb{Q}$ a particular finite etale cover will be defined over a finite extension of $\mathbb{Q}$. The varieties and maps will all have local polynomial expressions using coefficients in a number field L.

The effect of the profinite Galois group is only felt through the finite quotient

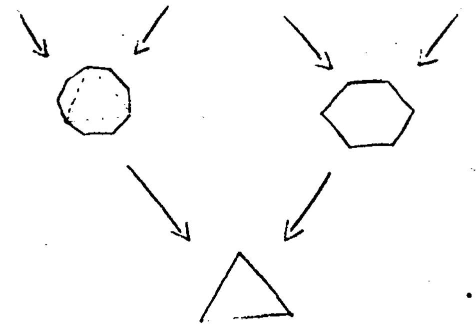
$$\mathrm{Gal}\,(L/\,\mathbb{Q})\,.$$

It follows that the inverse limit action gives homeomorphisms of the profinite homotopy sets

$$[\quad, \widehat{V}]\,.$$

iii) A consequence of this "finiteness at each level" phenomenon is that we can assume there is a cofinal collection of etale coverings each separately invariant under the Galois group. Thus we obtain a beautiful preview of the Galois action on the profinite completion –

an infinite scheme of complexes each with a finite symmetry and all these interrelated.



Action of $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ on Cohomology

$\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ acts on the multiplicative group of roots of unity contained in $\widetilde{\mathbb{Q}}$. The full group of automorphisms [23] of the roots of unity is the group of units $\widehat{\mathbb{Z}}^*$. The action of $\widehat{\mathbb{Z}}^*$ is naturally given by

$$\xi \mapsto \xi^a\,, \quad \xi^n = 1, \quad a \in \widehat{\mathbb{Z}}^*\,.$$

(ξ^a means ξ^k where k is any integer representing the "modulo n" residue of $a \in \widehat{\mathbb{Z}}^*$.)

This gives a canonical homomorphism

$$\begin{tikzcd}
\mathrm{Gal}\,(\widehat{\mathbb{Q}}/\,\mathbb{Q}) \arrow[r] \arrow[rr, bend left, "A"] & \left\{\begin{array}{c}\text{Galois group of the}\\ \text{field generated by}\\ \text{the roots of unity}\end{array}\right\} & \subseteq \widehat{\mathbb{Z}}^*\,.
\end{tikzcd}$$

Class field theory for $\mathbb{Q}$ says that A is onto and the kernel is the commutator subgroup of $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}, \mathbb{Q})$.

The Abelianization A occurs again in the etale homotopy type of $P^1(\mathbb{C})$.

PROPOSITION 5.3 *The induced action of the Galois group* $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}, \mathbb{Q})$ *on*

$$H_2\big((P^1(\mathbb{C}))\hat{\ };\mathbb{Z}\,\big) \cong \widehat{\mathbb{Z}}$$

is (after Abelianization) the natural action of $\widehat{\mathbb{Z}}^*$ *on* $\widehat{\mathbb{Z}}$.

PROOF: Consider the etale map $V \to P^1(\mathbb{C})$,

$$V = \mathbb{C}^* \xrightarrow[\mathbb{Z} \to \mathbb{Z}^n]{} \mathbb{C}^* \equiv \mathbb{C} - 0 \subseteq P^1(\mathbb{C})\,.$$

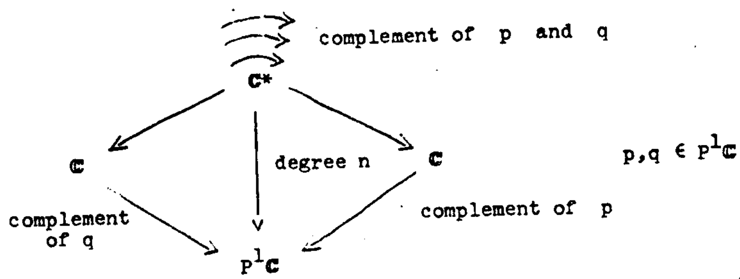
The covering group is generated by

$$z \mapsto \xi z \text{ where } \xi^n = 1\,.$$

Let σ be a field automorphism of $\widetilde{Q}$. We saw in the example of the preceding subsection that the action of σ corresponds to the action of σ on the roots of unity

$$\text{covering map } \xi \to \text{covering map } \xi\,.$$

We apply this calculation to the system of etale covers of $P^1(\mathbb{C})$



corresponding to the system of categories $\{C_n\}$

$$C_n = \{e \leftarrow \mathbb{Z}/n \to e\}\,.$$

The automorphism determined by σ on the entire system of etale covers of $P^1(\mathbb{C})$ is "homotopic" to an automorphism of the cofinal subsystem $\{C_n\}$. The above calculation[24] shows this automorphism of $\{C_n\}$ is

i) the identity on objects,

ii) sends $\xi \mapsto \xi^\sigma$, $\xi \in \mathbb{Z}/n$, for maps.

The realization or nerve of the covering category C_n is

$$\text{suspension } K(\mathbb{Z}/n, 1)$$

with 2-dimensional homology $\mathbb{Z}/n$.

The action of σ on the inverse limit of these homology groups

$$\lim_{\substack{\longleftarrow \\ n}} \mathbb{Z}/n \;=\; \lim_{\substack{\longleftarrow \\ n}} H_2(\text{suspension } K(\mathbb{Z}/n,1);\mathbb{Z}) \;=\; H_2\big((P^1(\mathbb{C}))\hat{}\,;\mathbb{Z}\big) \;=\; \widehat{\mathbb{Z}}$$

is built up from its action on the n^{th} roots of unity. This is what we wanted to prove.

Remark: One should notice the philosophically important point that the *algebraic variety* $P^1(\mathbb{C})$ does not have a canonical fundamental homology class. The algebraic automorphisms of $P^1(\mathbb{C})$ permute the possible generators freely.

The topology of the complex numbers determines a choice of orientation up to sign. The choice of a primitive fourth root of unity determines the sign.

Conversely, as we shall see in a later section (the Galois group in geometric topology) and in a later paper, choosing a K-theory orientation on a profinite homotopy type satisfying Poincaré Duality corresponds exactly to imposing a "homeomorphism class of nonsingular topological structures" on the homotopy type[25].

Corollary 5.4 *The induced action of* $G = \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ *on*

$$H^*\big((P^n(\mathbb{C}))\hat{\ };\widehat{\mathbb{Z}}\big) \cong \widehat{\mathbb{Z}}[x]/(x^{n+1}=0)$$

is (via Abelianization) the natural action of $\widehat{\mathbb{Z}}^*$*, namely*

$$x \mapsto ax\,,\ \ a \in \widehat{\mathbb{Z}}^*,\ \ x \textit{ any } 2 \textit{ dimensional generator}.$$

Proof: This is clear from Proposition 5.3 for $n = 1$, since

$$H^2\big((P^1(\mathbb{C}))\hat{\ };\widehat{\mathbb{Z}}\big) \;=\; \mathrm{Hom}\,\big(H_2(P^1(\mathbb{C}))\hat{\ };\widehat{\mathbb{Z}}\big)\,.$$

The general case follows from the naturality of the action and the inclusion

$$P^1(\mathbb{C}) \to P^n(\mathbb{C})\,.$$

Corollary 5.5 *G acts on*

$$H^*\big((BU_n)\hat{\ };\widehat{\mathbb{Z}}\big) \;=\; \widehat{\mathbb{Z}}[c_1, c_2, \ldots, c_n]$$

$$H^*\big((BU_n)^2;\widehat{\mathbb{Z}}\big) \;=\; \widehat{\mathbb{Z}}[c_1, c_2, \ldots, c_n]$$

by

$$c_i \mapsto a^i c_i\,,\ \ a \in \widehat{\mathbb{Z}}^*\,.$$

PROOF: For $n = 1$, this is true by passing to a direct limit in Corollary 5.4, using

$$P^\infty(\mathbb{C}) \;=\; \varinjlim_n P^n(\mathbb{C}) \;=\; BU_1\,.$$

For $n > 1$, use the naturality of the action, the map

$$\underbrace{P^\infty(\mathbb{C}) \times \cdots \times P^\infty(\mathbb{C})}_{n \text{ factors}} \to BU_n\,,$$

and the fact that the Chern classes go up to the elementary symmetric functions in the 2 dimensional generators of the factors of the product.

COROLLARY 5.6 *The action of* $G = \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ *on* $H^*(BSO_n;\widehat{\mathbb{Z}})$ *is determined by*

i) the action on $H^*(BSO_n;\mathbb{Z}\,/2)$ *is trivial.*

ii) If $g \in G$ *Abelianizes to* $a \in \widehat{\mathbb{Z}}^*$*,*

$$p_i \mapsto a^{2i} p_i \quad \dim p_i = 4i\,,$$

$$\chi \mapsto a^n \chi \quad \dim \chi = 2n\,,$$

where (modulo elements of order 2*)*

$$H^*\big((BSO_{2n+1})\hat{\ };\widehat{\mathbb{Z}}\big) \cong \widehat{\mathbb{Z}}[p_1, p_2, \ldots, p_n]\,,$$

$$H^*\big((BSO_{2n})\hat{\ };\widehat{\mathbb{Z}}\big) \cong \widehat{\mathbb{Z}}[p_1, p_2, \ldots, p_n, \chi]/(\chi^2 = p_n)\,.$$

PROOF: Any action on

$$H^*(P^n(\mathbb{R});\mathbb{Z}\,/2) \;=\; \mathbb{Z}\,/s[x]/(x^{n+1} = 0)$$

must be trivial. In the diagram of "algebraic" maps

$$\begin{tikzcd} & BSO_n \arrow[d] \\ \underbrace{P^\infty(\mathbb{R}) \times \cdots \times P^\infty(\mathbb{R})}_{n \text{ factors}} \arrow[r] & BO_n \end{tikzcd}$$

the horizontal map induces an injection and the vertical map a surjection on $\mathbb{Z}/2$ cohomology. This proves i).

For part ii) use the diagrams

$$BSO_{2n+1} \xrightarrow[\text{complexify}]{C_-} BU_{2n+1}\,,$$

$$\underbrace{BSO_2 \times \cdots \times BSO_2}_{n \text{ factors}} \xrightarrow{l} BSO_{2n} \xrightarrow[\text{complexify}]{C_+} BU_{2n}\,.$$

Modulo elements of order 2, C_-^* is onto, C_+^* is onto except for odd powers of χ, l^* is injective in dimension $2n$, so ii) follows from Corollary 5.5.

Remark: The nice action of Gal $(\widetilde{\mathbb{Q}}/\mathbb{Q})$ in the cohomology is related to some interesting questions and conjectures.

For each p there is a class of elements in Gal $(\widetilde{\mathbb{Q}}/\mathbb{Q})$ representing the Frobenius automorphism in characteristic p, $x \mapsto x^p$. Let $\mathscr{F}_p$ denote one of these.

In our Grassmannian case $\mathscr{F}_p$ acts on the q-adic part of the cohomology by

$$c_i \mapsto p^i c_i\,, \quad c_i \in H^{2i}(\quad, \widehat{\mathbb{Z}}_q)\,.$$

The famous Riemann Hypothesis in characteristic p – the remaining unproved Weil conjecture – asserts that for a large class of varieties[26] $\mathscr{F}_p$ acts on the cohomology in a "similar fashion" –

the action of $\mathscr{F}_p$ on the q-adic cohomology in dimension k has eigenvalues (if we embed $\widehat{\mathbb{Z}}_q$ in an algebraically closed field) which

a) lie in the integers of a finite extension of $\mathbb{Q}$,

b) are independent of $q \neq p$,

c) have absolute value $p^{k/2}$.

In our Grassmannian case the cohomology only appears in even dimensions, the eigenvalues are rational integers (p^i if $k = 2i$).

This simplification comes from the fact that the cohomology of the Grassmannians is generated by algebraic cycles (also in characteristic p) – the Shubert subvarieties,

$$S^k \overset{i}{\hookrightarrow} G\,. \quad (k = \text{complex dimension})$$

Looking at homology and naturality then implies

$$\begin{aligned}\mathscr{F}_p\,(i_* S^k) &= i_*\,\mathscr{F}_p\,S^k \ \text{ in homology}\\ &= i_* p^k S^k\\ &= p^k i_* S^k\end{aligned}$$

(the second equation follows since the top dimensional homology group is cyclic for S^k – the value of the eigenvalue can be checked in the neighborhood of a point).

Tate conjectures a beautiful converse to this situation – roughly, eigenvectors of the action of $\mathscr{F}_p$ in the $\widehat{\mathbb{Z}}_q$-cohomology for all q correspond to algebraic subvarieties (in characteristic p).

The action of $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ on K-theory, the Adams Operations, and the "linear" Adams Conjecture

$\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ also acts on the homotopy types

$$\begin{aligned}BU\hat{}\ &= \lim_{n\to\infty}\,(BU_n)\hat{} \quad \text{(infinite mapping cylinder)}\\ BO\hat{}\ &= \lim_{n\to\infty}\,(BO_n)\hat{}\end{aligned}$$

since it acts at each level n.

These homotopy types classify the group theoretic profinite completions of reduced K-theory,

$$\begin{aligned}\widetilde{K}U(X)\hat{}\ &\cong [X, BU\hat{}\,]\\ \widetilde{K}O(X)\hat{}\ &\cong [X, BO\hat{}\,]\end{aligned}$$

for X a finite dimensional complex.

For X infinite dimensional we take limits of both sides, e.g.

$$\widetilde{K}U(X)\hat{}\ \equiv \lim_{\substack{\longleftarrow\\ \text{finite subcomplexes}\\ \text{of } X}} \widetilde{K}U(\text{finite subcomplex})\hat{}\ \cong [X, BU\hat{}\,]\,.$$

Thus for any space X we have an action of $\mathrm{Gal}(\widetilde{\mathbb{Q}}/\mathbb{Q})$ on the profinite K-theory (real or complex)

$$K(X)\hat{}\,.$$

In the original K-theories, we have the beautiful operations of Adams,

$$k \in \mathbb{Z} \qquad KU(X) \xrightarrow{\psi_k} KU(X)\,,$$

$$KO(X) \xrightarrow{\psi^k} KO(X)\,,$$

In either the real or the complex case. The Adams operations satisfy

i) ψ^k (line bundle η) $= \eta^k = \underbrace{\eta \otimes \cdots \otimes \eta}_{k \text{ factors}}$,

ii) ψ^k is an endomorphism of the ring $K(X)$,

iii) $\psi^k \circ \psi^l = \psi^{kl}$.

ψ^k is defined by forming the Newton polynomial in the exterior powers of a vector bundle. For example,

$$\begin{aligned}
&\psi^1 V = \Lambda V = V \\
&\psi^2 V = V \otimes V - 2\Lambda^2 V \\
&\quad \vdots
\end{aligned}$$

The Adams operations determine operations in the profinite K-theory,

$$K(X)\hat{} \xrightarrow{(\psi^k)\hat{}} K(X)\hat{}$$

for finite complexes, and

$$\varprojlim_{\alpha} K(X_\alpha)\hat{} \xrightarrow{\varprojlim (\psi^k)\hat{}} \varprojlim_{\alpha} K(X_\alpha)\hat{} \quad X_\alpha \text{ finite}$$

for infinite complexes.

We wish to distinguish between the "isomorphic part" and the "nilpotent part" of the profinite Adams operations.

$K(X)\hat{}$ and the Adams operations naturally factor

$$\prod_p K(X)\hat{}_p \xrightarrow{\prod_p (\psi^k)\hat{}_p} \prod_p K(X)\hat{}_p$$

and $(\psi^k)\hat{}_p$ is an isomorphism iff k is prime to p.

If k is divisible by p, redefine $(\psi^k)\hat{}_p$ to be the identity. We obtain

$$K(X)\hat{} \xrightarrow[\cong]{\psi^k} K(X)\hat{}\,,$$

the "isomorphic part" of the Adams operations.

We note that $(\psi^k)\hat{}_p$ (before redefinition) was topologically nilpotent, the powers of $(\psi^k)\hat{}_p$ converge to the zero operation on the reduced group

$$\widetilde{K}(X)\hat{}\,, \ \text{ for } k \equiv 0 \pmod p\,.$$

One of our main desires is to study the ubiquitous nature of the "isomorphic part" of the Adams operations.

For example, recall the Abelianization homomorphism

$$G \ = \ \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q}) \to \widetilde{\mathbb{Z}}^*$$

obtained by letting G act on the roots of unity.

THEOREM 5.7 *The natural action of* $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ *in profinite* K*-theory* $K(X)\hat{}$ *reduces (via Abelianization) to an action of* $\widehat{\mathbb{Z}}^*$.

The "isomorphic part" of the Adams operation ψ^k

$$K(X)\hat{} \xrightarrow[\cong]{\psi^k} K(X)\hat{}$$

corresponds to the automorphism induced by

$$\mu_k \ = \ \prod_{(k,p)=1} (k) \prod_{p|k} (1) \ \in \ \prod_p \widehat{\mathbb{Z}}_p^* = \widehat{\mathbb{Z}}^*$$

Thus – if we think in terms of homotopy equivalences of the "algebraic varieties"

$$BU = \varinjlim_{n,k} G_{n,k}(\mathbb{C})\,, \quad BO = \varinjlim_{n,k} \text{“}G_{n,k}(\mathbb{R})\text{”}$$

we see that the "isomorphic part" of the Adams operation is compatible with the natural action of

$$\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$$

in the category of profinite homotopy types and maps coming from algebraic varieties defined over $\mathbb{Q}$*.*

Remark. We must restrict ourselves to the "isomorphic part" of the Adams operation to make this "algebraic" extension to the Adams operations to the Grassmannians and other varieties. For example,

Proposition 5.8 ψ^2 *cannot be defined on the* 2*-adic completion*

$$G_{2,n}(\mathbb{C})\hat{}_2\,,\ \ n \textit{ large}\,,$$

so that it is compatible with the nilpotent operation

$$\psi^2 \textit{ on } (BU)\hat{}_2\,.$$

We give the proof below. It turns out the proof suggests a way to construct some interesting new maps of quaternionic projective space.

We recall that an element γ in $K(X)\hat{}$ has a stable fibre homotopy type, for example in the real case the composition

$$X \to BO\hat{} \xrightarrow[\substack{\text{natural}\\ \text{map}}]{} BG\hat{} \cong BG\,.$$

Corollary 5.9 (Adams Conjecture). *For any space* X*, the stable fibre homotopy type of an element in*

$$K(X)\hat{} \quad \textit{(real or complex)}$$

is invariant under the action of the Galois group

$$\widehat{\mathbb{Z}}^* \ = \ \textit{Abelianized } \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$$

and therefore under the action of the "isomorphic part" of the Adams operations.

Proof of Corollary 5.9: We have filtered the homotopy equivalences of

$$(BU)\hat{} \text{ or } (BSO)\hat{}$$

corresponding to the elements of the Galois group by homotopy equivalences of

$$(BU_n)\hat{}\ \text{or}\ (BSO_n)\hat{}$$

(corresponding to elements in the Galois group).

By the inertia lemma of section 4 applied to complete spherical fibration theory $\{\widehat{S}^1, \widehat{S}^2, \dots, \widehat{S}^n, \dots\}$ we have the homotopy commutative diagrams

$$\begin{tikzcd} BU\hat{} \arrow[rr, "\alpha"] \arrow[dr, "n"'] & & BU\hat{} \arrow[dl] \\ & B\hat{}_\infty & \end{tikzcd} \qquad \begin{tikzcd} BO\hat{} \arrow[rr, "\alpha"] \arrow[dr] & & BO\hat{} \arrow[dl] \\ & B\hat{}_\infty & \end{tikzcd}$$

where $\alpha \in \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\mathbb{Q})$ and $B\hat{}_\infty$ classifies the "stable theory" of profinite spherical fibrations. (See section 4). But by Theorem 4.1

$$B\hat{}_\infty \cong K(\widehat{\mathbb{Z}}^*, 1) \times BSG\,.$$

Thus

$$\begin{tikzcd} BU\hat{} \arrow[rr, "\alpha"] \arrow[dr] & & BU\hat{} \arrow[dl] \\ & BSG & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} BSO\hat{} \arrow[rr, "\alpha"] \arrow[dr] & & BSO\hat{} \arrow[dl] \\ & BSG & \end{tikzcd} \quad \text{commute.}$$

This completes the proof for the complex case.

For the real case we only have to add the remark that the natural map

$$BO\hat{} \to BG$$

canonically factors

$$(BSO\hat{} \to BSG) \times (\mathbb{R}\,P^\infty \xrightarrow{\text{identity}} \mathbb{R}\,P^\infty)\,,$$

and α restricted to $\mathbb{R}\,P^\infty$ is the identity. Q. E. D.

Note: If we apply this to a finite complex X, recall that

$$[X, BG] \;\cong\; \prod_l [X(BG)\hat{}_l] \;\cong\; \text{product of finite } l\text{-groups}\,.$$

Thus $\psi^k x - x$ determines an element in

$$[X, BG]$$

which is concentrated in the p-components for p dividing k. This gives the conjectured assertion of Adams – "the element

$$k^{\text{high power}}(\psi^k x - x),\ \ x \in K(X)$$

determines a trivial element in

$$J(X) \subseteq [X, BG]\,.\text{''}$$

NOTE: The proof of the inertia lemma simplifies in these vector bundle cases – for example, the stable bundles

$$BO \xrightarrow{\substack{\text{natural}\\ \text{map}}} BG$$

is "exactly intrinsic" to the filtration $\{BO_n\}$. Namely,

$$\text{fibre} \to BO_{n-1} \to BO_n$$

is fibre homotopy equivalent to canonical spherical fibration over BO_n

$$S^{n-1} \to \begin{Bmatrix}\text{total space of}\\ \text{spherical}\\ \text{fibration}\end{Bmatrix} \to BO_n\,.$$

Thus the skeletal approximations used in that proof are not needed.

Furthermore, the proof then gives a much stronger result than the "classical Adams Conjecture". We obtain homotopy commutativities at each unstable level

$$\begin{tikzcd}
(BO_n)\hat{} \arrow[rr, "\alpha"] \arrow[dr] & & (BO_n)\hat{} \arrow[dl] \\
& \hat{B_n} &
\end{tikzcd} \qquad \alpha \in \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})\,.$$

In the next subsection we study certain α's by bringing in certain primes. This gives Frobenius operations on various p-adic components of the *profinite theory of n-dimensional vector bundles*

$$[\quad , (BO_n)\hat{}\,]\,.$$

In a later section we relate the action of the Galois group on n-dimensional profinite vector bundle theory to Galois actions on the analogous piecewise linear theory, topological theory, and the oriented spherical fibration theory. These are the Adams phenomena whose study inspired the somewhat extensive homotopy discussion of the earlier sections.

This discussion also yields the more natural statement of Corollary 5.9.

However, the Adams Conjecture for odd primes[27] was proved much earlier (August 1967) in Adams formulation using only the existence of an algebraically defined "space-like" object having the mod n cohomology of the complex Grassmannians plus an awkward cohomology argument with these space-like objects.

The use of etale homotopy at this time was inspired by Quillen – who was rumored to have an outline proof of the complex Adams conjecture "using algebraic geometry".

The "2-adic Adams Conjecture" came much later (January 1970) when a psychological block about considering non-projective varieties was removed [28]. Thus

$$G_{n,k}(\mathbb{R}) \underset{\substack{\text{homotopy}\\ \text{equivalent}}}{\sim} O_{n+k}(\mathbb{C})/O_n(\mathbb{C}) \times O_k(\mathbb{C})$$

could be treated.

The final subsection describes an intermediate attempt to treat real varieties directly. This was to be the first half of a hoped for proof of the 2-adic Adams Conjecture, the second half (required by lack of Galois symmetry) would depend on a step like that used in the next subsection and the simple structure of the cohomology of the commutator subgroup of Gal $(\widetilde{\mathbb{Q}}/\mathbb{Q})$.

Note (Higher order operations in K-theory) As in ordinary cohomology one can obtain *secondary operations* in K-theory by measuring the failure on the "cocycle level" of relations between operations on the "cohomology class level".

Thus the relation

$$\psi^2 \circ \psi^3 \;=\; \psi^3 \circ \psi^2$$

leads to a secondary operation in K-theory (Anderson). If we think of maps into $BU\hat{}$ as cocycles we have a representative

$$\varinjlim_{n,k} \big(\text{the etale type of } G_{n,k}(\mathbb{C})\big)$$

on which $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ acts by homeomorphisms.

Thus one can imagine that secondary operations are defined on the commutator subgroup – the elements in here correspond to the homotopies giving the relations among Adams operations.

Furthermore, if in some space like $K(\pi, 1)$, π finite, the K-theory is generated by "bundles defined over the field, $A_{\mathbb{Q}}$, "$\mathbb{Q}$ with the roots of unity adjoined" the secondary operations *should give zero*. For example

$$\psi^2 \circ \psi^3 \;=\; \psi^3 \circ \psi^2$$

holds on the "level of cocycles" for a generating set of bundles over $K(\pi, 1)$.

(We recall that for $K(\pi, 1)$, π finite, the homomorphisms of π into $\mathrm{GL}\,(n, \mathbb{C})$ which factor through $\mathrm{GL}\,(n, A_{\mathbb{Q}})$ generate the K-theory of $K(\pi, 1)$ topologically.)

Proof of 5.7: Let B denote BSG or BU.

Claim: two maps

$$\widehat{B} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \widehat{B}$$

are homotopic iff they "agree" on the "cohomology groups", $H^*(\widehat{B}; \widehat{\mathbb{Z}}) \otimes \mathbb{Q}$.

Assuming this we can deduce the theorem –

first the calculations of the preceding subsection (Corollary 5.5 and 5.6) show that G is acting on cohomology through its Abelianization.

Thus in the complex case we are done –

$\widehat{\mathbb{Z}}^*$ acts on complex K-theory and "$\mu_k \in \widehat{\mathbb{Z}}^*$" and "isomorphic part of ψ^k" agree on the cohomology of $BU\hat{}$ (see note below). Thus the action of μ_k agrees with that of ψ^k (see note below).

In the real case, recall that BO naturally splits

$$BO \cong BSO \times \mathbb{R}\,P^\infty$$

using the "algebraic maps"

$$\mathbb{R}P^\infty \cong BO(1,\mathbb{C}) \to \varinjlim_n BO(n,\mathbb{C}) \cong BO\,,$$
$$BSO \cong \varinjlim_n BSO(n,\mathbb{C}) \to \varinjlim_n BO(n,\mathbb{C}) \cong BO$$

and the "algebraic" Whitney sum operation in BO,

$$\varinjlim (BO_n \times BO_n \to BO_{2n})\,.$$

Thus the homotopy equivalences of $BO\hat{}$ corresponding to the elements in the Galois group naturally split.

The same is true for ψ^k. This uses

$$\psi^k \text{ (line bundle)} = \text{ line bundle}\,,$$

$$\psi^k \text{ (oriented bundle)} = \text{ oriented bundle}\,,$$

$$\psi^k \text{ is additive}\,.$$

A homotopy equivalence of $\mathbb{R}P^\infty$ must be homotopic to the identity so we are reduced to studying maps

$$BSO\hat{} \to BSO\hat{}\,.$$

Our claim and the previous calculations then show that $\widehat{\mathbb{Z}}^*$ acts in real K-theory (the elements of order two all act trivially,

$$t_p \in \mathbb{Z}/p-1 \oplus \widehat{\mathbb{Z}}_p = \widehat{\mathbb{Z}}_p^*$$
$$t_2 \in \mathbb{Z}/2 \oplus \widehat{\mathbb{Z}}_2 = \widehat{\mathbb{Z}}_2^*)$$

and ψ^k corresponds to the action of μ_k.

Note: One can repeat the arguments of the previous subsection to see the effect of ψ^k on the cohomology – the properties

$$\psi^k(\eta) = \eta^k \quad \eta \text{ line bundle}\,,$$

$$\psi^k \text{ additive}\,,$$

$$\psi^k \text{ commutes with complexification}$$

are all that are needed to do this.

We are reduced to proving the claim.

For the case $BU\hat{}$ or $BSO\hat{}$ away from 2 a simple obstruction theory argument proves this. For completeness we must take another tack. We concentrate on the case $B = BSO$.

It follows from work of Anderson and Atiyah that the group

$$[B, B]$$

i) is an inverse limit over the finite subcomplexes B_α of the finitely generated groups $[B_\alpha, B]$.

ii) There is a "cohomology injection"

$$[B, B] \xrightarrow{ph} \prod_{i=1} H^{4i}(B; \mathbb{Q}) \,.$$

We claim that B can be replaced by $\widehat{B}$ and ii) is still true.

Let T_α denote the (finite) torsion subgroup of $[B_\alpha, B]$, then

$$0 \to T_\alpha \to [B_\alpha, B] \xrightarrow{(ph)_\alpha} \prod_{i=1} H^{4i}(B_\alpha; \mathbb{Q})$$

is exact. If we pass to the inverse limit $(ph)_\alpha$ becomes an injection so

$$\varprojlim_\alpha T_\alpha \;=\; 0 \,.$$

So we tensor with $\widehat{\mathbb{Z}}$ and pass to the limit again. Then since

a) $\varprojlim_\alpha (T_\alpha \otimes \widehat{\mathbb{Z}}) \cong \varprojlim_\alpha T_\alpha = 0 \,,$

b) $$\begin{aligned} \varprojlim_\alpha ([B_\alpha, B] \otimes \widehat{\mathbb{Z}}) &\cong \varprojlim_\alpha [B_\alpha, \widehat{B}] \\ &\cong \varprojlim_\alpha [\widehat{B}_\alpha, \widehat{B}] \\ &\cong [\widehat{B}, \widehat{B}] \,, \end{aligned}$$

c) if we choose B_α to be the 4α skeleton of B it is clear that

$$\begin{aligned} \varprojlim_\alpha \prod_{i=0}^{\infty} H^{4i}(B_\alpha; \mathbb{Q}) \otimes \widehat{\mathbb{Z}} &\cong \varprojlim_\alpha \prod_{i=0}^{\infty} H^{4i}(B_\alpha) \otimes \mathbb{Q} \otimes \widehat{\mathbb{Z}} \\ &\cong \prod_{i=1}^{\infty} H^{4i}(\widehat{B}; \widehat{\mathbb{Z}}) \otimes \mathbb{Q} \end{aligned}$$

we obtain the "completed Pontrjagin character injection",

$$0 \to [\widehat{B}, \widehat{B}] \xrightarrow{ph \otimes \widehat{\mathbb{Z}}} \prod_{i=0}^{\infty} H^{4i}(\widehat{B}; \widehat{\mathbb{Z}}) \otimes \mathbb{Q} \, .$$

This proves our claim and the theorem.

Proof of Proposition 5.8. We first note a construction for "Adams operations on quaternionic line bundles and complex 2-plane bundles".

Suppose "ψ^p is defined" in $(BU_2)\hat{}_p$.[29]

Now ψ^p is defined on $(BU_2)\hat{}_p$ $q \neq p$ by choosing some lifting of $\mu_p \in \widehat{\mathbb{Z}}^*$ to the Galois group $\mathrm{Gal}\,(\widehat{\mathbb{Q}}/\mathbb{Q})$. Putting these ψ^p together gives ψ^p on

$$(BU_2)\hat{} = \prod_p (BU_2)\hat{}_p \, .$$

ψ^p can be easily defined on the localization,

$$(BU_2)_0 \;=\; K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \, .$$

Thus ψ^p is defined on BU_2, the fibre product of $(BU_2)_0$, $(BU_2)\hat{}$, and $(B\widehat{U}_2)_0 \cong K(\mathbb{Q} \otimes \widehat{\mathbb{Z}}, 2) \times K(\mathbb{Q} \otimes \widehat{\mathbb{Z}}, 4)$.

This operation has to be compatible (by a cohomology argument) with the map $BU_2 \xrightarrow{c_1} BU_1 \cong K(\mathbb{Z}, 2)$. Thus ψ^p is defined on the fibre,

$$BSU_2 \cong BS^3 \cong \text{"infinite quaternionic projective space"} \, .$$

For $p = 2$ we obtain a contradiction, there is not even a map of the quaternionic plane extending the degree four map of

$$\mathbb{H}P^1 \cong S^4 \, .$$[30]

Thus ψ^2 cannot be defined on a very large skeleton of $(BU_2)\hat{}_2$. Since BU_2 is approximated by $G_{2,n}(\mathbb{C})$ this proves the proposition.

For $p > 2$ we obtain the

Corollary 5.10 *For self maps of the infinite quaternionic projective space the possible degrees on $S^4 \cong \mathbb{H}P^1$ are precisely zero and the odd squares.*

Proof: The restriction on the degree was established by various workers.

(I. Bernstein proved the degree is a square using complex K-theory, R. Stong and L. Smith reproved this using Steenrod operations, G. Cooke proved the degree was zero or an odd square using real K-theory.)

We provide the maps. From the above we see that it suffices to find

$$\psi^p \text{ on } (BU_2)\hat{}_p\,, \quad p > 2\,.$$

This follows by looking at the normalizer of the torus in U_2,

$$N : 1 \to S^1 \times S^1 \to N \to \mathbb{Z}/2 \to 1\,.$$

$\mathbb{Z}/2$ acts by the flip of coordinates in $S^1 \times S^1$.

Consider the diagram of classifying spaces

$$\begin{tikzcd} BS^1 \times BS^1 \arrow[r] & BN \arrow[r] \arrow[d, "j"] & \mathbb{R}P^\infty \\ & BU_2 & \end{tikzcd} .$$

The horizontal fibration shows for $p > 2$

$$H^*(BN; \mathbb{Z}/p) \cong \text{ invariant cohomology of } BS^1 \times BS^1\,.$$

Thus j is an isomorphism of cohomology mod p. Sine $\pi_1 BN = \mathbb{Z}/2$, j becomes an isomorphism upon p-adic completion,

$$(BU_2)\hat{}_p \cong (BN)\hat{}_p\,.$$

Now N has a natural endomorphism induced by raising to the k^{th} power on the torus. Thus BN has self maps ψ^k for all k. These give ψ^k for all k on $(BU_2)\hat{}_p$, in particular we have ψ^p.

The proof actually shows

$$(BU_n)\hat{}_p \cong \big(B(\text{normalizer of torus})\big)\hat{}_p\,, \ \ p > n\,.$$

So we have

Corollary 5.11 *ψ^p exists in BU_n and BSU_n for $n < p$.*

Remark: For $n > 1$, these give the first examples of maps of classifying spaces of compact connected Lie groups which are not induced by homomorphisms of the Lie groups. ($SU_2 = S^3$ is fairly easy to check). This was a question raised by P. Baum.

Conjecture: *ψ^p does not exist in BU_p.*

Remark: It is tempting to note a further point. The equivalence

$$(BU_n)\hat{}_p \cong \big(B(\text{normalizer } T^n)\big)\hat{}_p$$

can be combined with the spherical topological groups of section 4 to construct exotic "p-adic analogues of U_n".

Namely

$$B(\text{normalizer } T^n) \;=\; \big(\prod_{i=1}^{\infty} BS^1 \times E\Sigma_n\big)/\Sigma_n$$

where Σ_n is the symmetric group of degree n and $E\Sigma_n$ is a contractible space on which Σ_n acts freely.

So consider for λ dividing $p-1$,

$$U(n,\lambda) \;=\; \Omega\Big(\prod_{i=1}^{n} B(S^{2\lambda-1})\hat{}_p \times E\Sigma_n/\Sigma_n\Big)\hat{}_p\,, \;\; n < p\,.$$

For $\lambda = 1$ we get the p-adic part of unitary groups up to degree $p-1$. For $\lambda = 2$ we get the p-adic part of the symplectic groups up to degree $p-1$. For λ another divisor we obtain "finite dimensional" groups (p-adic). (All this in the homotopy category.)

The Groups generated by Frobenius elements acting on the Finite Grassmannians

We saw in the previous subsection that the action of $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\mathbb{Q})$ was Abelian in the profinite K-theory – this action being densely populated by the "isomorphic part of the Adams operations".

Thus we have a homomorphism

$$\widehat{\mathbb{Z}}^* \xrightarrow[\psi]{\left\{\begin{array}{c}\text{closure of}\\ \text{“Adams isomorphisms”}\end{array}\right\}} \pi_0 \left\{\begin{array}{c}\text{space of}\\ \text{homotopy equivalences}\\ \text{of } B\end{array}\right\} \equiv \pi(B)$$

$$B = \begin{cases} BO\hat{\ } = \varinjlim_{n,k} G_{n,k}(\mathbb{R})\hat{\ } \\ BU\hat{\ } = \varinjlim_{n,k} G_{n,k}(\mathbb{C})\hat{\ }\,, \end{cases}$$

so that

$$\begin{tikzcd} \mathrm{Gal}\,(\widehat{\mathbb{Q}}/\,\mathbb{Q}) \arrow[rr, "\begin{array}{c}\text{Galois}\\ \text{action}\end{array}"] \arrow[dr, "\text{Abelianization}"'] & & \pi(B) \\ & \widehat{\mathbb{Z}}^* \arrow[ur, "\begin{array}{c}\text{“closure of Adams}\\ \text{isomorphisms”}\end{array}"'] & \end{tikzcd}$$

commutes.

This Galois symmetry in the infinite Grassmannians is compatible with the Galois symmetry in the “finite Grassmannians” (k and/or n finite). [31]

So we have compatible “symmetry homomorphisms”

$$\begin{tikzcd} \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q}) \arrow[r, "\text{Galois}"] \arrow[dr, "\text{Galois}"'] & \pi\big(G_{n,k}(\mathbb{R})\hat{\ }\big) \\ & \pi\big(G_{n,k}(\mathbb{C})\hat{\ }\big)\,. \end{tikzcd}$$

The action on cohomology Abelianizes as we have seen so it is natural to ask to what extent the “homotopy representation” of $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ Abelianizes.

We can further ask (assuming some Abelianization takes place) whether or not there are natural elements (like the Adams operations in the infinite Grassmannian case) generating the action.

We note for the first question that this is a homotopy problem of a different order of magnitude than that presented by the calculations of the last subsection. There is no real theory[32] for proving the existence of homotopies, between maps into such spaces as finite

Grassmannians,

$$X \underset{g}{\overset{f}{\rightrightarrows}} (\text{finite Grassmannian})\hat{}\ .$$

One has to construct the homotopies or nothing. In this case the first part of the construction depends on arithmetic properties of the equations defining the Grassmannians. The second part comes essentially from the contractibility of

$$\big(K(\pi,1)\big)\hat{}_p$$

where π is a non-p group.

Thus these considerations must be sifted through the sieve of primes. We do this by considering only one p-adic component of the profinite completion of the finite Grassmannians.

We find an infinite number of groups of symmetries (one for each prime not equal to p)

$$\begin{tikzcd} U_q \arrow[r, "\substack{\text{Frobenius map}\\ \text{at } q}"] \arrow[dr] & \pi(p\text{-adic finite Grassmannian}) \\ & \widehat{\mathbb{Z}}^* \end{tikzcd} .$$

U_q is the subgroup of $\widehat{\mathbb{Z}}_p^*$ generated by the prime q.

Each of these Frobenius actions is compatible with the single action of the p-adic component of the Abelianized Galois group $\widehat{\mathbb{Z}}^*$, $\widehat{\mathbb{Z}}_p^*$ on the p-adic component of the infinite Grassmannians,

$$\left\{\begin{matrix}\text{Galois group of all}\\ p^{n\text{-th}}\text{ roots of unity}\\ n=1,2,3,\ldots\end{matrix}\right\} = \widehat{\mathbb{Z}}_p^* \xrightarrow{\substack{\text{Galois}\\ \text{action}}} \pi(p\text{-adic infinite Grassmannian})\,.$$

However at the finite level we don't know if the different Frobenius actions commute or are compatible – say if the powers of one prime q converge p-adically to another prime l does

$$(q\text{-Frobenius})^n \to (l\text{-Frobenius})$$

in the profinite group

$$\pi(p\text{-adic finite Grassmannian})\,?$$

It seems an interesting question – this compatibility of the different Frobenius operations on the finite Grassmannians. The group they generate might be anywhere between an infinite free product of $\widehat{\mathbb{Z}}_p^*$'s and one $\widehat{\mathbb{Z}}_p^*$. The answer depends on the geometric question of finding sufficiently many etale covers of these Grassmannians to describe the etale homotopy. Then one examines the field extensions of $\mathbb{Q}$ required to define these, their Galois groups, and the covering groups to discover commutativity or non-commutativity relations between the different homotopy Frobenius operations. The "intuitive" examples above[33] and the "nilpotent" restriction of the last section imply that "non-Abelian groups have to enter somewhere".

The simplest non-commutative Galois group G that might enter into this question[34] is that for the field extension of $\mathbb{Q}$ obtained by adjoining the "p^{th} roots of p", the twisted extension

$$1 \to F_p \to G \to F_p^* \to 1$$

F_p the prime field.

We close this summary with the remark that the arithmetic of the defining equations of $O(n, \mathbb{C})$ (as we understand it) forces us to omit the case $q = 2$, p odd for some of the finite (odd) real Grassmannians.

This omission provides a good illustration of the arithmetic involved. We describe the situation in Addendum 1.

The essential ingredients of the theorem are described in Addendum 2.

We describe the theorem.

The Galois group $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ contains infinitely many (conjugacy classes of) subgroups

$$\text{“decomposition group”} \;=\; G_q \;\text{“}\subseteq\text{”}\; \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})\,.$$

G_q is constructed from an algebraic closure of the q-adic completion of $\mathbb{Q}$. G_q has as quotient the Galois group of the algebraic closure of F_q,

$$G_q \xrightarrow{\text{“reduction mod } p\text{”}} \mathrm{Gal}\,(\widetilde{F}_q/F_q) \cong \widehat{\mathbb{Z}}$$

with canonical generator the Frobenius at q.

There is a natural (exponential) map

$$\widehat{\mathbb{Z}} \xrightarrow[\text{Frobenius} \mapsto q]{\exp} \widehat{\mathbb{Z}}_p^*$$

with image $U_q \subseteq \widehat{\mathbb{Z}}_p^*$.

THEOREM 5.12 *For each prime* q[35] *not equal to* p *there is a natural Frobenius diagram*

$$\begin{tikzcd}
\mathrm{Gal}\,(\widehat{\mathbb{Q}}/\,\mathbb{Q}) \arrow[rr, "\substack{\textit{Galois}\\ \textit{action}}"] & & \pi\begin{pmatrix} \textit{p-adic finite} \\ \textit{Grassmannian} \end{pmatrix} \\
G_q \arrow[u, "\substack{\textit{conjugacy}\\ \textit{class of}\\ \textit{inclusions}}"] \arrow[r, "\substack{\textit{``reduction}\\ \textit{mod q''}}"'] & \widehat{\mathbb{Z}} \arrow[r, "\textit{Frobenius} \mapsto q"] & U_q \subseteq \widehat{\mathbb{Z}}_p^* \arrow[u, "\textit{to be constructed}"']
\end{tikzcd}$$

provides a partial Abelianization of the Galois action.

The image of the canonical generator Frobenius in $\widehat{\mathbb{Z}}$ gives a canonical Frobenius (or Adams) operation ψ^q, on the p-adic theory of vector bundles with a fixed fibre dimension and codimension.

CAUTION: As discussed above we do not know that

$$\psi^q \circ \psi^l \;=\; \psi^l \circ \psi^q\,.$$

However we are saying that the subgroup generated by a single ψ^q is 'correct', $U_q \subseteq \widehat{\mathbb{Z}}_p^*$.

PROOF: First one constructs an algebraic closure of the q-adic numbers $\mathbb{Q}_q$ which has Galois group G_q and contains (non-canonically) the algebraic closure $\widetilde{\mathbb{Q}}$ of $\mathbb{Q}$. Then G_q acts on $\mathbb{Q}$ and we have a map

$$G_q \to G = \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})\,.$$

This map is defined up to conjugation and is an injection. G_q also acts on the integers in this extension of $\mathbb{Q}_q$ and their reductions modulo q which make up $\widetilde{F}_q$ the algebraic closure of the prime field. Thus we have a map

$$G_q \to \widehat{\mathbb{Z}} = \mathrm{Gal}\,(\widetilde{F}_q/F_q)\,.$$

The $\widehat{\mathbb{Z}}$ has a canonical generator, the Frobenius for the prime q; and the map is a surjection.[36]

These two maps are part of the commutative diagram

$$
(I)\qquad
\begin{tikzcd}
 & G \arrow[rr, "\text{Abelianization}"] & & \widehat{\mathbb{Z}}^* \arrow[d, "\substack{p\text{-adic}\\ \text{projection}}"] \\
G_q \arrow[ur, "\text{into}"] \arrow[dr, "\text{onto}"'] & & & \\
 & \widehat{\mathbb{Z}} \arrow[rr, "\text{Frobenius} \mapsto q"] & & \mathbb{Z}_p^*
\end{tikzcd}
$$

Now Artin and Mazur show under certain circumstances that the p-adic part of the etale homotopy type of the variety can be constructed just from the variety reduced modulo q, $q \neq p$. For example, for complex Grassmannians it is enough to observe that the definition of "the Grassmannian in characteristic q" gives a non-singular variety of the same dimension –

$$\mathrm{GL}\,(n+k, \widetilde{F}_q)/\,\mathrm{GL}\,(n, \widetilde{F}_q) \times \mathrm{GL}\,(k, \widetilde{F}_q)$$

is non-singular and has dimension $2nk$.

In this (good reduction mod q) case it follows from Artin-Mazur that the action of $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ restricted to G_q factors through an action of $\widehat{\mathbb{Z}}$ generated by the Frobenius at q,

$$
\begin{tikzcd}
 & G \arrow[dr, "\text{Galois}"] & \\
G_q \arrow[ur] \arrow[dr] & & \pi \begin{pmatrix} p\text{-adic finite} \\ \text{Grassmannian} \end{pmatrix} \\
 & \widehat{\mathbb{Z}} \arrow[ur, "\text{Frobenius}"'] &
\end{tikzcd}
.
$$

The action of $\widehat{\mathbb{Z}}$ on cohomology (because of diagram (I) factors through $\widehat{\mathbb{Z}}_p^*$,

$$\widehat{\mathbb{Z}} \xrightarrow[\substack{\text{Frobenius}\\ \text{for } q}\ \to\ q]{r} \widehat{\mathbb{Z}}_p^*\,.$$

We claim that the action of $\pi = \text{kernel } r$ is trivial in the p-adic homotopy type (denote it by X). Then the natural map

$$G_q \to \widehat{\mathbb{Z}} \xrightarrow[\text{Frobenius}]{} \pi(X)$$

factors through $U_q \subseteq \widehat{\mathbb{Z}}_p^*$,

$$\begin{array}{ccccc} G_q & \longrightarrow & U_q & \xrightarrow{\text{Frobenius}} & \pi(X) \\ & & \cap\!\!| & & \\ & & \widehat{\mathbb{Z}}_p^* & & \end{array}$$

as desired.

Now the action of $\widehat{\mathbb{Z}}$ (and therefore π) is built up from an inverse system of actions on simply connected nerves[37] on which π is acting via cellular homeomorphisms.

In each nerve N_α the action factors through the action of a finite Galois quotient group π_α. Let E_α denote the universal cover of $K(\pi_\alpha, 1)$ and form the new inverse system

$$\{N'_\alpha\} \;=\; \{(N_\alpha \times E_\alpha)pi_\alpha\}\,.$$

We have a fibration sequence

$$N_\alpha \to N'_\alpha \to K(\pi_\alpha, 1)$$

of spaces with finite homotopy groups. We can form the homotopy theoretical inverse limit as in section 3 to obtain the fibration sequence

$$X \to X' \xrightarrow{\pi} K(\pi, 1)\,.$$

The action of π on the mod p cohomology of the fibre is trivial by construction. Also, the mod p cohomology of π is trivial – for we have the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & U_q & \longrightarrow & 1 \\ & & & & || & & \cap\!\!| & & \\ & & & & \Big(\prod\limits_{l \neq p} \widehat{\mathbb{Z}}_l\Big) \times \widehat{\mathbb{Z}}_p & \dashrightarrow & \widehat{\mathbb{Z}}_p = \mathbb{Z}/p-1 \times \widehat{\mathbb{Z}}_p & & \end{array}$$

Thus by an easy spectral sequence argument X' has the same mod p cohomology as X. Also, π, the fundamental group of X' has a trivial p-profinite completion. It follows that the composition

$$\left\{\begin{matrix} p\text{-adic part of} \\ \text{finite} \\ \text{Grassmannian} \end{matrix}\right\} = X \to X' \xrightarrow[\substack{p\text{-profinite} \\ \text{completion}}]{c} (X')\hat{}_p$$

is a homotopy equivalence. This means that $\pi \times c$ gives an equivalence

$$X' \xrightarrow{\pi \times c} X \times K(\pi, 1)\,.$$

The action of π was trivial in X' by construction. It follows from

$$X' \cong X \times K(\pi, 1)$$

that the action is homotopy trivial in X. This is what we set out to prove.

Remark: From the proof one can extract an interesting homotopy theoretical fact which we roughly paraphrase – if a map

$$X \xrightarrow{f} X$$

is part of a "transformation group" π acting on X where

i) the mod p cohomology of π is trivial,

ii) π acts trivially on the mod p cohomology of X,

then f is homotopic to the identity on the p-adic component of X.

Addendum 1. In the case of the real Grassmannians we sometimes exclude $q = 2$ for p odd. The reason is that our description of the complex orthogonal group

$$O(n, \mathbb{C}) \subseteq \mathrm{GL}\,(n, \mathbb{C})$$

used the form

$$x_1^2 + x_2^2 + \cdots + x_n^2\,.$$

In characteristic $q = 2$

$$x_1^2 + \cdots + x_n^2 \;=\; (x_1 + \cdots + x_n)^2\,.$$

So the subgroup of $\mathrm{GL}\,(n, F_q)$ preserving $x_1^2 + \cdots + x_n^2$ can also be described as the subgroup preserving the linear functional $x_1 + \cdots + x_n$. This defines a subgroup which is "more like $\mathrm{GL}\,(n-1)$" than the orthogonal group. It has dimension $n^2 - n$ instead of $\frac{1}{2}n(n-1)$ so our description of $O(n, \mathbb{C})$ does not reduce well mod 2.

We can alter the description of the complex orthogonal group for $n = 2k$. Consider the subgroup of $\mathrm{GL}(n,\mathbb{C})$ preserving the "split form"

$$x_1x_2 + x_3x_4 + \cdots + x_{2k-1}x_{2k}\,.$$

This defines a conjugate subgroup

$$O(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})\,.$$

This description reduces well modulo 2 – the dimension of this subgroup of $\mathrm{GL}(n,\widetilde{F}_q)$ is $\frac{1}{2}n(n-1)$.

Thus, we can include the cases of the "even" real Grassmannians

$$\widetilde{G}_{2n,2k}(\mathbb{R}) \;=\; SO(2n+2k,\mathbb{C})/SO(2n,\mathbb{C}) \times SO(2k,\mathbb{C})$$

for $q = 2$ in Theorem 5.12.

For example,

$$BSO_{2n} \;=\; \lim_{k\to\infty} \widetilde{G}_{2n,2k}(\mathbb{R})$$

is included for $q = 2$.

BSO_{2n+1} may also be included for $q = 2$ using the following device. At an odd prime p the composition

$$BSO_{2n+1} \to BU_{2n+1} \to B^1 \to (B^1)\hat{}_p$$

is p-profinite completion. Here B^1 may be described in either of two ways –

i) $B^1 = (BU_{2n+1} \times S^\infty)/(\mathbb{Z}/2)$ where $\mathbb{Z}/2$ acts by

$$\text{complex conjugation} \times \text{antipodal map}\,,$$

ii) $(B^1)\hat{}$ is the limit of the etale homotopy types of the "real Grassmannians"

$$\lim_{k\to\infty} \mathrm{GL}(n+k,\mathbb{R})/\,\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(k,\mathbb{R})\,.$$

(For a further discussion of this see the next subsection.)

So the subgroup of $\widehat{\mathbb{Z}}_p^*$ generated by 2 acts on the homotopy type

$$(BSO_{2n+1})\hat{}_p$$

via its action on $(BU_{2n+1})\hat{}_p$.

In all cases ($p = 2, 3, 5, \dots$), the element of order 2 in $\widehat{\mathbb{Z}}_p^*$ corresponds to complex conjugation and acts trivially on the p-adic component of the homotopy type of the real Grassmannians.

Addendum 2. We might indicate the essential ingredients of the proof.

Suppose V is any variety defined over $\mathbb{Q}$. If the cohomology of V is generated by algebraic cycles then the action of $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ on the cohomology (which then only occurs in even dimensions) factors through its Abelianization $\widehat{\mathbb{Z}}^*$.

If V can be reduced well mod q the action of $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ restricted to G_q simplifies to a $\widehat{\mathbb{Z}}$ action.

This action of $\widehat{\mathbb{Z}}$ can then be further simplified to a $U_q \subseteq \widehat{\mathbb{Z}}_p^*$ action if $(\pi_1 V)\hat{}_p = 0$ using the homotopy construction described in the proof.

The etale homotopy of real varieties – a topological conjecture

Suppose the complex algebraic variety $V_{\mathbb{C}}$ can be defined over the real numbers $\mathbb{R}$ – the equations defining $V_{\mathbb{C}}$ can be chosen to have real coefficients. Let $|V_{\mathbb{R}}|$ denote the variety (possibly vacuous) of real solutions to these equations.

$V_{\mathbb{C}}$ has a natural involution c, complex conjugation, induced locally by

$$(x_0, x_1, \dots, x_n) \overset{c}{\mapsto} (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n)\,.$$

c is an algebraic homeomorphism of V with fixed points $|V_{\mathbb{R}}|$.

A natural question is – can one describe some aspect of the homotopy type of the real variety $|V_{\mathbb{R}}|$ algebraically?

We have the etale homotopy type of the complex points $V_{\mathbb{C}}$,V_{et}, and V_{et} has an involution σ corresponding to complex conjugation in $V_{\mathbb{C}}$. The pair (V_{et}, σ) is an algebraically constructed homotopy model of the geometric pair $(V_{\mathbb{C}}, \text{conjugation})$.

Thus we are led to consider the related question of "geometric homotopy theory" –

What aspect of the homotopy type of the fixed points F of an involution t on a finite dimensional locally compact space X can be recovered from some homotopy model of the involution?

In the examples of section 1 we saw how certain 2-adic completions of the cohomology or the K-theory of the fixed points could be recovered from the associated "fixed point free involution"

$$(X', t') \equiv (X \times S^\infty, t \times \text{antipodal map}), \quad S^\infty = \text{infinite sphere}$$

with orbit space

$$X_t = X'/(x \sim t'x) \equiv X \times S^\infty/(\mathbb{Z}/2).$$

Now (X', t') is a good homotopy model of the geometric involution (X, t). For example, the projection on the second factor leads to the useful fibration

$$X \cong X' \to X_t \to \mathbb{R}P^\infty \equiv S^\infty/\text{antipodal}.$$

Let (geometric, $\mathbb{Z}/2$) denote the category of locally compact finite dimensional topological spaces with involution and equivariant maps between them. let (homotopy, free $\mathbb{Z}/2$) denote the category of CW complexes with involution and homotopy classes of equivariant maps between them.

We have two constructions for objects in (geometric, $\mathbb{Z}/2$):

a) the geometric operation of taking fixed points,

b) the homotopy theoretical operation of passing to the associated free involution

$$(X', t') = (X \times S^\infty, t \times \text{antipodal}).$$

These can be compared in a diagram

$$\begin{tikzcd}
(\text{geometric}, \mathbb{Z}/2) \arrow[r, "\times(S^\infty,\text{antipodal})"] \arrow[d, "\substack{\text{fixed}\\ \text{points}}"'] & (\text{homotopy}, \text{free } \mathbb{Z}/2) \arrow[d, dashed, "\mathcal{F}"] \\
\{\text{topological spaces}\} \arrow[r, "\text{forget}"] & \{\text{homotopy category}\}.
\end{tikzcd}$$

Our basic question then becomes – can we make a construction $\mathcal{F}$, "the homotopy theoretical fixed points of a fixed point free involution", so that the above square commutes?

Note we should only ask for the 2-adic part of the homotopy type of the fixed point set. This is natural in light of the above examples in cohomology and K-theory. Also examples of involutions on spheres show the odd primary part of the homotopy type of the fixed points can vary while the equivariant homotopy type of the homotopy model (X', t') stays constant.

There is a natural candidate for the homotopy theoretical fixed points. We make an analogy with the geometric case – in (geometric, $\mathbb{Z}/2$) the fixed points of (X, t) are obtained by taking the space

$$\{\text{equivariant maps } (\text{point} \to X)\}\,.$$

In (homotopy, free $\mathbb{Z}/2$), the role of a point should be played by any contractible CW complex with free involution, for example S^∞ – the infinite sphere.

Definition. *If X is a CW complex with an involution t define the* homotopy theoretical fixed points $\mathcal{F}$ *of (X, t) by*

$$\mathcal{F}(X,t) \;=\left[\begin{matrix}\textit{singular complex of}\\ \textit{equivariant maps}\end{matrix}\;(S^\infty \to X)\right].$$

We list a few easy properties of $\mathcal{F}$:

i) $\mathcal{F}(S^\infty, \text{antipodal map})$ is contractible ($\simeq$ point).

ii) $\mathcal{F}(X, t) \simeq \mathcal{F}(X', t')$.

iii) $\mathcal{F}(X \times X, \text{flip}) \simeq X$.

iv) $\mathcal{F}(X', t') \simeq$ singular complex of cross sections $(X_t \to P^\infty(\mathbb{R}))$ (generalization of i).)

We make the

Fixed Point Conjecture *If (X, t) is a triangulable involution on a locally compact space, then*

$$\big(\textit{fixed points } (X,t)\big)\hat{}_2 \simeq \big(\mathcal{F}(X',t')\big)\hat{}_2\,,$$

that is,

$$\begin{tikzcd}[column sep=small]
 & (\text{geometric}, \mathbb{Z}/2) \arrow[rr, "\substack{\times\text{contractible}\\ \text{free } \mathbb{Z}/2\text{-space}\\ (\text{say } S^\infty)}"] \arrow[d, "\substack{\text{“geometric}\\ \text{fixed points”}}"] & & (\text{homotopy}, \text{free } \mathbb{Z}/2) \arrow[d, "\substack{\text{“homotopy theoretical}\\ \text{fixed points”}}"'] \\
 & \{\text{triangulable topological spaces}\} \arrow[dr, "\substack{\text{2-adic}\\ \text{completion}}"'] & & \{\text{homotopy category}\} \arrow[dl, "\substack{\text{2-adic}\\ \text{completion}}"] \\
 & & \{\text{homotopy category}\} &
\end{tikzcd}$$

commutes for triangulable involutions on locally compact spaces.

Paraphrase – “We can make a homotopy theoretical recovery of the 2-adic homotopy type of the geometric fixed point set from the associated (homotopy theoretical) fixed point free involution”.

One can say a little about the question.

i) The conjecture is true for the natural involution on $X \times X$ – this follows by direct calculation – property iii) above.

ii) The conjecture is true if *the set of fixed points of X is vacuous* – for then X_t is cohomologically finite dimensional. So there are no cross sections on $X_t \to \mathbb{R}P^\infty$. (This has the interesting Corollary E below.)

iii) If the involution is trivial – the fixed points are all of X. However, the homotopy fixed points is the space of all maps of $\mathbb{R}P^\infty$ into X. For these to agree 2-adically it must be true that *the space of all based maps of $\mathbb{R}P^\infty$ into a finite dimensional space X is contractible.*

iv) The latter conjecture seems quite difficult from a direct point of view. The fixed point conjecture seems to hinge on it –

In fact suppose (X, t) is a smooth manifold with smooth involution with fixed submanifold F – say F connected.

Let ν denote the projectivized normal bundle of F, then one can determine that

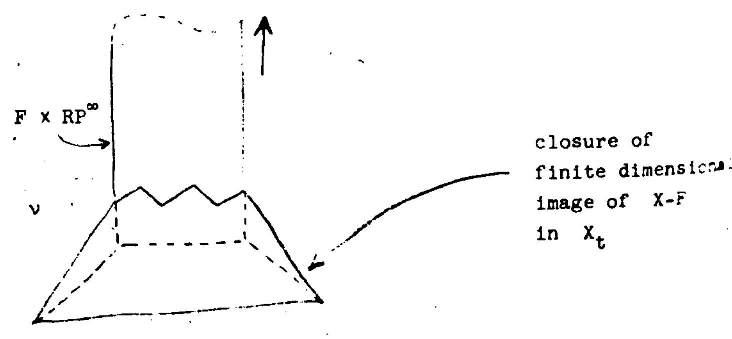
$$X_t \cong F \times \mathbb{R}P^\infty \cup_\nu (X - F)/(x \sim tx)$$

where

$$\nu \to (X - F)/(x \sim tx) \text{ is the natural inclusion},$$

$$\nu \to F \times \mathbb{R}\, P^\infty \text{ is the}$$

$$(\text{projection onto } F) \times (\text{canonical line bundle over } \nu)$$



The conjecture of iii) indicates that any map of $\mathbb{R}\, P^\infty$ into X_t cannot involve the "locally compact part of X_t" too seriously so we need only consider maps into the infinite part of $F \times \mathbb{R}\, P^\infty$. Applying the conjecture of iii) again gives the result.

The Real Etale Conjecture

Let us return to the original question about the homotopy of real algebraic varieties.

Let (X, t) denote the etale homotopy type of the *complex points* with involution corresponding to complex conjugation. We first think of (X, t) as approximated by an inverse system of complexes X^α each with its own involution. We can do this by restricting attention to the cofinal collection of etale covers of $V_{\mathbb{C}}$ which are invariant (though not fixed) by conjugation.

We form the fibration sequence

$$X^\alpha \to X_t^\alpha \to P^\infty(\mathbb{R}) \;\; (= \mathbb{R}\, P^\infty)$$

and pass to limit to obtain the complete fibration sequence

$$X \to X_t \to P^\infty(\mathbb{R})\,.$$

This sequence has a direct algebraic description.

To say that a variety $V_{\mathbb{R}}$ is defined over $\mathbb{R}$ means that we have a scheme built from the spectra of finite $\mathbb{R}$-algebras. We have a map

$$V_{\mathbb{R}} \to \operatorname{Spec} \mathbb{R} .$$

The complex variety $V_{\mathbb{C}}$ is the fibre product of natural diagrams

$$\begin{tikzcd} & V_{\mathbb{R}} \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{R} . \end{tikzcd}$$

Applying the complete etale homotopy functor gives a fibre product square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \text{etale homotopy type } (V_{\mathbb{R}}) \arrow[d] \\ * \arrow[r] & \text{etale homotopy type } \operatorname{Spec} \mathbb{R} = K(\mathbb{Z}/2,1) , \end{tikzcd}$$

i.e. the sequence

$$X \to X_t \to P^\infty(\mathbb{R}) .$$

Thus we can conjecture three equivalent descriptions of the 2-adic homotopy type of the variety of real points $|V_{\mathbb{R}}|$ –

i) The 2-adic completion of the space of equivariant maps of the infinite sphere into the etale homotopy type of the associated complex variety. (We first form the inverse system of equivariant maps into each nerve in the system, then 2-adically complete and form a homotopy theoretical inverse limit as in section 3).

ii) The space of cross sections of the etale realization of the defining map

$$V_{\mathbb{R}} \to \operatorname{Spec} \mathbb{R} .$$

(Again complete the etale homotopy type of $V_{\mathbb{R}}$. We get a map of CW complexes

$$X_t \to P^\infty(\mathbb{R}) .$$

Then take the singular complex of cross sections and 2-adically complete this.)

NOTE: This last description is analogous to the definition of the "geometric real points": a real point of $V_{\mathbb{R}}$ is an $\mathbb{R}$-morphism

$$\operatorname{Spec} \mathbb{R} \to V_{\mathbb{R}} \,,$$

namely, a "cross section" of the defining map

$$V_{\mathbb{R}} \to \operatorname{Spec} \mathbb{R} \,.$$

iii) A set of components of the space of all maps

$$\{\text{etale type } (\operatorname{Spec} \mathbb{R}) \to \text{etale type } (V_{\mathbb{R}})\}$$

2-adically completed.

(The space of cross sections of

$$X_t \to \mathbb{R}\, P^\infty = \text{etale type } (\operatorname{Spec} \mathbb{R})$$

may be described (up to homotopy) as the subset of the function space of all maps

$$\{\mathbb{R}\, P^\infty \to X_t\}$$

consisting of those components which project to the non-trivial homotopy class of maps of $\mathbb{R}\, P^\infty$ to $\mathbb{R}\, P^\infty$. This description is perhaps easier to compute theoretically.)

We first prove a subjunctive theorem and then some declarative corollaries.

THEOREM 5.13 *The topological fixed point conjecture implies the above etale descriptions of the* 2*-adic homotopy type of a real variety are correct.*

PROOF: Let U_n be a linearly ordered inverse system of locally directed, finite etale covers of the variety of complex points $V_{\mathbb{C}}$ so that

i) U_n is invariant by complex conjugation,

ii) $\hat{V_{\mathbb{C}}} \cong \varprojlim_n \text{nerve } C(U_n)$ where the nerve of the category of smallest neighborhoods is as discussed above.

However instead of using the little category $C(U)$ discussed above use the (larger) homotopy equivalent category $C(U, V)$ (introduce labels).

U_x is an object of $C(U,V)$ if "U_x is a smallest neighborhood of x". The morphisms are the same diagrams as before

$$\begin{tikzcd} U_x \arrow[rr] \arrow[dr] & & U_y \arrow[dl] \\ & V_{\mathbb{C}} & \end{tikzcd}$$

– we ignore the "labels" x and y.

Enlarging the category like this makes the map between nerves

$$\text{nerve } C(U^1,V) \to \text{nerve } C(U,V)$$

canonical if U^1 refines U.

Inductively choose a linearly ordered system of locally directed finite topological coverings V_n so that

i) V_n refines U_n and V_i for $i < n$
each $V \in V_n$ is contractible.

ii) V is invariant by conjugation.

Then we have a canonical diagram of equivariant maps

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] \\ & \text{nerve } \big(C(V_n,V)\big) \arrow[r] \arrow[d] & \text{nerve } \big(C(U_n,V)\big) \arrow[d] \\ V_{\mathbb{C}} \arrow[ur, "f_n"] \arrow[r, "f_{n-1}"'] & \text{nerve } \big(C(V_{n-1},V)\big) \arrow[r] \arrow[d] & \text{nerve } \big(C(U_{n-1},V)\big) \arrow[d] \\ & {} & {} \end{tikzcd} .$$

(To construct f_n we assume as above for each n that $V_{\mathbb{C}}$ is triangulated so that

i) complex conjugation is piecewise linear,

ii) $V_{\mathbb{C}} - U_\alpha$ is a subcomplex, $U_\alpha \in V_n$.)

By the proposition above each f_n is an equivalence. By etale homotopy theory the right hand column converges to the profinite completion of $V_{\mathbb{C}}$.

From this it follows that the profinite completion of the map

$$V_{\mathbb{C}} \times S^\infty/(\mathbb{Z}/2) \xrightarrow{t} \mathbb{R}\,P^\infty$$

is equivalent to

$$\varprojlim_n \big(\text{nerve } C(U_n) \times S^\infty/(\mathbb{Z}/2)\big)^{\hat{}} \xrightarrow{a} \mathbb{R}\,P^\infty$$

$\big($or,

$$\text{etale type } V_{\mathbb{R}} \to \text{etale type } (\text{Spec } \mathbb{R})\big)\,.$$

Thus the 2-adic completion of the space of cross sections of these maps agree. The topological fixed point conjecture asserts we obtain the 2-adic completion of the homotopy type of the real points in the first instance. This proves the theorem.

The proof does not show that some aspect of the cohomology and K-theory of a 'real variety' $|V_{\mathbb{R}}|$ can be described algebraically.

Cohomology: Let $\mathcal{R}$ denote the "cohomology ring of $\mathbb{Z}/2$",

$$\mathbb{Z}/2[x] \cong H^*\big(P^\infty(\mathbb{R}); \mathbb{Z}/2\big)\,.$$

The $\mathbb{Z}/2$ etale cohomology of $V_{\mathbb{R}}$ is a module over $\mathcal{R}$ – using the etale realization of the defining map

$$V_{\mathbb{R}} \to \text{Spec } \mathbb{R}\ .$$

Then P. A. Smith theory (which follows easily from the "picture of X_t" above) implies

Corollary H

$$H^*(\text{variety of real points}; \mathcal{R}_x) \cong \begin{array}{l}\text{etale cohomology}\\ V_{\mathbb{R}} \text{ localized with}\\ \text{respect to the}\\ \text{prime ideal } (x)\end{array}$$

$$\cong H^*(V_{\mathbb{R}}; \mathbb{Z}/2) \otimes_{\mathcal{R}} \mathcal{R}_x\,,$$

$\mathcal{R}_x = \mathbb{Z}/2[x, x^{-1}] = \mathcal{R}$ localized at the prime ideal (x).

K-THEORY: Let $\widehat{\mathcal{R}}$ denote the group ring of $\mathbb{Z}/2$ over the 2-adic integers,

$$\widehat{\mathcal{R}} = \widehat{\mathbb{Z}}_2[x]/(x^2-1)\,.$$

The work of Atiyah and Segal on equivariant K-theory may be used to deduce

COROLLARY K *The K-theory of the variety of real points $|V_{\mathbb{R}}|$ satisfies*

$$K(|V_{\mathbb{R}}|) \otimes \widehat{\mathcal{R}} \cong K\big((\textit{etale homotopy type } V_{\mathbb{R}})\hat{}_2\big)\,.$$[38]

A final corollary is interesting.

COROLLARY E *If $\{f_i\}$ is a finite collection of polynomials with real coefficients, then the polynomial equations*

$$\{f_i = 0\}$$

have a real solution iff the etale cohomology of the variety defined by $\{f_i\}$ has mod 2 cohomology in infinitely many dimensions.

PROOF: We actually prove a variety $V_{\mathbb{R}}$ defined over $\mathbb{R}$ has a real point if the etale cohomology (with $\mathbb{Z}/2$ coefficients) is non-zero above twice complex dimension $V_{\mathbb{C}}$.

If there is a real point p,

$$\begin{tikzcd} \operatorname{Spec}\mathbb{R} \arrow[r, "p"] \arrow[dr, "\text{identity}"'] & V_{\mathbb{R}} \arrow[d, "\text{“defining equation”}"] \\ & \operatorname{Spec}\mathbb{R} \end{tikzcd}$$

the etale realization shows

$$H^*(\text{etale type } V_{\mathbb{R}}; \mathbb{Z}/2) \supseteq H^*\big(P^\infty(\mathbb{R}); \mathbb{Z}/2\big)\,.$$

If there is no real point,

$$H^*(\text{etale type } V_{\mathbb{C}} \times S^\infty/(\mathbb{Z}/2); \mathbb{Z}/2)$$

vanishes above $2n$, $n =$ complex dimension $V_{\mathbb{C}}$,

$$\big(\text{etale type } V_{\mathbb{C}} \times S^\infty/(\mathbb{Z}/2) \cong \text{etale type } V_{\mathbb{C}}/(\mathbb{Z}/2)\big)\,.$$

But

$$\text{etale type } V_{\mathbb{C}} \times S^{\infty}/(\mathbb{Z}/2) \underset{\text{2-adically}}{\cong} \text{etale type of } V_{\mathbb{R}}$$

as shown above.

NOTE: The "picture of X_t" above shows

$$H^i(\text{etale type } V_{\mathbb{R}}; \mathbb{Z}/2)$$

is constant for large i.

Moreover this stable cohomology group is the direct sum of the mod 2 cohomology of the real points.

Notes

1 "etale" equals smooth.

2 (For precise definitions see for example I. G. MacDonald "Introduction to Schemes" Benjamin, N. Y.)

3 For a simple description of this construction see Lubkin, "On a Conjecture of Weil", American Journal of Mathematics", p. 456, 1967.

4 "Etale Homotopy" Springer Verlag Lecture Notes.

5 One has to assume that X satisfies some further condition so that the homotopy groups of the "nerve" are finite – normal or non-singular suffices. Anyway, if they are not one can profinitely complete the nerve and then take an inverse limit to construct $\hat{X}$ algebraically.

6 Only finite coverings are algebraic. Riemann began the proof of the converse.

7 The subset of U_α consisting of those points for which U_α is the "smallest neighborhood" is obtained by removing a finite number of U_β's from U_α. It this consists of open simplices.

8 f may be so constructed that this is possible.

9 We are allowing infinite covers.

10 Using the compact topology on $[\quad, \Sigma K(\mathbb{Z}/n, 1)]$ as in section 3.

11 First few pages of Colloquium volume on algebraic geometry.

12 Under some mild assumption for example X normal.

13 $[\quad, Y]$ denotes the functor homotopy classes of maps into Y.

14 Recall that if in some case the nerve does not have finite homotopy groups, then we first profinitely complete it.

15 We use this variety to study the homotopy of X because an analogy works for the real Grassmannian.

16 Chevalley, "Theory of Lie Groups", last chapter, Vol. 1.

17 Thus we can apply this etale homotopy discussion to any compact Lie group. I am indebted to Raoul Bott for suggesting this.

18 Actually a homeomorphism of the homotopy functor $[\quad, \widehat{V}]$.

19 I think some analogy should be made between this "inertia" of $\mathbb{Q}$ and the "inertia" of stable fibre homotopy types of section 4.

20 For the purpose of the moment we think only of ordinary covers.

21 Although it is an open question for the Grassmannians. For partial results see the subsection "The Groups generated by Frobenius elements acting on the Finite Grassmannians".

22 I am indebted to G. Washnitzer for explaining this point to me very early in the game.

23 This is easily seen using the isomorphisms

$$\text{group of all roots of unity} \cong \varinjlim_{n} \mathbb{Z}/n\,,$$
$$\widehat{\mathbb{Z}}^{*} \cong \varprojlim_{n} (\mathbb{Z}/n)^{*}\,.$$

($\widetilde{\mathbb{Q}}$ is the field of all algebraic numbers).

24 The example $\mathbb{C} - 0$ of the previous subsection.

25 Actually this works well in the simply connected case away from the prime 2. (Other choices are required at 2 to "impose a topology").

26 Those with good mod p reductions.

27 Reported to the AMS Winter Meeting, Berkeley, January 1968.

28 In a heated discussion with Atiyah, Borel, De Ligne and Quillen – and all present were required.

29 "ψ^p is defined" means an operation compatible under the natural inclusions into K-theory with the original Adams operation.

30 A cell argument (Arkowitz and Curjel) or a transversality argument plus the signature formula shows the degree would have to satisfy a congruence ruling out degree 4, $d(d-1) \equiv 0 \pmod{24}$.

31 In fact this is the essential point of the Adams Conjecture.

32 Besides the "tautological bundle theory".

33 The categories of projective spaces.

34 Informal oral communication from J. Tate.

35 We make certain exceptions for $q = 2$, p odd in the real case. These are discussed below. Note these exceptions do not affect the study of the "2-adic real Grassmannians".

36 I am indebted to Barry Mazur for explaining this situation and its connection to etale homotopy.

37 In the real case we consider only the oriented Grassmannian.

38 The right hand side may be described for example by

$$\varprojlim_{n} \varinjlim_{\substack{\text{etale coverings} \\ \text{of } V_{\mathbb{R}}}} K(\text{nerve}; \mathbb{Z}/2^n)\,.$$

Section 6. The Galois Group in Geometric Topology

We combine the Galois phenomena of the previous section with the phenomenon of geometric periodicity that occurs in the theory of manifolds. We find that the Abelianized Galois group acts compatibly on the completions of Grassmannians of k-dimensional *piecewise linear* subspaces of $\mathbb{R}^\infty$.

We study this symmetry in the theory of piecewise linear bundles and other related geometric theories.

To motivate this study consider the problem of understanding what invariants determine a compact manifold.

There is the underlying homotopy type plus some extra geometric invariant. In fact define a map

$$\begin{Bmatrix}\text{compact}\\ \text{manifolds}\end{Bmatrix} \xrightarrow{\Delta} \begin{Bmatrix}\text{homotopy} & \text{“tangent}\\ \text{types} & \text{bundles”}\end{Bmatrix}.$$

With the appropriate definitions and hypotheses Δ is an injection and the “equations defining the image” are almost determined.

We remark that the notion of isomorphism on the right is somewhat subtle – we should have a diagram

$$\begin{tikzcd} TM \arrow[r, "dg"] \arrow[d] & TL \arrow[d] \\ M \arrow[r, "g"'] & L, \end{tikzcd}$$

g is any homotopy equivalence between the underlying homotopy types M and L and dg is a *bundle isomorphism* of the "tangent bundles" TM and TL which is properly homotopic to the *homotopy theoretical derivative* of g, a naturally defined proper fibre homotopy equivalence between TM and TL.[1]

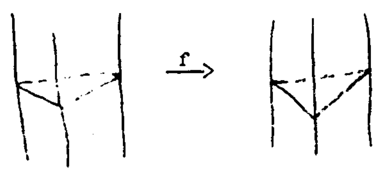
So to understand manifolds we must understand bundles, bundle isomorphisms, and deformations of fibre homotopy equivalences to bundle isomorphisms.

There are questions of a homotopy theoretical nature which we can decompose and arithmetize according to the scheme of sections two and three. (We pursue this more strenuously in a sequel.)

It turns out that the odd primary components of the piecewise linear or topological questions have a surprisingly beautiful structure – with Galois symmetry and four-fold periodicity in harmonic accord.

Piecewise linear bundles

Consider a "block of homeomorphisms",



, $\sigma \times R^n \longrightarrow \sigma \times R^n$.

σ a simplex

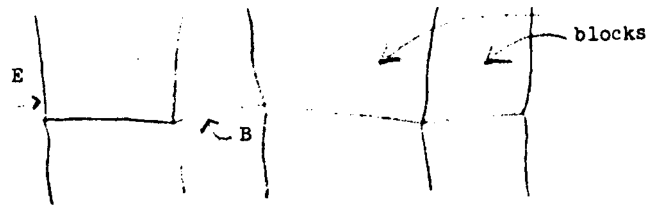
f satisfies

i) f is a piecewise linear homeomorphism,

ii) for every face $\tau < \sigma$, f keeps $\tau \times \mathbb{R}^n$ invariant, "f preserves the blocks".

This defines a simplex of the piecewise linear group PL_n.

This group determines a theory of "$\mathbb{R}^n$-block bundles",

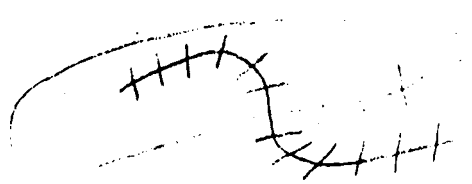


we have a "total space" E which admits a decomposition into blocks, $\sigma \times \mathbb{R}^n$, one for each simplex in the "base", B which is embedded in E as the "zero section", $\sigma \times \{0\} \subseteq \sigma \times \mathbb{R}^n$.

The notion of isomorphism reduces to piecewise linear homeomorphism of the pair of polyhedra (E, B). The isomorphism classes form a proper "bundle theory" classified by homotopy classes of maps into the "PL Grassmannians", $G_n(PL)$ (or BPL_n – the classifying space of the group PL_n).

The fact that block bundles are functorial is a non-trivial fact because there is no "geometrical projection", only a homotopy theoretical one.

The lack of a "geometric projection" however enables the tubular neighborhood theorem to be true. Submanifolds have neighborhoods which can be "uniquely blocked",



and transversality constructions can be made using the blocks.

In fact, it seems to the author that the category of polyhedra with the concomitant theory of block bundles provides the most general an natural setting for performing the *geometrical* constructions associated with transversality and intersection.

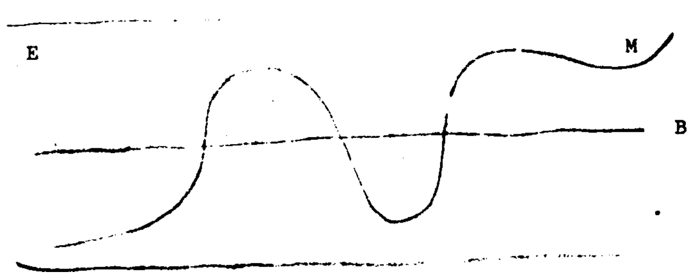
We note the fact that a homotopy theoretical "non-zero section" of a block bundle cannot necessarily be realized geometrically precisely because there is no "geometric projection".

The Transversality Construction

Consider a closed submanifold M^{n+l} of the block bundle E. It is possible [2] (if B is compact) to perform a compactly supported isotopy so that

$$M \text{ is transversal to } B\,,$$

M intersects a neighborhood of B in a union of blocks intersect the neighborhood.



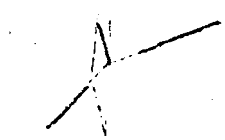
The intersection with B is a compact manifold V^l which is a subpolyhedron of B. There are certain points to be noted.

i) If M^{n+l} varies by a proper cobordism W^{n+l+1} in $E \times$ unit interval then V varies by a cobordism.

ii) A proper map $M \xrightarrow{f} E$ can be made transversal to B by making the graph of f transversal to $M \times B$. We obtain a proper map

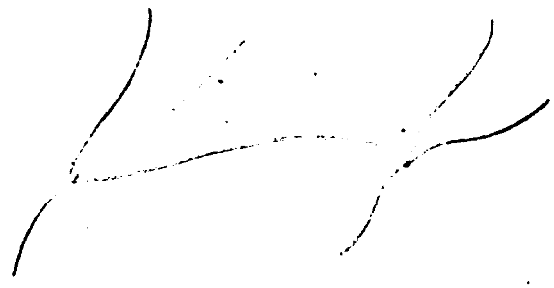
$$V \to B\,.$$

iii) More general polyhedra X may be intersected with B. The intersection has the same singularity structure as X. For example, consider



a $\mathbb{Z}/n$-manifold in E, that is, a polyhedron formed from a manifold with n-isomorphic collections of boundary components by identification. The transversal intersection with B is again a

$\mathbb{Z}/n$-manifold. We can speak of $\mathbb{Z}/n$-cobordism,



and property ii) holds.

iv) Passing to cobordism classes of proper maps

$$M \to E$$

and cobordism classes of maps

$$V \to B$$

yields Abelian groups (disjoint union)

$$\Omega_* E \text{ and } \Omega_* B$$

and we have constructed the Thom intersection homomorphism

$$\Omega_* E \xrightarrow{\cap B} \Omega_* B\,,$$

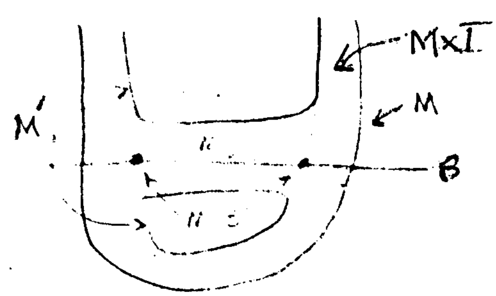
plus a $\mathbb{Z}/n$ analogue

$$\Omega_*(E; \mathbb{Z}/n) \xrightarrow{\cap B} \Omega_*(B; \mathbb{Z}/n)\,.$$

PROPOSITION 6.1 *Intersection gives the Thom isomorphism*

$$\Omega_* E \xrightarrow[\cong]{\cap B} \Omega_* B\,.$$

PROOF: If $M \cap B = \partial W$, then we can "cobord to M' off of B" using $M \times I$ union "closed blocks over W", schematically



Now use that fact that the one point compactification of E, E^+ with B removed is contractible to "send M' off to infinity", using a proper map $M' \times \mathbb{R} \to E - B$. This proves $\cap B$ is injective. $\cap B$ is clearly onto since

$$(\text{blocks over } V) \cap B \ = \ V\,.$$

Note if E were oriented by a cohomology class

$$U \in H^n(E, E - B; \mathbb{Z})$$

the isomorphism above would hold between the oriented cobordism groups (also denoted Ω_*).

We define $\mathbb{Z}/n$-manifolds by glueing together oriented isomorphic boundary components and obtain the $\mathbb{Z}/n$-Thom isomorphism

$$\Omega_*(E; \mathbb{Z}/n) \xrightarrow[\cong]{\cap B} \Omega_*(B; \mathbb{Z}/n)\,.$$

The Signature Invariants and Integrality

The Thom isomorphism

$$\Omega_* E \xrightarrow[\cong]{\cap B} \Omega_* B$$

is the basic *geometric* invariant of the bundle E. To exploit it we consider the numerical invariants which arise in surgery on manifolds.

If $x \in H_{4i}(V; \mathbb{Q})$, define "the signature of the cycle x" by

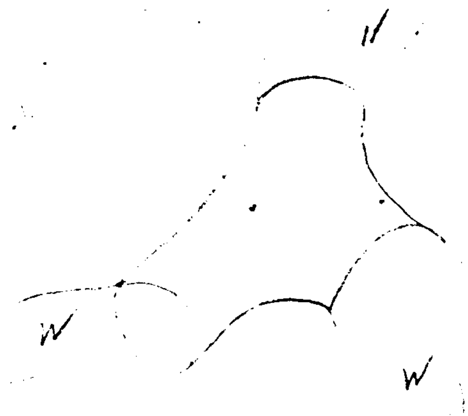
$$\text{signature } x \ = \ \text{signature of quadratic form on } H^{2i}(V; \mathbb{Q})$$

given by $\langle y^2, x\rangle$, $y \in H^{2i}(V; \mathbb{Q})$.

An oriented manifold V has a signature in $\mathbb{Z}$. An oriented $\mathbb{Z}/n$-manifold has a signature in $\mathbb{Z}/n$,

$$\text{signature } V \ = \ \text{signature}\,(V/\text{singularity } V) \ \bmod n\,.$$

These signatures are cobordism invariants. For example, if V is cobordant to zero, we can unfold the cobordism



where Q and W are obtained from the cobordism of V to $\emptyset$ and see that

$$0 \overset{\text{Thom}}{=} \text{signature } \partial Q \overset{\text{Novikov}}{=} \text{signature } V + 3 \text{ signature } W$$

using the addition lemma (Novikov) for the signature of "manifolds with boundary", (= signature $W/\partial W$) and the cobordism invariance of the ordinary signature (Thom).

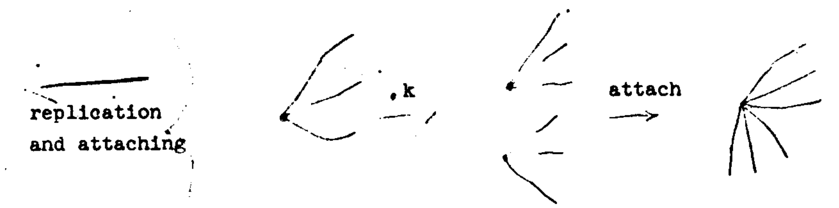
These signature relations are proved by pleasant little duality arguments.

Now note that we have

i) coefficient homomorphism

$$\Omega_*(\quad ; \mathbb{Z}/kn) \rightleftarrows \Omega_*(\quad ; \mathbb{Z}/n)$$

defined by



ii) an exact ladder

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_i(\quad) \arrow[r, "\cdot n"] \arrow[d] & \Omega_i(\quad) \arrow[r] \arrow[d, "k"] & \Omega_i(\quad;\mathbb{Z}/n) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Omega_i(\quad) \arrow[r, "\cdot kn"] & \Omega_i(\quad) \arrow[r] & \Omega_i(\quad;\mathbb{Z}/kn) \arrow[r] & \cdots
\end{tikzcd}$$

derived geometrically.

We can form

$$\mathbb{Q}/\mathbb{Z}\text{-bordism} \quad \Omega_*(\quad;\mathbb{Q}/\mathbb{Z}) = \varinjlim_n \Omega_*(\quad;\mathbb{Z}/n)$$
$$\mathbb{Q}\text{-bordism} \quad \Omega_*(\quad;\mathbb{Q}) = \varinjlim_k \left(\Omega_*(\quad) \xrightarrow{k} \Omega_*(\quad)\right)$$

and an exact sequence

$$\cdots \to \Omega_*(\quad) \to \Omega_*(\quad;\mathbb{Q}) \xrightarrow{t} \Omega_*(\quad;\mathbb{Q}/\mathbb{Z}) \to \ldots .$$

Definition (signature invariant of E) *Compose the operations*

i) intersect with the zero section,

ii) take the signature of the intersection to obtain the "signature invariant of E"

$$\sigma(E) = \begin{tikzcd} \Omega_*(E;\mathbb{Q}) \arrow[rr] \arrow[d, "t"'] & & \mathbb{Q} \arrow[d] \\ \Omega_*(E;\mathbb{Q}/\mathbb{Z}) \arrow[rr] \arrow[urr, phantom, "\substack{\text{signature}\\ \text{of intersection}}"] & & \mathbb{Q}/\mathbb{Z}\,. \end{tikzcd}$$

We assume E is oriented.

The rational part of the signature invariant carries precisely the same information as "rational characteristic classes of E".

$$1 + L_1 + L_2 + \cdots \in \prod H^{4i}(B;\mathbb{Q})\,. \quad \text{(Thom).}$$

The extension of the rational signature to a $\mathbb{Q}/\mathbb{Z}$ signature can be regarded as a *canonical* integrality theorem for the rational characteristic classes of E.

To explain this for bundles consider the localizations

$$K^n(E)_l = KO^n(E)_c \otimes \mathbb{Z}_l \qquad n = \text{fibre dimension}$$

$$H^{4*}(B)_2 = \prod_{i=0}^{\infty} H^{4i}(B;\mathbb{Z}_{(2)})\,.$$

Recall

$\mathbb{Z}_{(2)}$ = integers localized at 2,

$\mathbb{Z}_l$ = integers localized at l, l = set of odd primes,

$KO^n_{\text{compact support}} = KO^n_c = KO^n$ (one point compactification of E),

$\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}/p^\infty = \bigoplus_p \varinjlim_n \mathbb{Z}/p^n$.

Theorem 6.1 *The rational characteristic class of a piecewise linear block bundle E over B (oriented)*

$$1 + L_1 + L_2 + \cdots \in \prod H^{4i}(B;\mathbb{Q})$$

determined by the rational part of the signature invariant of E satisfies two canonical integrality conditions

i) (at the prime 2) there is a canonical "integral cohomology class"

$$\mathscr{L}_E \in H^{4*}(B)_2$$

so that

$$\mathscr{L}_E \to L_E$$

under the coefficient homomorphism induced by

$$\mathbb{Z}_{(2)} \to \mathbb{Q}\,.^3$$

ii) (at odd primes) there is a canonical "K-theory orientation class"

$$\Delta_E \in K(E)_l \quad l = \{odd\ primes\}$$

so that

$$\text{Pontrjagin character } \Delta_E \in H_c^{4*}(E)$$

is related to L by the Thom isomorphism

$$ph\,\Delta_E \;=\; L_E \cdot (\text{Thom class})\,,$$

ph is the Chern character of "Δ_E complexified".

iii) $\mathscr{L}_E$ is determined by L_E and the 2-adic part of the $\mathbb{Q}\,/\,\mathbb{Z}$-signature of E.

Δ_E is determined by L_E and the l-adic part of the $\mathbb{Q}\,/\,\mathbb{Z}$ signature of E.

Remarks:

The invariants Δ_E and $\mathscr{L}_E$ are invariants of the "stable isomorphism class of E", the isomorphism class of

$$E \times \mathbb{R}^k\,.$$

If B is a closed manifold and the fibre dimension of E is even, then

$$B \cap B$$

represents the Poincaré dual of the Euler class in homology, an "unstable invariant". If this class is zero then B is homologous to a cycle in $E - B$ If $B \cap B$ is cobordant to zero in $B \times I$ then B is cobordant in $E \times I$ to a submanifold of $E - B$ (by the argument above).

We will see that

i) L and the rational Euler class form a complete set of rational invariants for E (fibre dimension E even). The set of bundles is almost a "lattice" in the set of invariants.

ii) Δ_E and the fibre homotopy type of $E - B \to B$ form a complete set of invariants at odd primes.

At the prime 2, $\mathscr{L}_E$, the fibre homotopy type of $E - B \to B$, and a certain additional 2 torsion invariant $\mathscr{K}$ determine E. The precise form and geometric significance of $\mathscr{K}$ is not yet clear.

We pursue the discussion of the odd primary situation and the K-theory invariant Δ_E. We will use Δ_E to construct the Galois symmetry in piecewise linear theory. The construction of Δ_E follows from the discussion below.

Besides the "algebraic implication" for piecewise linear theory of Δ_E, we note here that for smooth bundles Δ_E will be constructed from the "Laplacian in E". We hope to pursue this "analytical implication" of Δ_E in PL theory in a later discussion.

Geometric Characterization of K-theory

We consider a marvelous geometric characterization of elements in K-theory (real K-theory at odd primes).

Roughly speaking, an element in $K(X)$ is a "geometric cocycle" which assigns a residue class of integers to every smooth $\mathbb{Z}/n$-manifold in X. The cocycle is subject to certain conditions like cobordism invariance, amalgamation of residue classes, and a periodicity formula.[4]

Actually there is another point of view besides that of the title. Geometric properties of manifold theory force a four-fold periodicity into the space classifying fibre homotopy equivalences between PL bundles.

This four-fold periodic theory is the germane theory for studying the geometric invariants of manifolds beyond those connected to the homotopy type or the action of π_1 on the universal cover.

The obstruction theory in this geometric theory has the striking "theory of invariants" property of the "geometric cocycle" above (at all primes). This can be seen by interesting geometric arguments using manifolds with "join-like singularities".[5]

The author feels this is the appropriate way to view this geometric theorem about real K-theory at odd primes – where "Bott periodicity" coincides with the "geometric periodicity".

However, the proof of the geometric cocycle theorem (at odd primes) is greatly facilitated using K-theory. Moreover the Galois group of $\mathbb{Q}$ acts on K-theory.

In summary, the "geometric insight" into this theorem about K-theory comes from the study of manifolds, the Galois symmetry in manifold theory comes from K-theory.[6]

We describe the theorem.

Let $\Omega^l_*(X;\mathbb{Q}/\mathbb{Z})$ denote the *odd part* of the $\mathbb{Q}/\mathbb{Z}$-bordism group defined by *smooth manifolds*,

$$\Omega^l_*(X;\mathbb{Q}/\mathbb{Z}) \;=\; \varinjlim_{n \text{ odd}} \left\{\begin{matrix}\text{cobordism class of}\\ \text{smooth } \mathbb{Z}/n \text{ manifolds}\\ \text{in } X\end{matrix}\right\} .$$

Let

$$\left\{\begin{matrix}\text{finite}\\ \text{geometric}\\ \text{cocycles}\end{matrix}\right\}^0 \subseteq \{\Omega^l_{4*}(X;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\lambda} \mathbb{Q}/\mathbb{Z}\}$$

be the subgroup of homomorphisms satisfying the "periodicity relation"

$$\lambda\big((V \xrightarrow{4i} X) \times (M \xrightarrow{4l} \mathrm{pt})\big) \;=\; \text{signature } M \cdot \lambda(V \to X)\,, \;\; M \text{ closed}\,.$$

Similarly let

$$\left\{\begin{matrix}\text{geometric}\\ \text{cocycles}\\ \text{over } \mathbb{Q}\end{matrix}\right\}^0 \subseteq \{\Omega_{4*}(X;\mathbb{Q}) \xrightarrow{\lambda} \mathbb{Q}\}$$

be the subgroup of homomorphisms[7] satisfying this periodicity relation.

Theorem 6.3 *Let X be a finite complex. Then there are natural isomorphisms*

$$K(X)_0 \cong \left\{\begin{matrix} \textit{geometric} \\ \textit{cocycles on } X \\ \textit{over } \mathbb{Q} \end{matrix}\right\}^0$$

$$K(X)\hat{} \cong \left\{\begin{matrix} \textit{finite} \\ \textit{geometric} \\ \textit{cocycles on } X \end{matrix}\right\}^0$$

$K(X)_0$ means the localization of $K(X)$ at $\{0\}$, $KO(X) \otimes \mathbb{Q}$. $K(X)\hat{}$ is the profinite completion of $KO(X)$ (with respect to groups of odd order).

Remarks:

a) The isomorphisms above are determined by the case $X = \text{pt}$, where

$$(\Omega_{4*}^{(\text{pt})} \xrightarrow{\text{signature}} \mathbb{Z}) \otimes \mathbb{Q} \cong 1 \in K(\text{pt})_0 = \mathbb{Q}\ .$$

b) The proof shows that the isomorphism holds for cohomology theories – that is, if we consider *geometric cocycles* with values on $4i+j$-manifolds subject to cobordism and the periodicity relation, then we are describing elements in

$$K^j(X)_0 \ \text{ or } \ K^j(X)\hat{}\ .$$

c) Such elements provide essentially independent information. However an integrality condition relating the rational and finite geometric cocycles

$$\begin{array}{ccc} \Omega_*(X;\mathbb{Q}) & \xrightarrow{\sigma_{\mathbb{Q}}} & \mathbb{Q} \\ \Big\downarrow t & & \Big\downarrow \\ \Omega_*(X;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\sigma_f} & \mathbb{Q}/\mathbb{Z} \end{array}$$

implies we can construct an element in

$$K(X) = KO(X) \otimes \mathbb{Z}\left[\tfrac{1}{2}\right].$$

(For we have the exact sequence

$$0 \to K(X) \xrightarrow{i \oplus j} K(X)\hat{} \oplus K(X)_0 \xrightarrow{l-c} K(X)_{(\text{Adele})_l} \to 0$$

corresponding to the arithmetic square of section 1 for groups, $l = \{\text{odd primes}\}$.

The integrality condition means that $\sigma_{\mathbb{Q}}$ is integral on the lattice

$$\Omega_*(X) \subseteq \Omega_*(X;\mathbb{Q})\,.$$

This in turn implies that the element determined by σ_f in

$$K(X)\hat{}\otimes\mathbb{Q} = K(X)_{(\text{Adele})_l}$$

is "rational" and is in fact the image of the element determined by $\sigma_{\mathbb{Q}}$ (upon tensoring with $\widehat{\mathbb{Z}}_l$).)

Corollary 6.4 *The signature invariant of a PL block bundle E over a finite complex*

$$\sigma(E) = \begin{array}{ccc} \Omega_*(E;\mathbb{Q}) & \longrightarrow & \mathbb{Q} \\ \big\downarrow & & \big\downarrow \\ \Omega_*(E;\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \end{array}$$

determines a canonical element in the K-theory with compact supports of E,

$$\Delta_E \in K_c^d(E)\,, \quad d = \text{dimension } E\,.$$

Proof of Corollary: We restrict the signature invariant to the subgroup of $\Omega_*(E^+;\mathbb{Q}/\mathbb{Z})$ generated by smooth $\mathbb{Z}/n$-manifolds of dimension $4i+d$,[8] n odd. The periodicity relation is clear. We have a geometric cocycle of "degree d".

Similarly for $\mathbb{Q}$.

We obtain elements in

$$K_c(E)\hat{} \;\text{ and }\; K_c(E)_0\,.$$

The integrality condition implies these can be combined to give a canonical element in

$$K_c(E)\,.$$

We proceed to the proof of the theorem.

First there is the construction of a map

$$K(X) \xrightarrow{\Delta} \begin{Bmatrix}\text{geometric} \\ \text{cocycles}\end{Bmatrix} .$$

In effect, the construction amounts to finding the Δ_E of the corollary when E is a vector bundle.

Let γ denote the canonical bundle over the Grassmannian BSO_{4n}.

There is a natural element

$$\Delta_{4n} \ = \ \frac{\Lambda^+ - \Lambda^-}{\Lambda^+ + \Lambda^-} \in KO_c(\gamma) \otimes \mathbb{Z}\left[\tfrac{1}{2}\right].$$

$\Lambda = \Lambda^+ \oplus \Lambda^-$ is the canonical splitting of the exterior algebra of γ into the eigenspaces of the "$*$ operator" for some Riemannian metric on γ.

Explanation:

a) $KO_c(\gamma) \equiv KO(\text{Thom space } \gamma) = KO(MSO_{4n})$ is isomorphic to the kernel of the restriction

$$KO(BSO_{4n}) \to KO(BSO_{4n-1}) .$$

b) Elements in $KO(BSO_{4n})$ can be defined be real representations of SO_{4n}. Λ denotes the exterior algebra regarded as a representation of SO_{4n}. Λ^+ and Λ^- are the ± 1 eigenspaces of the involution

$$\alpha : \Lambda \to \Lambda$$

given by Clifford multiplication with the volume element in Λ^n (i.e. $\alpha : \Lambda^i \to \Lambda^{n-i}$ is $(-1)^i *$ where $*$ is Hodge's operator).

c) $\Lambda^+ \oplus \Lambda^-$ has dimension 2^{4n} so that it is invertible in $KO(BSO_{4n})$ with $1/2$ adjoined.

d) Λ^+ and Λ^- are isomorphic as representations of SO_{4n-1} so

$$\frac{\Lambda^+ - \Lambda^-}{\Lambda^+ + \Lambda^-} \text{ lies in the kernel } \big(KO(BSO_{4n}) \to KO(BSO_{4n-1})\big) \otimes \mathbb{Z}\left[\tfrac{1}{2}\right].$$

e) The element $\Delta_{4n} \in KO_c(\gamma_{4n}) \otimes \mathbb{Z}\left[\tfrac{1}{2}\right]$ restricts to a generator of $KO_c(\mathbb{R}^{4n})$ which is 2^{-2n} times the "natural integral generator" (defined using $\Delta_+ - \Delta_-$ where $\Delta = \Delta_+ + \Delta_-$ is the "basic spin representation" defined using the Clifford algebra of $\mathbb{R}^{4n}$).

f) We shall think of K-theory $K^0, K^1, K^2, \dots$ (KO tensor $\mathbb{Z}[\frac{1}{2}]$) as being a cohomology theory – periodic of order four with the periodicity isomorphism defined by Δ_4 in $KO_c(\mathbb{R}^4) \otimes \mathbb{Z}[\frac{1}{2}]$

$$KO(X) \xrightarrow[x \mapsto x \cdot \Delta_4]{\cong} KO_c(X \times \mathbb{R}^4)\,.$$

g) The elements Δ_{4q} are multiplicative with respect to the natural map

$$BSO_{4q} \times BSO_{4r} \to BSO_{q+r}\,;$$

$$\Delta_{4(q+r)}/(\gamma_{4q} \times \gamma_{4r}) = \Delta_{4q} \times \Delta_{4r}$$

in $KO_c(\gamma_{4q} \times \gamma_{4r}) \otimes \mathbb{Z}[\frac{1}{2}]$.

(I am indebted to Atiyah and Segal for this discussion.)

The elements Δ_{4n} are defined in the universal spaces for bordism,

$$\{MSO_{4n}\} \;=\; \{\text{Thom space } \gamma_{4n}\}\,.$$

They define then a natural transformation

$$\left\{\begin{matrix}\text{smooth}\\ \text{bordism}\\ \text{theory}\end{matrix}\right\} \xrightarrow{\Delta} \{K\text{-theory}\}$$

either on the homology level, Δ_* or on the cohomology level, Δ^* and these are related by Alexander duality.

Thus smooth manifolds can be regarded as *cycles in K-theory*. Then any element in $K^i(X)$, ν may be evaluated in a manifold M of dimension n in X and we obtain

$$(M \to X) \cap \nu \in K^{n-i}(\mathrm{pt})\,.$$

If M is a $\mathbb{Z}/k$-manifold we obtain an element in $K^{n-i}(\mathrm{pt}) \otimes \mathbb{Z}/k$.[9]

Since the transformation is multiplicative $\big(\Delta_{4n} \cdot \Delta_{4l} \sim \Delta_{4(n+l)}\big)$,

$$\big((M \to X) \times (V \to \mathrm{pt})\big) \cap_\nu \;=\; \Delta(V) \cdot \big(\nu \cap (M \to X)\big)$$

where $\Delta(V)$ is in $K_v(\mathrm{pt})$, $v = \dim V$.

To calculate $\Delta(V)$ we recall the "character of Δ", the germ for calculating the characteristic classes of Δ is

$$\begin{aligned}
\text{germ of } \phi^{-1} ph\,\Delta &= \left(\phi^{-1} ph \tfrac{\Lambda^+ - \Lambda^-}{\Lambda^+ + \Lambda^-}\right)_{\text{germ}} \qquad (\phi = \text{Thom isomorphism}) \\
&= \left(\phi^{-1} ch \tfrac{\Lambda^+ - \Lambda^-}{\Lambda^+ + \Lambda^-} \otimes \mathbb{C}\right)_{\text{germ}} \\
&= \frac{1}{x} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\tanh x}{x} \\
&= \text{germ of } 1/L\text{-genus}\,.
\end{aligned}$$

To calculate $\Delta(V)$ we encounter the characteristic classes of the normal bundle, thus

$$\begin{aligned}
\Delta(V) &= \langle 1/L(\text{normal bundle of } V), V\rangle \\
&= \langle L(\text{tangent bundle of } V), V\rangle \\
&= \text{signature of } V\,,
\end{aligned}$$

by the famous calculation of Hirzebruch.

This proves each element in $K(X)$ determines[10] geometric cocycles,

$$K(X) \xrightarrow{\Delta_f} \left\{\begin{matrix}\text{finite} \\ \text{geometric} \\ \text{cocycles}\end{matrix}\right\}$$

$$K(X) \xrightarrow{\Delta_{\mathbb{Q}}} \left\{\begin{matrix}\text{geometric} \\ \text{cocycles over} \\ \mathbb{Q}\end{matrix}\right\}\,.$$

To analyze Δ_f consider again the natural transformation Δ_* from bordism $\otimes\,\mathbb{Z}\,[1/2]$ to K-theory. By obstruction theory in the universal spaces one sees directly that Δ^* is onto for zero dimensional cohomology.[11]
Namely: We have a map of universal spaces

$$M \xrightarrow{(\Delta)} B \quad (B = (BO)_{\{\text{odd primes}\}})$$

and

i) (Δ) is onto in homotopy,

$$\pi_* M = \Omega_* \otimes_\Omega \mathbb{Z}\left[\tfrac{1}{2}\right] \xrightarrow{\text{signature}} \pi_* B = K_*(\mathrm{pt})$$

$$V \to \Delta V$$

ii)

$$\begin{aligned}
&H^*\big(B; \pi_{*-1} \text{ fibre } (\Delta)\big) \\
&\cong H^*\big(B; (\text{ideal of signature zero manifolds})_{-1} \otimes \mathbb{Z}\left[\tfrac{1}{2}\right]\big) \\
&\cong 0\,.
\end{aligned}$$

We can find then a universal cross section.

It follows by Alexander duality when X is a finite complex that Δ_* is onto for all dimensions.

In the kernel of Δ_* we have the elements

$$\{(V \to \mathrm{pt}) \times (M \to X), \text{signature } V = 0\}\,.$$

In the elegant expression of Conner-Floyd we then have a map,

$$\Omega_*(X) \otimes_{\Omega_*} \mathbb{Z}\left[\tfrac{1}{2}\right] \xrightarrow{\Delta_*} K_*(X)\,,$$

which is a naturally split surjection.

One passes again to cohomology in dimension zero to see that this is injective. One sees by a rational calculation that Δ_* is injective for the case

$$X = (MSO_N)_{\text{large skeleton}}\,.$$

The result follows for any X because we can calculate $\Delta_*(\mu)$ considering μ as a map

$$h\text{-fold suspension } X \xrightarrow{\mu} (MSO_N)_{\text{skeleton}}\,.$$

The natural splitting of Δ^* implies

$$\mu^*(\text{kernel } \Delta^*) = (\text{kernel } \Delta^*) \cap \text{image } \mu^*$$

in $\Omega^* \otimes_{\Omega_*} \mathbb{Z}\,[1/2]$.

The left hand side is zero (Δ^* is injective for MSO_N) while image μ^* is a general element. Thus kernel $\Delta^* = 0$, and

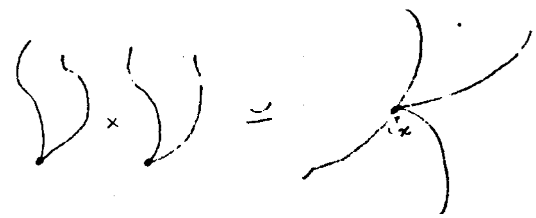
$$\Omega_*(X) \otimes_{\Omega_*} \mathbb{Z}\left[\tfrac{1}{2}\right] \xrightarrow[\cong]{\Delta_*} K_*(X)\,.$$[12]

From the definition of mod n homology (the ordinary homology of X smash the Moore space) we see that

$$\Omega_*(X;\mathbb{Z}/n)\otimes_{\Omega_*} \cong K_*(X;\mathbb{Z}/n)\,.$$

Now $\Omega_*(X;\mathbb{Z}/n)$ is a "multiplicative theory". We can form the product of two $\mathbb{Z}/n$-manifolds.

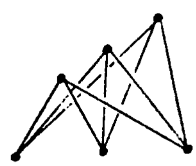
For example



and,

one obtains a $\mathbb{Z}/n$-manifold except at points like x. The normal link at x is

$$\mathbb{Z}/n * \mathbb{Z}/n\,, \quad \text{a } \mathbb{Z}/n\text{-manifold of dimension one}\,.$$



(Z/3 * Z/3)

Now $\mathbb{Z}/n*\mathbb{Z}/n$ bounds a $\mathbb{Z}/n$-manifold (the cobordism is essentially unique) so we can remove the singularity by replacing the cone over $\mathbb{Z}/n * \mathbb{Z}/n$ with this cobordism at each bad point x.

We obtain then a product of $\mathbb{Z}/n$-manifolds.

Thus

$$K_*(X;\mathbb{Z}_n) \quad \Omega_*(X;\mathbb{Z}/n) \otimes_{\Omega_*} \mathbb{Z}/n$$

has a natural co-multiplication[13]. In particular, it is a $\mathbb{Z}/n$-module.

We can then define a natural evaluation

$$K^*(X;\mathbb{Z}/n) \xrightarrow{e} \mathrm{Hom}\left(K_*(X;\mathbb{Z}/n);\mathbb{Z}/n\right).$$

The right hand side is a cohomology theory – "Hom of $\mathbb{Z}/n$-modules into $\mathbb{Z}/n$ is an exact functor". We have an isomorphism for the point – this is Bott periodicity mod n. Thus e is an isomorphism, *Pontrjagin Duality for K-theory.*

Therefore

$$\begin{aligned} K^*(X;\mathbb{Z}/n) &\cong \operatorname{Hom}\big(K_*(X;\mathbb{Z}/n),\mathbb{Z}/n\big) \\ &\cong \operatorname{Hom}\big(K_*(X;\mathbb{Z}/n),\mathbb{Q}/\mathbb{Z}\big) \\ &\cong \operatorname{Hom}\big(\Omega_*(X;\mathbb{Z}/n)\otimes_{\Omega_*}\mathbb{Z}/n,\mathbb{Q}/\mathbb{Z}\big) \end{aligned}$$

So

$$\begin{aligned} \varprojlim_{n \text{ odd}} K(X;\mathbb{Z}/n) &\cong \varprojlim_{n \text{ odd}} \operatorname{Hom}\big(\Omega_*(X;\mathbb{Z}/n)\otimes_{\Omega_*}\mathbb{Z},\mathbb{Q}/\mathbb{Z}\big) \\ &\cong \operatorname{Hom}\big(\varinjlim_{n}\Omega_*(X;\mathbb{Z}/n)\otimes_{\Omega_*}\mathbb{Z},\mathbb{Q}/\mathbb{Z}\big) \\ &\subseteq \operatorname{Hom}\big(\varinjlim_{n}\Omega_*(X;\mathbb{Z}/n),\mathbb{Q}/\mathbb{Z}\big) \\ &= \operatorname{Hom}\big(\Omega^l_*(X;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}\big)\,. \end{aligned}$$

Now $K(X)$ is finitely generated so we can tensor the $\mathbb{Z}/n$ coefficient sequence with $\widehat{\mathbb{Z}}$, and pass to the inverse limit over odd n (the groups are compact) to obtain the isomorphism

$$K^*(X)\hat{} \cong \varprojlim_{n \text{ odd}} K^*(X;\mathbb{Z}/n)\,.$$

But then the isomorphism above identifies the profinite K-theory in dimension i with the finite geometric cocycles of degree i.

This proves the profinite statement.

We note that the rational statement is a rational cohomology calculation.

One checks that there is a unique element Σ in $K(X)\otimes\mathbb{Q}$ so that

$$\sigma_{\mathbb{Q}}(M \xrightarrow{f} X) = \langle L_M \cdot ph\, f^*\Sigma, M\rangle$$

for each geometric cocycle $\sigma_{\mathbb{Q}}$. (L_M is the L genus of the tangent bundle of M.) Q. E. D.

Remark: Atiyah has an interesting formulation of the problem of giving a direct *geometric* construction of Δ_E for a PL bundle. Consider a compact polyhedral submanifold of $\mathbb{R}^N$.

$M \subset \mathbb{R}^N$ with "PL normal bundle E". If X is a non-singular point (x in the interior of a top dimensional simplex) we have the natural volume form γ_x on the normal space of M at x. γ_x may be regarded as an element in the Clifford algebra of the tangent space to $\mathbb{R}^N$ at these points of M.

γ_x satisfies $\gamma_x^2 = 1$ an defines a splitting of $\Lambda(\mathbb{R}^N_x)$,

$$\Lambda(\mathbb{R}^N_x) \;=\; (\Lambda\tau_x \otimes \Lambda^+\nu_x) \oplus (\Lambda\tau_x \otimes \Lambda^-\nu_x)\,,$$

τ_x the tangent space to M at x, Λ^+, Λ^- as above.

The formal difference of these vector spaces is

$$\begin{aligned}\Lambda\tau_x \otimes \Lambda^+\nu_x - \Lambda\tau_x \otimes \Lambda^-\nu_x &= \Lambda\tau_x \otimes (\Lambda^+\nu_x - \Lambda^-\nu_x) \\ &\text{“="}\; \frac{\Lambda^+ - \Lambda^-}{\Lambda^+ + \Lambda^-}(\nu_x)\,,\end{aligned}$$

the local form of the element above.

So the problem of constructing Δ_E may be "reformulated" –

extend the function γ_x over all of M to the following –

i) for each point y there is a unit γ_y in the Clifford algebra of $\mathbb{R}^N$ at y.

ii) γ_y satisfies $\gamma_y^2 = 1$ (or at least γ_y^2 lies in a contractible region about the identity in the units of the Clifford algebra).

iii) γ_y is constructed by connecting the local geometry of M at y with the homotopy of regions in the Clifford algebra of $\mathbb{R}^N$ at y.

We note that the case (manifold, normal bundle in Euclidean space) is generic for the problem of constructing Δ_E, so this would give the construction for any PL bundle E.

The profinite and rational theory of PL bundles

We continue in this mode of considering a problem in terms of its rational and profinite aspects and the compatibility between them.

This approach is applicable to the theory of piecewise linear bundles because of the existence of a classifying space,

$$BPL_n \;=\; \text{Grassmannian of "}PL\ n\text{-planes" in } \mathbb{R}^\infty\,.$$

Thus we can define the profinite completion of the set of n-dimensional PL bundles over X by

$$[X, \text{profinite completion } BPL_n]\,.$$

For each locally finite polyhedron we obtain a natural compact Hausdorff space. In the oriented theory the "geometric points" corresponding to actual bundles form a dense subspace[14]. The topological affinities induced on these points correspond to subtle homotopy theoretical connections between the bundles.

To interpret the general point recall we have the natural map of classifying spaces,

$$(BSPL_n)\hat{} \to (BSG_n)\hat{}\,.^{15}$$

So a profinite PL bundle determines a completed spherical fibration. We can think of a completed PL bundle as having its underlying spherical homotopy type plus some extra (mysterious) geometrical structure.

The connectivity of SPL_n implies the splitting

$$\left\{\begin{matrix}\text{oriented}\\ \text{profinite}\\ PL_n\text{-theory}\end{matrix}\right\} \cong \prod_p \left\{\begin{matrix}\text{oriented}\\ p\text{-adic}\\ PL_n\text{-theory}\end{matrix}\right\}\,.$$

We will consider only the p-adic components for p odd – the 2-adic component is still rather elusive.

In the discussion the profinite completion $X\hat{}$ will always mean with respect to groups of odd order. We occasionally include the prime 2 for points relating to the 2-adic linear Adams Conjecture.

We can also define "rational PL_n bundles" over X

$$\cong [X, (BPL_n)_{\text{localized at zero}}]\,.$$[16]

Intuitively, the rationalization of a bundle contains the "infinite order" information in the bundle.

Recall that the arithmetic square tells us we can assemble an actual PL bundle from a rational bundle and a profinite bundle which satisfy a "rational coherence" condition.

In the case of PL_n bundles this coherence condition is expressible in terms of characteristic classes as for spherical fibrations in section 4. This follows from the

Theorem 6.5 ($\mathbb{Q}$) *The rational characteristic classes*

$$L_1, L_2, \ldots, L_i \in H^{4i}(B;\mathbb{Q})$$

and the 'homotopy class'

$$\begin{array}{lll} \textit{Euler class} & \chi \in H^n(B;\mathbb{Q}) & n \textit{ even} \\ \textit{Hopf class} & \mathscr{H} \in H^{2n-2}(B;\mathbb{Q}) & n \textit{ odd} \end{array}$$

form a complete set of rational invariants of the n-dimensional oriented PL bundle E over B.

In fact the oriented rational PL_n bundle theory is isomorphic to the corresponding product of cohomology theories

$$\begin{array}{ll} n \textit{ even} & H^n(\quad;\mathbb{Q}) \times \prod H^{4*}(\quad;\mathbb{Q}) \\ n \textit{ odd} & H^{2n-2}(\quad;\mathbb{Q}) \times \prod H^{4*}(\quad;\mathbb{Q})\,. \end{array}$$

Remarks:

i) We note the unoriented rational theory described in the footnote is obtained for n even by twisting the Euler class using homomorphisms

$$\pi_1(\text{base}) \to \mathbb{Z}/2 = \{\pm 1\} \subseteq \mathbb{Q}^*\,.$$

For n odd no twisting is required because (-1) acts trivially on the Hopf class.

ii) We also note that the relations for even dimensions $2n$

$$\chi^2 = \text{“}n^{\text{th}} \text{ Pontrjagin class”}$$

e.g. $\chi^2 = 3L_1 \qquad (n=1)$

$$\chi^2 = \frac{45L_2 + 9L_1^2}{7} \quad (n=2)$$

$$\vdots$$

do not hold for block bundles, as they do for vector bundles. It seems reasonable to conjecture that these relations do hold in the intermediate theory of PL microbundles – where we have a geometric projection.

The information carried by the rational L classes is precisely the integral signatures of closed manifold intersections with the zero section.

The Euler class measures the transversal intersection of B with itself.

The invariants for the profinite PL theory

Consider the two invariants of a PL bundle E over B,

i) the K-theory orientation class

$$\Delta_E \in \widehat{K}_c(E)\,,$$

ii) the completed fibre homotopy type

$$(E - B) \rightsquigarrow B$$

after fibrewise completion.

We have already remarked that the second invariant is defined for a profinite PL bundle over B.

The first is also defined – to see this appeal again to the universal example E_n over $BSPL_n$.

It is easy to see that $\widehat{K}_c E_n$ is isomorphic to the corresponding group for the completed spherical fibration over $(BSPL_n)\hat{}$.

THEOREM 6.5 ($\widehat{\mathbb{Z}}$) *The two invariants of a profinite PL bundle of dimension n*

i) fibre homotopy type

ii) natural K-orientation

form a complete set of invariants, $n > 2$.[17]

In fact we have an isomorphism of theories

$$\left\{\begin{matrix} \textit{profinite} \\ PL_n \ \textit{bundles} \end{matrix}\right\} \cong \left\{\begin{matrix} K \ \textit{oriented} \\ \widehat{S}^{n-1} \ \textit{fibrations} \end{matrix}\right\} .$$

We see that the different ways to put (profinite) geometric structure in the homotopy type $E - B \to B$ correspond precisely to the different $\widehat{K}$-orientations.

We recall that an orientation determines a Thom isomorphism

$$\widehat{K}(B) \overset{\Delta}{\cong} \widehat{K}(E)\,.$$

Since Δ_E was constructed from the signature invariant $\sigma(E)$, the K-theory Thom isomorphism is compatible with the geometric Thom isomorphism discussed above.

We then see the aesthetically pleasing point that at the odd primes the pure geometric information[18] in a PL bundle is carried by the $\mathbb{Z}/n$ signature of $\mathbb{Z}/n$ intersections with the zero section (n odd).

Moreover we see that any assignment of signatures to "virtual intersections" (with a subsequent geometric zero section) for a spherical fibration satisfying cobordism invariance and the periodicity relation can be realized by a homotopy equivalent profinite PL bundle.

From the homotopy theoretical point of view we see that the obstruction to realizing a completed spherical fibration as the "complement of the zero section" in a PL bundle is precisely the obstruction to K-orientability.

Remark: The obstruction to K-orientability can be measured by a canonical characteristic class

$$k_1(\xi) \in \widehat{K}^1(B)\,.$$

where ξ is any complete spherical fibration over B.

Thus the image of profinite PL bundles in spherical fibrations is the locus

$$k_1(\xi) \;=\; 0\,,$$

the collection of K-orientable fibrations.

We discuss the proof in the "K-orientation sequence" subsection.

Galois symmetry in profinite PL theory

We use the K-theory characterization to find Galois symmetry in the profinite piecewise linear theory.

Recall the action of the Abelianized Galois group

$$\widehat{\mathbb{Z}}^* \text{ in } \widehat{K}(X)\,, \quad x \mapsto x^\alpha \ \ \alpha \in \widehat{\mathbb{Z}}^*\,.$$

This action extends to the cohomology theory – K^*-theory

$$\widehat{K}^n(X) \to \widehat{K}^n(X)\,.$$

Define

$$(x)^\alpha = \alpha^{-n/2} x^\alpha \ \text{ if } n = 4k$$

using the periodicity isomorphism

$$\widehat{K}^n(X) \cong \widehat{K}^0(X)\,, \quad (x) \sim x\,.$$

For other dimensions the action of $\widehat{\mathbb{Z}}^*$ is defined by the suspension isomorphism. The factor $\alpha^{-n/2}$ insures the action commutes with suspension isomorphism in K-theory for all values of n.

The group $\widehat{\mathbb{Z}}^* \times \widehat{\mathbb{Z}}^*$ acts on the profinite PL_n theory by acting on the invariants,

$$\big(\Delta_E, (E-B) \to B\big) \overset{(\alpha,\beta)}{\longmapsto} \big(\beta\Delta_E^\alpha, (E-B) \to B\big)\,,$$

$$\beta, \alpha \in \widehat{\mathbb{Z}}^*\,, \quad \Delta_E \in \widehat{K}^n(E^+)\,,$$

E^+ the Thom space of the spherical fibration.

We compare this action with the action of $\mathrm{Gal}\,(\bar{\mathbb{Q}}/\,\mathbb{Q})$ in profinite vector bundle theory. Also recall the action of $\widehat{\mathbb{Z}}^*$ on oriented $\widehat{S}^{n-1}$-fibrations,

$$(\xi, U_\xi) \mapsto (\xi, \alpha U_\xi)\,, \ \ \alpha \in \widehat{\mathbb{Z}}^*\,.$$

Theorem 6.6 (Generalized Adams Conjecture)
Consider the natural map of oriented profinite theories with actions

$$\begin{array}{ccccc} \left\{\begin{matrix}\textit{vector bundles}\\ \textit{of dimension } n\end{matrix}\right\} & \xrightarrow{t} & \left\{\begin{matrix}\textit{profinite}\\ PL_n \textit{ bundles}\end{matrix}\right\} & \xrightarrow{h} & \left\{\begin{matrix}\textit{completed}\\ S^{n-1}\textit{-fibrations}\end{matrix}\right\} \\ \mathrm{Gal}\,(\bar{\mathbb{Q}}/\,\mathbb{Q}) & & \widehat{\mathbb{Z}}^* \times \widehat{\mathbb{Z}}^* & & \widehat{\mathbb{Z}}^* \end{array}\,.$$

Let σ belong to $\mathrm{Gal}\,(\bar{\mathbb{Q}}/\,\mathbb{Q})$ with Abelianization $\alpha \in \widehat{\mathbb{Z}}^*$. Let V and E denote n-dimensional vector and PL bundles respectively. Then

$$t(V^\sigma) = \begin{cases} \alpha^{n/2} \cdot t(V)^\alpha & n \textit{ even} \\ \alpha^{(n-1)/2} \cdot t(V)^\alpha & n \textit{ odd} \end{cases}$$

$$h(\beta E^\alpha) = \beta h(E)\,.$$

Corollary 1 *The action of* $\mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$ *on the "topological type" of vector bundle is Abelian.*

Corollary 2 *The "diagonal action" in* PL_n *theory,*

$$E \mapsto \alpha^{n/2} E^{\alpha} \qquad n \text{ even}$$

$$E \mapsto \alpha^{(n-1)/2} E^{\alpha} \quad n \text{ odd}$$

is "algebraic". It is compatible with the action of the Galois group on vector bundles.

Note:

i) The signature invariant together with a 2-primary (Arf) invariant were used by the author to show the topological invariance of geometric structures (triangulations) in bundles under a "no 3-cycle of order 2" hypothesis (1966). Since we are only concerned with odd primes here "homeomorphism implies PL homeomorphism". This explains the wording of Corollary 1.

   These invariants also gave results about triangulations of simply connected manifolds. This fundamental group hypothesis was essentially removed[19] by the more intrinsic arguments of Kirby and Siebenmann (1968, 1969) who also *constructed* triangulations.

ii) We would like to reformulate Corollary 2 in the following way. Think of the symmetries in these theories of bundles as being described by group actions in the classifying spaces. Then the group action on the PL Grassmannian corresponding to

$$V \mapsto \alpha^{n/2} V^{\alpha} \quad \alpha \in \widehat{\mathbb{Z}}^* \;\; n \text{ even}$$

   keeps the image of the linear Grassmannian "invariant" and there gives the action of the Galois group, $\mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$.

If we let n approach $infty$ these symmetries in profinite bundle theory simplify,

i) the action of $\mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$ on stable vector bundle Abelianizes,

$$\widehat{\mathbb{Z}}^* \text{ in } K(X)\hat{}\,.$$

ii) The $\widehat{\mathbb{Z}}^* \times \widehat{\mathbb{Z}}^*$ action in PL_n bundles becomes a $\widehat{\mathbb{Z}}^*$ action

$$\big(\Delta_E, (E{-}B) \to B\big)^{(\alpha,\beta)} = \big(\alpha\Delta_E^{\beta}, (E{-}B) \to B\big) \cong \big(\Delta_E^{\beta}, (E{-}B) \to B\big)\,.$$

iii) The action of $\widehat{\mathbb{Z}}^*$ on $\widehat{S}^{n-1}$-fibrations becomes trivial.

Theorem 6.7 *We obtain an equivariant sequence of theories*

$$\begin{tikzcd}
\left\{\begin{matrix}\textit{stable vector}\\ \textit{bundles}\end{matrix}\right\} \arrow[r] & \left\{\begin{matrix}\textit{stable PL}\\ \textit{bundles}\end{matrix}\right\} \arrow[r] & \left\{\begin{matrix}\textit{stable spherical}\\ \textit{fibrations}\end{matrix}\right\} \\
& \widehat{\mathbb{Z}}^*\textit{-action} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where

α *is induced by the action of* $\mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$ *on the "real Grassmannian". This action becoming Abelian in the limit of increasing dimensions.*

β *is induced by the action of* $\widehat{\mathbb{Z}}^*$ *on the signature invariant* $\sigma(E)$ *via its identification with the* K*-theory class* Δ_E.

γ *is induced by changing orientations and is the trivial action stably.*

Corollary *Stably, all the natural symmetry in PL is "algebraic (or Galois) symmetry".*

Problem It would be interesting to calculate the effect of the Galois group on the signature invariant $\sigma(E)$ per se,

$$\sigma(E^\alpha) \;=\; ?\big(\sigma(E)\big)\,.$$

Note These theorems give a rather explicit formula for measuring the effect of the Galois group on the topological type of a vector bundle at odd primes.

This might be useful for questions about algebraic vector bundles in characteristic 2.

The proofs are discussed in the "Equivariance" section below.

Normal Invariants (Periodicity and the Galois Group)

We consider for a moment the theories of fibre homotopy equivalences between bundles (normal invariants).

These theories are more closely connected to the geometric questions about manifolds which motivated (and initiated) this work. Moreover, the two real ingredients in the ensuing calculations find their most natural expression here.

Geometrically a normal invariant over the compact manifold M (with or without boundary) is a cobordism class of normal maps, a degree one map

$$L \xrightarrow{f} M$$

covered by a bundle map

$$\begin{pmatrix}\text{normal bundle of } L\\ \text{in euclidean space}\end{pmatrix} \xrightarrow{bf} \begin{pmatrix}\text{any bundle}\\ \text{over } M\end{pmatrix}.$$

We may talk about smooth, PL, or topological normal invariants, sN, plN or tN. Normal invariants form a group (make $f \times f'$ transversal to the diagonal in $M \times M$) and have natural geometric invariants. For example for each $\mathbb{Z}/n$-manifold V in M consider any "cobordism invariant" of the quadratic form in the transversal inverse image submanifold $f^{-1}(V)$.

From the homotopy theoretical point of view normal invariants correspond to fibre homotopy equivalences between bundles over M,

$$E \xrightarrow{f} F.$$

An equivalence is a fibre homotopy commutative diagram (over M)

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\cong"'] & F \arrow[d, "\cong"] \\ E' \arrow[r, "f'"] & F' \end{tikzcd}$$

where the vertical maps are bundle isomorphisms. These homotopy normal invariants can be added by Whitney sum and the Grothendieck groups are classified by maps into a universal space

$$sN \sim [\quad, G/O] \qquad plN \sim [\quad, G/PL] \qquad tN \sim [\quad, G/Top]$$

depending on the category of bundles considered.

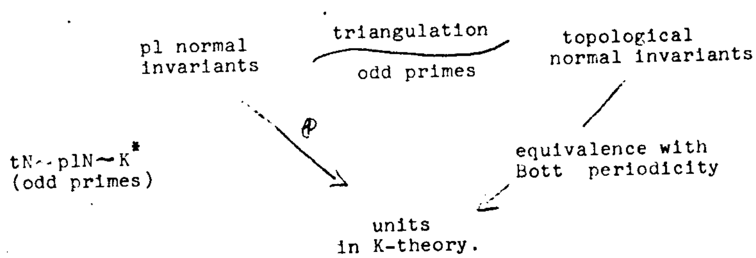
The geometric description of normal invariants leads to a parametrization of the manifolds (smooth, PL, or topological) within a simply connected homotopy type. This uses the "surgery on a map" technique of Browder and Novikov. [20]

The homotopy description may now be used to study these invariants of manifolds.

First there is an interplay between geometry and homotopy theory – it turns out that the geometric invariants suggested above (signatures of forms over $\mathbb{R}$ and Arf invariants of forms over $\mathbb{Z}/2$) *give a complete set of numerical invariants of a normal cobordism class in either the PL or topological context.* (This is contained in Sullivan, "Geometric Topology Notes" Princeton 1967 for PL case, and uses the more recent work of Kirby and Siebenmann constructing triangulations for the topological case.)

The relations between the invariants are described by cobordism and a periodicity formula. This is the *geometric periodicity*, which is a perfect four-fold periodicity in the topological theory (even at 2).

We will use this at odd primes where the periodicity may be interpreted in terms of natural equivalences with K-theory



This isomorphism implies that we have an action of the Galois group $\widehat{\mathbb{Z}}^*$ in the topological normal invariants.

The map $\mathscr{P}$ is constructed from the transversality invariants (the signature invariant, again) using the geometric characterization of K-theory above.

The K-orientation of a PL bundle developed as a generalization of this odd primary calculation of PL normal invariants.

The isomorphism $\mathscr{P}$ is the first "ingredient" in our calculations below. It is discussed in more detail below.

The second ingredient was constructed in section 5.

The homotopy description of normal invariants can be profinitely completed. The discussion of the Adams Conjecture above shows the following.

For each profinite vector bundle v over M and $\alpha \in \widehat{\mathbb{Z}}^*$ we have a canonical element

$$(v^\alpha \sim v)$$

in the group of profinite smooth normal invariants over M.[21]

(Recall that the fibre homotopy equivalence

$$v^\alpha \to v \quad \dim v = n$$

was canonically determined by the isomorphism induced by α on BO_{n-1}.)

We only require the existence of these natural "Galois elements" for the calculation below. However, the precise structure of these elements should be important for future 'twisted calculations'.

We hope to pursue this structure and the quasi action of the Galois group in a later discussion of manifolds.

Some Consequences of Periodicity and Galois Symmetry

We will use the work up to now to study the stable geometric theories.

Recall we have the signature invariant of a PL bundle leading to the K-orientation theorem[22] and the odd primary periodicity for normal invariants.

We also have the Galois symmetry in the homotopy types of the algebraic varieties,

Grassmannian of k-planes in n-space

leading to an action of the Galois group $\widehat{\mathbb{Z}}^*$ on vector bundles.

We combined these to obtain a compatible action of the Galois group on the linear and the piecewise linear theory and the canonical fibre homotopy equivalence between conjugate bundles

$$x^\alpha \sim x\,.$$

Consider the stable profinite theories

$$\begin{tikzcd}
\left\{\begin{matrix}\text{smooth}\\ \text{bundles}\end{matrix}\right\} \arrow[rr] \arrow[dr] & & \left\{\begin{matrix}\text{spherical}\\ \text{fibrations}\end{matrix}\right\} \\
& \left\{\begin{matrix}\text{topological}\\ \text{bundles}\end{matrix}\right\} \arrow[ur] &
\end{tikzcd}.$$

We may appeal to the homotopy equivalences (linearizations)

$$\left\{\begin{matrix}\text{group of}\\ \text{diffeomorphisms}\\ \text{of } \mathbb{R}^n\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{group of linear}\\ \text{isomorphisms}\\ \text{of } \mathbb{R}^n\end{matrix}\right\} \qquad \text{(Newton)}$$

$$\left\{\begin{matrix}\text{group of}\\ \text{homeomorphisms}\\ \text{of } \mathbb{R}^n\end{matrix}\right\} \simeq^{23} \left\{\begin{matrix}\text{group of}\\ \text{piecewise linear}\\ \text{isomorphisms}\\ \text{of } \mathbb{R}^n\end{matrix}\right\} \qquad \text{(Kirby-Siebenmann)}$$

to identify this array with the more computable[24] array

$$\begin{tikzcd}
\left\{\begin{matrix}\text{vector}\\ \text{bundles}\end{matrix}\right\} \arrow[rr] \arrow[dr] & & \left\{\begin{matrix}\text{spherical}\\ \text{fibrations}\end{matrix}\right\} \\
& \left\{\begin{matrix}\text{piecewise}\\ \text{linear}\\ \text{bundles}\end{matrix}\right\} \arrow[ur] &
\end{tikzcd}$$

we have been studying.

We consider these interchangeably.

THEOREM 6.8 *The kernels of the natural maps of the stable profinite bundle theories (the J-homomorphisms)*

$$\begin{tikzcd}
& \begin{matrix}\text{smooth}\\ \text{bundles}\end{matrix} \arrow[ddr] \arrow[dl, phantom, "\cong"] & & \begin{matrix}\text{topological}\\ \text{bundles}\end{matrix} \arrow[ddl] \arrow[dr, phantom, "\cong"] & \\
\begin{matrix}\text{vector}\\ \text{bundles}\end{matrix} \arrow[drr] & & & & \begin{matrix}\text{piecewise}\\ \text{linear}\\ \text{bundles}\end{matrix} \arrow[dll] \\
& & \begin{matrix}\text{spherical}\\ \text{fibrations}\end{matrix} & &
\end{tikzcd}$$

consist in each case[25] *of the subgroups generated by differences of conjugates* $\{x - x^{\alpha},\ \alpha \in$ *Galois group*, $\widehat{\mathbb{Z}}^{*}\}$.

Note on content

We have seen in section 5 that $x^{\alpha} - x$ has a (canonical) fibre homotopy trivialization for vector bundles.

Assuming this Adams essentially proved Theorem 6.8 for vector bundles by a very interesting K-theory calculation.

On the other hand it is clear from our definition of the Galois action for PL or Top that conjugate elements are fibre homotopy equivalent.

The burden of the proof is then the other half of the statement.

It turns out that the point is compatibility of the Galois action. This neatly reduces the PL case to the linear case and a modified form of Adams calculations.

To proceed to the proof of this theorem and to further study the interrelationship between the stable theories we decompose everything using the roots of unity in $\widehat{\mathbb{Z}}$.

Consider the p-adic component for one odd prime p. Recall from section 1, $\widehat{\mathbb{Z}}_p^{*}$ has torsion subgroup

$$F_p^{*} \cong \mathbb{Z}/(p-1)\,.$$

Let $\xi_p = \xi$ be a primitive $(p-1)^{\mathrm{st}}$ root of unity in $\widehat{\mathbb{Z}}_p$ and consider any $\widehat{\mathbb{Z}}_p$-module K in which $\widehat{\mathbb{Z}}_p^{*}$ acts by homomorphisms. Denote the operation on K

$$x \mapsto x^{\xi}$$

by T and consider

$$\pi_{\xi^i} \;=\; \prod_{j \neq i} \frac{T - \xi^j}{\xi^i - \xi^j} \qquad i = 0, \ldots, p-2\,.$$

These form a system of orthogonal projections which decompose K

$$K \;=\; K_1 + (K_{\xi^1} + \cdots + K_{\xi^{p-2}})\,.$$

K_{ξ^i} is the eigenspace of T with eigenvalue ξ^i ($K_{\xi^i} = \pi_{\xi^i} K$). We group the ξ-eigenspaces to obtain the invariant splitting

$$K \;=\; K_1 + K_{\xi_p}\,, \text{ independent of choice of } \xi_p\,.$$

If K is a $\widehat{\mathbb{Z}}$-module on which the Galois group $\widehat{\mathbb{Z}}^*$ acts by a product of actions of $\widehat{\mathbb{Z}}^*$ on K_p, we can form this decomposition at each odd prime, collect the result and obtain a natural splitting

$$K \;=\; K_1 + K_\xi\,,$$

where $K_1 = \prod_p (K_p)_1$, $K_\xi = \prod_p (K_p)_{\xi_p}$.

We obtain in this way *natural* splittings of

profinite vector bundles

profinite topological bundles

topological normal invariants (profinite)

$$\begin{array}{ll} K_0 & K_0 \cong (K_0)_1 + (K_0)_\xi \\ K_{top} & K_{top} \cong (K_{top})_1 + (K_{top})_\xi \\ tN & tN \cong (tN)_1 + (tN)_\xi\,. \end{array}$$[26]

To describe the “interconnections between these groups” we consider another natural subgroup of K_{top}, “the subgroup of Galois equivariant bundles”.

Let $\mathscr{C}^1$ denote the subgroup of topological bundles $\{E\}$ whose natural Thom isomorphism

$$K(\text{base } E) \xrightarrow[\cong]{\cup\Delta_E} K(\text{Thom space } E)$$

is *equivariant with respect to the action of the Galois group.*

Note that this means

$$(x \cdot \Delta_E)^\alpha \;=\; x^\alpha \cdot \Delta_E$$

or $\Delta_E^\alpha = \Delta_E$. Thus the identity map provides an isomorphism between E and E^α, i.e.

$$\mathscr{C}^1 \subseteq \left\{\begin{array}{c}\text{fixed points of}\\ \text{Galois action on}\\ \text{topological bundles}\end{array}\right\}.$$

In particular $\mathscr{C}^1 \subseteq (K_{top})_1$, the subgroup fixed by elements of finite order.

Consider now the more geometric diagram of theories

$$\begin{tikzcd}
 & \left\{\begin{matrix}\text{vector}\\ \text{bundles}\end{matrix}\right\} \arrow[dr, "\theta"] & \\
 & & \left\{\begin{matrix}\text{topological}\\ \text{bundles}\end{matrix}\right\} \\
\left\{\begin{matrix}\text{theory of}\\ \text{fibre homotopy}\\ \text{equivalences}\end{matrix}\right\} \arrow[r, phantom, "\cong"] & \left\{\begin{matrix}\text{topological normal}\\ \text{invariants}\end{matrix}\right\} \arrow[ur, "j"'] &
\end{tikzcd}$$

$$\theta(\text{vector bundle}) = \text{underlying } \mathbb{R}^n \text{ bundle}$$

$$j\begin{pmatrix}\text{fibre homotopy}\\ \text{equivalence } E \sim F\end{pmatrix} = E - F\,.$$

By definition image θ consists of the smoothable bundles and image j consists of fibre homotopically trivial bundles.

Recall that the Galois group $\widehat{\mathbb{Z}}^*$ acts compatibly on this diagram of theories.

Theorem 6.9 (decomposition theorem)

a) θ is an injection on the 1 *component and*

$$(K_{top})_1 \ = \ \theta(K_0)_1 \oplus \mathscr{C}^1\,.$$

b) j is an injection on the ξ component and

$$(K_{top})_\xi \ = \ j(tN)_\xi\,.$$

c) $(\text{image}\,\theta)_\xi \subset (\text{image}\,j)_\xi$, *"a vector bundle at ξ is fibre homotopy trivial".*

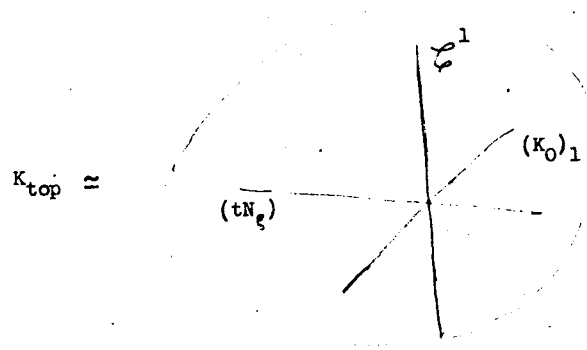
d) $(\text{image}\,j)_1 \subseteq (\text{image}\,\theta)_1$, *"a homotopy trivial bundle at* 1 *is smoothable".*

In other words the subgroups

$$\mathscr{C}^1 = \{\text{equivariant bundles}\}$$

$$\text{image}\,t + \text{image}\,\theta \ = \ \{\text{smoothable or homotopy trivial bundles}\}$$

are complementary. Moreover, the latter subgroup canonically splits into a subgroup of smoothable bundles and a subgroup of homotopy trivial bundles



NOTE: Since K_{top}, $(tN)_\xi$, and $(K_0)_1$ are representable functors so is $\mathscr{C}^1$. The homotopy groups of $\mathscr{C}^1$ are the torsion subgroups of $K_{top}(S^i)$. These can be described in the symmetrical fashion

$$\mathscr{C}^1 = \frac{\text{stable } (i-1) \text{ stem}}{\text{image } j} \cong \frac{\text{group of } (i-1) \text{ exotic spheres}}{\partial \text{ parallelizable}}.$$

For example $(\mathscr{C}^1)_p$ is $\big(2p(p-1)-2\big)$-connected. This follows directly from the K-orientation sequence below. Calculations like these were also made by G. Brumfiel – in fact these motivated the splittings.

COROLLARY 1 *$\mathscr{C}^1$ injects into homotopy theory and into "topological theory mod smooth theory".*

Analogous to this statement that the equivariant bundles are homotopically distinct we have

COROLLARY 2 *Distinct stable vector bundles fixed by all the elements of finite order in the Galois group are also topologically distinct.*

These corollaries follow formally from Theorem 6.9.

The K-theory characteristic class of a topological bundle

In order to prove the decomposition theorem we measure the "distance" of a topological bundle from the subgroup of bundles with an equivariant Thom isomorphism.

The key invariant is the K-theory characteristic class Θ_E defined for any topological bundle E.[27]

Consider the function

$$\widehat{\mathbb{Z}}^* \xrightarrow{\Theta_E} K^*(\text{Base})$$

defined by the equation

$$\Theta_\alpha \cdot \Delta_E = \Delta_E^\alpha \qquad \alpha \in \widehat{\mathbb{Z}}^* .$$

Θ_E measures the non-equivariance of the Thom isomorphism defined by Δ_E

$$x \mapsto x \cdot \Delta_E .$$

We note that

o) Θ_E is a product of functions

$$(\Theta_E)_p : \widehat{\mathbb{Z}}_p^* \to K^*(\text{Base})_p .$$

i) Θ_E satisfies a "cocycle condition".

$$\Theta_{\alpha\beta} = (\Theta_\alpha)^\beta \Theta_\beta .$$[28]

ii) Θ_E is continuous.

iii) Θ_E is exponential

$$\Theta_{E\oplus F} = \Theta_E \cdot \Theta_F .$$

iv) If k is an integer prime to p then (see K-lemma below)

$$\Theta_k \begin{pmatrix}\text{oriented 2-plane} \\ \text{bundle } \eta\end{pmatrix}_p = \frac{1}{k}\left(\frac{\eta^k - \bar\eta^k}{\eta - \bar\eta}\right)\left(\frac{\eta + \bar\eta}{\eta^k + \bar\eta^k}\right)_p .$$

(We regard η and $\bar\eta$ as complex line bundles for the purposes of computing the right hand side. The element obtained is fixed by conjugation so lies in the real K-theory.)

The multiplicative group of (diagonal) 1-cocycles functions satisfying o), i) and ii) is denoted $Z_d^1(\widehat{\mathbb{Z}}^*, K^*)$, the group of *continuous crossed homomorphisms* from the group $\widehat{\mathbb{Z}}^*$ into the $\widehat{\mathbb{Z}}^*$-module K^*.

Among these we have the principal crossed homomorphisms the image of

$$K^* \xrightarrow{\delta} Z^1_d(\widehat{\mathbb{Z}}^*, K^*)$$

$$u \mapsto \delta u\,, \ \text{defined by } \delta u_\alpha = u^\alpha / u\,.^{29}$$

The quotient of crossed homomorphisms by principal crossed homomorphisms is the 1-dimensional cohomology of the group $\widehat{\mathbb{Z}}^*$ with coefficients in K^*

$$H^1_d(\widehat{\mathbb{Z}}^*; K^*)\,.$$

now consider the problem (at odd primes) of classifying a *vector bundle* E up to

linear isomorphism

fibrewise homeomorphism

fibre homotopy type.

The geometric characterization above gave a 'geometric cocycle' which *determined the stable isomorphism type* of E,

$$\Delta \in \mathrm{Hom}\left(\Omega^l_*(\quad; \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}\right).$$

These numerical invariants can be calculated analytically given a Riemannian geometry on E.

The K-theory Thom class Δ_E can also be determined by the Hodge complex on E. Δ_E and the fibre homotopy type of E determine the topological type of E. The Θ invariant is then calculated from the action of the Galois group on Δ_E.

THEOREM 6.10 *A vector bundle E is topologically trivial iff the cocycle Θ_E is identically* 1.

THEOREM 6.11 *E is fibre homotopy trivial iff Θ_E is cohomologous to* 1.

Any cohomology class contains the Θ invariant of some vector bundle E.

Let J_0 and J_{top} denote the images of the passage to fibre homotopy type.

Corollary[30]

$$\begin{aligned} J_0 &\cong H^1_d(\widehat{\mathbb{Z}}^*; K^*) \\ J_{top} &\cong \mathscr{C}^1 \oplus H^1_d(\widehat{\mathbb{Z}}^*; K^*) \\ K_{top} &\cong \mathscr{C}^1 \oplus Z^1_d(\widehat{\mathbb{Z}}^*, K^*) \end{aligned}$$

(all isomorphisms are canonical.)

Corollary *Any cocycle is the* Θ *invariant of some topological bundle.*

Note: The natural map

$$\left\{\begin{matrix}\text{topological}\\ \text{bundles}\end{matrix}\right\} \xrightarrow{\Theta} Z^1_d(\widehat{\mathbb{Z}}^*, K^*)$$

is split using the decomposition above. In fact, the group of cocycles is isomorphic to the subgroup of topological bundles

$$\left\{\begin{matrix}\text{smooth}\\ \text{bundle}\end{matrix}\right\}_1 \oplus \left\{\begin{matrix}\text{homotopy}\\ \text{trivial bundle}\end{matrix}\right\}_\xi \cong K_1 \oplus K^*_\xi\,.$$

In these natural K-theory coordinates for topological bundles,

i) a vector bundle V has topological components

$$\left(V_1, \delta^{-1}(\Theta_V)_\xi, 0\right),$$

ii) a topological normal invariant $\sim u \in K^*$ has components

$$\left(\Theta^{-1}(\delta u)_1, u_\xi, 0\right)$$

in K_{top}.

Note: Because Θ_E is a product and $\widehat{\mathbb{Z}}^*_p$ is cyclic Θ_E is determined by its value at one "generating" point $\alpha \in \widehat{\mathbb{Z}}^*$.[31] Thus the single invariant $\Theta_\alpha(E) \in K^*(B)$ is a complete topological invariant of the vector bundle E. In these terms we can then say that E is fibre homotopically trivial iff $\Theta_\alpha(E) = \delta u(\alpha) = u^\alpha/u$ for some $u \in K^*(B)$.

we stated the theorem in the *invariant form* above to show the rather striking analogy between the "problem" and the "form" of the solution.

We pursue this "form" and add a fourth "K-group" to the collection

$$K\,,\ \ K^*\,,\ \ Z^1_d(\widehat{\mathbb{Z}}^*, K^*)\,.$$

Consider the fibre product $\mathscr{K}$ in the diagram

$$\begin{tikzcd} \mathscr{K} \arrow[r, "p_1"'] \arrow[d, "p_2"] & K \arrow[d, "\Theta"] \\ K^* \arrow[r, "\delta"] & Z^1_d \end{tikzcd} \qquad K\text{-theory square}$$

$$\mathscr{K} = \{(V,u) \in K^* \times K \mid \Theta_V = \delta u\}\,.$$

An element of $\mathscr{K}$ is a vector bundle together with a cohomology of its Θ invariant to 1.

Recall the theory of smooth normal invariants or fibre homotopy equivalences $E \sim F$ between vector bundles. we have the natural diagram

$$\begin{array}{ccccc} \left\{\begin{matrix}\text{smooth}\\ \text{normal}\\ \text{invariants}\end{matrix}\right\} = sN & \longrightarrow & K_0 & = & \left\{\begin{matrix}\text{smooth}\\ \text{bundles}\end{matrix}\right\} \\ \downarrow & & \downarrow & & \\ \left\{\begin{matrix}\text{topological}\\ \text{normal}\\ \text{invariants}\end{matrix}\right\} = tN & \longrightarrow & K_{top} & = & \left\{\begin{matrix}\text{topological}\\ \text{bundles}\end{matrix}\right\} \end{array} \qquad \begin{matrix}\text{geometric}\\ \text{square}\end{matrix}$$
.

The periodicity $tN \sim K^*$, the identity $K_0 \sim K$, and the Θ invariant $K_{top} \xrightarrow{\Theta} Z^1$ define a canonical map of this geometric square into the K-theory square.

THEOREM 6.12 *The induced additive morphism*

$$\left\{\begin{matrix}\textit{homotopy}\\ \textit{equivalences}\\ \textit{between}\\ \textit{smooth bundles}\end{matrix}\right\} = sN \xrightarrow{\textit{formalization}} \mathscr{K} = \{(V,u) : \Theta_V = \delta_u\}$$

is onto. *Thus any deformation of the* Θ *invariant of a vector bundle* V *to zero is realized by a fibre homotopy trivialization.*

NOTE: The proof will show that $\mathscr{K}$ is naturally isomorphic to the representable theory

$$K_\xi \oplus K^*_1\,.$$

Thus sN splits as theories. The splitting is not necessarily additive. In fact, it seems reasonable to believe that the obstructions to an additive splitting are non zero and central.

THE PROOFS.

We work our way backwards making the K-theory calculation and proving Theorem 6.12 first.

Let $\alpha \in \widehat{\mathbb{Z}}^*$ be such that $\alpha_p \in \widehat{\mathbb{Z}}_p^*$ is a topological generator for each p.

K-LEMMA *The diagram* $\big(\theta_\alpha(x) = \Theta$ *invariant of* x *evaluated at* $\alpha\big)$

$$\begin{array}{ccc} K & \xrightarrow{x \mapsto \theta_\alpha(x)} & K^* \\ {\scriptstyle x \mapsto x^\alpha - x}\big\downarrow & & \big\downarrow{\scriptstyle u \mapsto u^\alpha/u} \\ K & \xrightarrow[x \mapsto \theta_\alpha(x)]{} & K^* \end{array} \quad \textit{commutes}.$$

Moreover the horizontal map is an isomorphism at 1*, and the vertical maps are isomorphisms at* ξ*.*

PROOF:

a) The diagram commutes because

i) any homomorphism $K \to K^*$ is determined by its effect in the reduced K-theory of the $4i$-spheres. To see this note that we have an H-map of classifying spaces

$$O \times BO \to 1 \times BO$$

with given action on the primitive homology (rational coefficients) – the spherical homology classes. We have already seen (the claim in Theorem 3.7 or by elementary obstruction theory here in the odd primary case) that this determines the homotopy class.

ii) The reduced K-theory of the sphere S^{4i} is cyclic so if θ_α maps a generator ν of $\widetilde{K}(S^{4i})$ to $(1+\nu)^{\theta_i}$ in $1 + \widetilde{K}(S^{4i}) = K^* S^{4i}$, then

$$\begin{aligned} (\theta_\alpha \nu)^\alpha / \theta_\alpha \nu &= (1+\theta_i \nu)^\alpha/(1+\theta_i\nu) \ \text{ since } \nu^2 = 0 \\ &= (1+\theta_i \alpha^{2i}\nu)/(1-\theta_i\nu) \\ &= 1 + \theta_i(\alpha^{2i}-1)\nu\,, \end{aligned}$$

but

$$\begin{aligned}\theta_\alpha(\nu^\alpha - \nu) &= \theta_\alpha(\nu^\alpha)/\theta_\alpha(\nu)\\ &= (1+\nu^\alpha)^{\theta_i}(1-\theta_i\nu)\end{aligned}$$

which also equals $1+\theta_i(\alpha^{2i}-1)\nu$. So they agree on spheres.

b) We have used the formulae

$$\nu^\alpha = \alpha^{2i} \in \widetilde{K}S^{4i}\,,$$

$$(1+\nu)^\alpha = 1+\alpha^{2i}\nu \in K^*S^{4i}\,.$$

The second follows from the first. The first follows from our calculations on homology in section 5. Together they imply the vertical maps act on the $4i$-spheres by multiplication by $\alpha^{2i}-1$, $i = 1, 2, \ldots$.

These are isomorphisms if $i \not\equiv 0 \bmod (p-1/2)$, the region of dimensions where 'K_ξ of the spheres' is concentrated. This proves the vertical maps are isomorphisms for all spaces.

c) To study the horizontal maps recall the formula

$$\theta_k(\eta) \;=\; \frac{1}{k}\left(\frac{\eta^k-\eta^{-k}}{\eta-\eta^{-1}}\right)\left(\frac{\eta+\eta^{-1}}{\eta^k+\eta^{-k}}\right).$$

This is proved by noting the *Laplacian K-theory Thom class* of an oriented 2-plane bundle η is

$$U_\Delta \;=\; \frac{\eta-\bar\eta}{\eta+\bar\eta} \in KO^2(\mathbb{C}P^\infty) \subseteq K^0_U(\mathbb{C}P^\infty)$$

where we calculate in the complex K-theory of $\mathbb{C}P^\infty$. (We can regard KO^2 as the subspace reversed by complex conjugation, and KO^0 is subspace preserved by conjugation). U_Δ is correct because it has the right character

$$\frac{e^x-e^{-x}}{e^x+e^{-x}} = \tanh x\,.$$

The formula for $\theta_k(\eta)$ follows.

We relate our θ_k to Adams's ρ_k-operation

$$\rho_k(\eta) = \frac{1}{k}\left(\frac{\eta^k-1}{\eta-1}\right)$$

which he calculated in S^{4i},

$$\rho_k(\nu) = 1 + \rho_i \nu$$

and as far as the power of p is concerned for k a generator of $\widehat{\mathbb{Z}}_p^*$ ρ_i is the numerator of $(B_i/4i)$. [32] Thus it gives an isomorphism between the 1 components $\widetilde{K}_1$ and K_1^*.

Recall $\bar{\eta} = \eta^{-1}$, thus

$$\begin{aligned}\theta_k(\eta) &= \tfrac{1}{k}\left(\tfrac{\eta^k - \eta^{-k}}{\eta - \eta^{-1}}\right)\left(\tfrac{\eta + \eta^{-1}}{\eta^k + \eta^{-k}}\right) \\ &= \tfrac{1}{k}\left(\tfrac{\eta^{2k}-1}{\eta^2-1}\right)\left(\tfrac{\eta^2+1}{\eta^{2k}+1}\right) \\ &= \tfrac{1}{k^2}\left(\tfrac{\eta^{2k}-1}{\eta^2-1}\right)^2 \tfrac{k(\eta^4-1)}{\eta^{4k}-1} \\ &= \left(\rho_k(\eta^2)\right)^2 \cdot \left(\rho_k(\eta^4)\right)^{-1}.\end{aligned}$$

Thus

$$\theta_\alpha(x) = \left(\rho_\alpha(x^\beta)\right)^2 \rho_\alpha(x^{\beta^2})^{-1}, \quad \begin{aligned} \forall\, x &\in \widetilde{K} \\ \alpha &\in \widehat{\mathbb{Z}}^* \\ \beta = 2 &\in \widehat{\mathbb{Z}}^{*} \end{aligned}$$ [33]

using the fact that these two exponential operations agree on complex line bundles. If we calculate $\widetilde{K}(S^{4i})$ then,

$$\begin{aligned}\theta_\alpha(\nu) &= (1 + 2^{2i}\rho_i\nu)^2 (1 - 4^{2i}\rho_i\nu) \\ &= 1 + 2^{2i+1}(1 - S^{2i-1})\rho_i\nu\,.\end{aligned}$$

Since $(1 - 2^{2i-1}) \not\equiv 0 \bmod p$ if $i \equiv 0 \bmod (p-1)/2$ we see that θ_k also induces an isomorphism on the eigenspace of 1.

Note: The calculation here and that of Adams show that the primes p for which smooth theory is not a direct summand in topological theory are precisely the irregular primes and those for which 2 has odd order in the multiplicative group of F_p^*. More explicitly,

$$\underset{\text{irregular}}{\{37, 59, 67, 101, \dots\}} \cup \underset{8k-1}{\{7, 23, 31, \dots\}} \cup \underset{\substack{\text{some of the}\\ \text{primes of}\\ \text{the form}\\ 8k+1\,.}}{\{73, 89, \dots\}}$$ [34]

There are infinitely many primes in each of these sets.

Proof that $sN \xrightarrow{f} \mathscr{K}$ is onto.

Consider the canonical elements $x^\alpha \sim x$ in sN, This gives rise to a diagram

$$\begin{tikzcd}
 & & & & K_\xi & \\
K_\xi \arrow[urrrr, "d_\xi"] & & & & K_\xi \arrow[u] \arrow[dr, "\theta_\alpha"] & \\
K \arrow[u, phantom, sloped, "\subseteq"] \arrow[urrrr, "x \mapsto (x^\alpha - x)"] \arrow[r, "g"] \arrow[drrrr, "x \mapsto \theta_\alpha(x)"'] & sN \arrow[rr, "f"] & & \mathscr{K} \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & K \\
K_1 \arrow[u, phantom, sloped, "\subseteq"] \arrow[drrrr, "d_1"] & & & & K^* \arrow[ur, "\delta_\alpha"'] \arrow[d] & \\
 & & & & K_1^* &
\end{tikzcd}$$

where $g(x) = x^\alpha \sim x$. (Note we don't have to check any compatibility since we only need the map g on any arbitrarily large skeleton of the classifying space for K-theory.)

A check of homotopy groups shows d_ξ and d_1 are isomorphisms up to some large dimension. This shows f is onto $\mathscr{K}$ and

$$\mathscr{K} \cong K_\xi \oplus K_1^*$$

for complexes up to any large dimension.

(If we wish the theory of compact representable functors then shows f has an actual cross section.)

This proves Theorem 6.12.

Proof of Theorem 6.9 We have the canonical map of diagrams,

$$\begin{tikzcd} sN \arrow[r] \arrow[d] & K_0 \arrow[d, "\theta"] \\ tN \arrow[r, "j"'] & K_{top} \end{tikzcd} \quad \begin{array}{cc} \xrightarrow{f} & \xrightarrow{\mathrm{Id}} \\ \xrightarrow{p} & \xrightarrow{\Theta} \end{array} \quad \begin{tikzcd} \mathscr{K} \arrow[r] \arrow[d] & K \arrow[d, "\theta_\Delta"] \\ K^* \arrow[r, "\delta"'] & Z_d^1 \end{tikzcd}$$

θ_Δ means the Θ cocycle associated to our Laplacian Thom class).

a) Id is the identity,

b) p is an isomorphism by the construction below,

c) f is onto by what we just proved,

d) $(\theta_\Delta)_1$ is isomorphic to $\big(x \mapsto \theta_\alpha(x)_1\big)$ so it is an isomorphism by the K-lemma,

e) δ_ξ is an isomorphism similarly,

f) $\mathscr{C}^1 = \text{kernel}\ \theta$ by definition.

Theorem 6.9 now follows formally – the crucial point being the naturality of the eigenspaces under the maps j and θ.

Proof of Theorems 6,10, 6.11 We also see that

$$\theta(K_0)_1 \oplus j(tN)_\xi \underset{\cong}{\xrightarrow{\Theta}} Z^1_d(\widehat{\mathbb{Z}}^*, K^*)\,,$$

or

$$K_{top} \cong \mathscr{C}^1 \oplus Z^1_d(\widehat{\mathbb{Z}}^*, K^*)\,,$$

and under this identification the maps

$$tN \to K_{top}\,, \quad K_0 \to K_{top}$$

are just $0 + \delta$ and $0 + \theta_\Delta$ respectively.

This proves 6.10 and parts ii) and iii) of the corollary. Part i) is a restatement of Theorem 6.11.

Theorem 6.11 part i) follows from the definition of Θ and the ontoness of f.

Part ii) follows from the fact that H^1_d is "concatenated at 1" since δ_ξ is an isomorphism. But the smooth bundles generate because $(\theta_\Delta)_1$ is an isomorphism.

Proof of Theorem 6.7 We have proved $x^\alpha - x \sim 0$. For the converse first consider the vector bundle case:

V is fibre homotopy trivial iff $\theta_V \sim 0$ by Theorem 6.11.

$\theta_\Delta(V) \sim 0$ means there is a u so that

$$\theta_\Delta(V)_\alpha = \delta_u(\alpha)\,.$$

By the K-lemma V also in K_1 implies $V = x^\alpha - x$. This takes care of the eigenspace of 1.

At ξ every element is a difference of conjugates. The proof is completed by adding two cases 1 and ξ.

The topological case:

The Galois action in $(K_{top})_\xi$ is isomorphic to that in K_ξ^* by Theorem 6.8 and periodicity. Thus any element in $(K_{top})_\xi$ is a difference of conjugates.

The eigenspace of 1 is all smooth except for $\mathscr{C}^1$ which doesn't enter by 6.8. Thus we are reduced to the smooth case just treated. Q. E. D.

The K-orientation sequence and PL_n theory

To study K-oriented fibration theory and its relation to PL theory we us the K-orientation sequence on the classifying space level

$$\cdots \to K\hat{G_n} \to S\hat{G_n} \to \hat{B_\otimes} \to (BKG_n)\hat{} \to (BSG_n)\hat{}\ .$$

$(BSG_n)\hat{}$ classifies the oriented fibration theory with fibre $\widehat{S}^{n-1}$. $(BKG_n)\hat{}$ classifies the theory of K-oriented S^{n-1} fibrations. (The argument of Dold and Mayer-Vietoris sequence for K-theory implies $(BKG_n)\hat{}$ exists and classifies the K-oriented theory.)

$\hat{B_\otimes}$ classifies the special units in $\widehat{K}$. The group of K-units acts on the K-oriented fibrations over a fixed base.

Then we have the loop spaces $S\hat{G_n}$, $K\hat{G_n}$, ...

The maps are the natural ones, for example,

i) $\hat{B_\otimes} \to BKG\hat{}$ is obtained by letting the K-theory units act on the trivial orientation of the trivial bundle.

ii) If $n \to \infty$ $S\hat{G_n} \to \hat{B_\otimes}$ is induced by the natural transformation

$$\begin{Bmatrix}\text{stable cohomotopy} \\ \text{theory}\end{Bmatrix}\hat{} \to \{K\text{-theory}\}\hat{}$$

by looking at the multiplicative units.

The sequence is exact, for any consecutive three spaces there is a long exact sequence of homotopy.

Note: There is an orientation sequence for any multiplicative cohomology theory h

$$\to KG_n \to SG_n \to H_\otimes \to BhG_n \to BSG_n\,,$$

where

$$[\quad, H_\otimes] \cong \text{special units } h^0(\ )\,.$$

In case the multiplication in the theory is associative enough in the cocycle level the obstruction to h-orientability is measured by an element in $h^1_\otimes(BSG)$ – *the h^* analogue of the 1st Stiefel Whitney class.*

This is true for K-theory although the details of this discussion are not secured.

Another way to find the first Stiefel Whitney class for K-theory is to use the identification (below) of the orientation sequence (stable) with the sequence

$$\cdots \to G \to G/PL \to BPL \to BG \rightsquigarrow B(G/PL) \rightsquigarrow \ldots$$

This sequence has been extended to the right (infinitely) by Boardman. The space $B(G/PL)$ may be infinitely de-looped and its loop space is equivalent to $BO\hat{}$ at odd primes. Postnikov arguments (taught to me by Frank Peterson) show $B(G/PL)$ is equivalent at odd primes to the classifying space for K^1, and $\widetilde{K}^0 \cong K^*$.

Another construction of the 1st Stiefel Whitney class in K-theory (for odd primes) can be given by constructing a "signature invariant in BG" using recently developed techniques for surgery on Poincaré Duality spaces. [35] This makes the construction canonical.

We compare the K-orientation sequence with the $(PL_n \subseteq G_n)$-sequence

$$\begin{tikzcd}
\cdots \arrow[r] & KG_n \arrow[r] \arrow[d] & SG_n \arrow[r] \arrow[d, "\text{completion}"] & G_n/PL_n \arrow[r] \arrow[d, "p"] & BSPL_n \arrow[r] \arrow[d, "\Delta_{PL}"] & BSG_n \arrow[d, "\text{completion}"] \\
\cdots \arrow[r] & KG_n\hat{} \arrow[r] & SG_n\hat{} \arrow[r] & B_\otimes\hat{} \arrow[r] & (BKG_n)\hat{} \arrow[r] & (BSG_n)\hat{}\,.
\end{tikzcd}$$

Δ_{PL} is defined by mapping an oriented bundle E to its canonically associated K-oriented fibration

$$\big((E-B) \to B, \Delta_E\big)\,.$$

p is induced by commutativity in the right hand square (and is unique). (Or historically by its own signature invariant.)[36]

Claim: *p is the completion map at odd primes for G_n/PL_n for $n > 2$.*

The surgery techniques of Kervaire, Milnor and Levine reformulated[36] *give the periodic homotopy groups of G_n/PL_n, $n > 2$.*

$$0\,,\ \mathbb{Z}/2\,,\ 0\,,\ \mathbb{Z}\,,\ 0\,,\ \mathbb{Z}/2\,,\ 0\,,\ \mathbb{Z}\,, \ldots$$ [36]

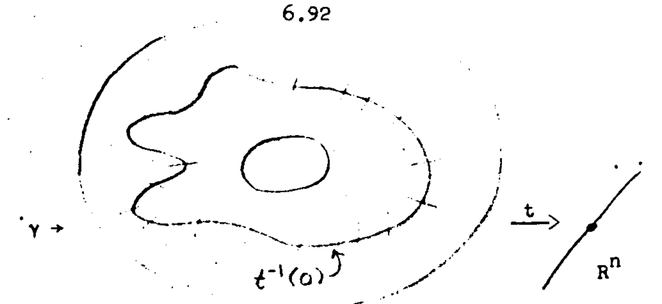
The generator of pi_{4k} is represented by

$$\gamma \xrightarrow{t} \mathbb{R}^n$$

where γ is a block bundle over S^{4k} and t is a proper "degree one" map which is transversal to $0 \in \mathbb{R}^n$ and $t^{-1}(0)$ is a $4k$-manifold with signature a power of 2

$$(16, 8, 8, 8, \ldots\,)\,.$$ [36]

6.92



If we unravel the definition of Δ_{PL} defined in terms of transversality and signatures of manifolds we see the claim is true on homotopy in dimension $4k$

$$\mathbb{Z} \to \widehat{\mathbb{Z}}$$

$$\text{generator} \mapsto \text{power of } 2\,.$$

All other groups are zero at odd primes.

Corollary Δ_{PL} *is profinite completion at odd primes. Thus profinite PL_n theory is isomorphic to K-oriented $\widehat{S}^{n-1}$ theory.*

Proof: We tensor the upper homotopy sequence with $\widehat{\mathbb{Z}}$ to obtain an exact sequence isomorphic by the five lemma to the lower sequence.

This completes the proof of Theorem 6.5 ($\widehat{\mathbb{Z}}$).

Theorem 6.5 ($\mathbb{Q}$) follows from the discussion of BSG_n in section 4, the diagram

$$\begin{tikzcd} G_n/PL_n \arrow[r] \arrow[d, "\cong"'] & BSPL_n \arrow[r] \arrow[d] & BSG_n \arrow[d] \\ G/PL \arrow[r] & BSPL \arrow[r] & BSG \end{tikzcd} \qquad n > 2$$

and the finiteness of $\pi_i BSG$. These imply

$$\begin{aligned}(BSPL_n)_0 &\cong (BSPL)_0 \times (BSG_n)_0 \\ &\cong \prod_i H^{4i}(\quad;\mathbb{Q}) \times H^d(\quad;\mathbb{Q})\end{aligned}$$

where $d = n$ or $2n-2$ if n is even or odd respectively.

Equivariance

We obtain group actions in the three oriented theories

$$\begin{array}{ccccc} \left\{\begin{matrix} n\text{-dimensional} \\ \text{vector bundles}\end{matrix}\right\} & \to & \left\{\begin{matrix} n\text{-dimensional} \\ PL \text{ bundles}\end{matrix}\right\} & \to & \left\{\begin{matrix} \widehat{S}^{n-1} \\ \text{fibrations}\end{matrix}\right\} \\ \mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q}) & & \widehat{\mathbb{Z}}^* \times \widehat{\mathbb{Z}}^* & & \widehat{\mathbb{Z}}^* \end{array}$$

using

i) etale homotopy of section 5,

ii) the identification of PL theory with K-oriented theory just discussed,

iii) the completion construction of section 4 for fibrations.

The first formula of Theorem 6.6 follows easily from the cofibration

$$(BSO_{n-1})\hat{} \to (BSO_n)\hat{} \to \text{Thom space } \widehat{\gamma}_n$$

where $\widehat{\gamma}_n$ is the completed spherical fibration associated to the canonical bundle over BSO_n.

An element α in $\mathrm{Gal}\,(\widetilde{\mathbb{Q}}/\,\mathbb{Q})$ determines a homotopy equivalence (α) of the Thom space and obstruction theory shows

$$(\alpha) * x \;=\; \beta \cdot x^{\alpha}$$

where x is an element in $\widehat{K}$ (Thom space), and

$$\beta \;=\; \begin{cases} \alpha^{n/2} & n \text{ even}\,, \\ \alpha^{(n-1)/2} & n \text{ odd}\,. \end{cases}$$

(The diagram

$$\begin{tikzcd}
\text{Thom space } \widehat{\gamma}_n \arrow[r, "x"] \arrow[d, "(\alpha)"] & \widehat{B} \arrow[d, "\text{operation } \beta\cdot(\ \ )^{\alpha}"] \\
\text{Thom space } \widehat{\gamma}_n \arrow[r, "x"] & \widehat{B}
\end{tikzcd}$$

commutes on the cohomology level.)

The second formula is immediate.

For the *stable equivariance* of Theorem 6.7 note that there is an automorphism of

$$\big(\widehat{\gamma} \text{ fibre join } K(\widehat{\mathbb{Z}}, 1)\big)^{\text{fibrewise completion}}$$

which multiplies any orientation by an arbitrary unit in $\widehat{\mathbb{Z}}$.

The actions simplify as stated and the equivariance follows from the unstable calculation.

Notes

1 See Sullivan, 1966 Princeton Thesis for a definition of dg.

2 Rourke and Sanderson, Block Bundles I, II, III, Annals of Mathematics 1967, 1968.

3 In this work we only treat ii).

4 There is a good analytical interpretation of these residues.

5 The author hopes a young, naive, geometrically minded mathematician will find and develop these arguments.

6 Today, anyway.

7 Defined for smooth or PL manifolds in X. Over $\mathbb{Q}$ the theories are isomorphic.

8 All other values are zero anyway. $E^+ =$ one point compactification of E.

9 Technically, this uses the fact that $\mathbb{Z}/k$-manifolds represent "bordism with $\mathbb{Z}/k$ coefficients" which is easy to prove. One defines a map by transversality and the coefficient sequence above gives the isomorphism.

10 We note that there is a more direct evaluation of Δ obtained by embedding the split $\mathbb{Z}/k$-manifold M in $D^2_* \times \mathbb{R}^N$ equivariantly. D^2_* means D^2 with $\mathbb{Z}/k$ rotating the boundary. One combines this embedding with the map of M into X and the classifying map of ν into BSO_{4k}. One can then pull Δ_{4k} back to the mod k Moore space $\left(D^2_*/((\mathbb{Z}/k)\partial) \times \mathbb{R}^N, \infty\right)$ with K-group $\mathbb{Z}/k$. Following this construction through allows an analytical interpretation of the residue classes in $\mathbb{Z}/k$ in terms of the "elliptic operator for the signature problem".

11 We assume all groups are localized at odd primes.

12 This kernel argument and the tensor product formulation are due to Conner-Floyd. See Springer Verlag Lecture Notes "The relation of Cobordism to K-theory", for their beautiful "complex and symplectic formulae".

13 The existence of some multiplication can be seen by general homotopy theory. The cycle proof above illustrates in part the spirit of the author's arguments in the geometric context sans K-theory.

14 This follows for finite complexes by induction over the cells.

15 For the stable theory this means a profinite PL bundle has a classical fibre homotopy type.

16 We are thinking primarily of the oriented case. However BPL_{2n} can be localized using the fibration

$$BSPL_n \to BPL_{2n} \to K(\mathbb{Z}/2, 1)$$

and the fibrewise localization of section 4. BPL_{2n+1} should be localized at zero because the action of $\pi_1 = \mathbb{Z}/2$ is rationally trivial.

17For $n = 1$, the oriented theory is trivial. For $n = 2$, a complete invariant is the Euler class in cohomology.

18Beyond homotopy theoretical information.

19The manifold $S^3 \times S^1 \times S^1$ illustrates the interdependence.

20See author's thesis. C. T. C. Wall extended this theory to non-simply connected manifolds and another "second order" invariant associated with the fundamental group comes into play.

21We note here that a compact manifold with boundary has an arbitrary (finite) homotopy type.

22Proved below in the K-orientation sequence paragraph.

23As n approaches infinity and for odd primes.

24We can appeal to the combinatorics of Lie theory or rectilinear geometry to find invariants.

25Recall that the prime 2 is excluded in the piecewise linear or topological case.

26Our notation refers to the reduced, completed groups from now on. For example we denote the special units in K-theory by K^* $(= 1 + \widetilde{K})$. We write K_0 to emphasize the fact that we are speaking of stable profinite vector bundles. K by itself refers to the canonically isomorphic reduced cohomology theory $[\quad, B_0]$.

27The 'idea' of the invariant is due to Thom. A related Θ for vector bundles was used by Adams and Bott.

28This condition was precociously emphasized by Bott in his study of the related invariant. In the completed context above Θ_E has exactly the power and "form" for the questions we ask. We prove property iii) in a later paper. It is true for vector bundles (in fact ii) iii) and iv) determine Θ on vector bundles) – the extension to general topological bundles only requires a product formula for the signature of $\mathbb{Z}/m$-manifolds.

29Note δu is a diagonal cocycle because the action is a product action. We consider only this diagonal context throughout.

30The homotopy statement is true at 2 for vector bundles.

31Note $\widehat{\mathbb{Z}}^*$ is not cyclic thus our emphasis on diagonal cocycles, coboundaries, etc. One wonders at the significance of the full cohomology group (and even the full cohomology theory) in connection with fibre homotopy types and Michael Boardman's new cohomology theories.

32J. F. Adams "$J(X)$ II, II, III" Topology 1966.

33That is, $\theta_\alpha(x) = \rho_\alpha\big(2x^{(2)} - x^{(4)}\big)$.

34Namely $p = 8k + 1$ and 2 has odd order in F_p^*.

35 Norman Levitt and Lowell Jones.

36See D. Sullivan Thesis, Princeton 1966, Geometric Topology Seminar Notes Princeton 1967.