

OPEN AND CLOSED STRING FIELD THEORY INTERPRETED IN CLASSICAL ALGEBRAIC TOPOLOGY

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Dedicated to Graeme Segal on his 60th birthday

Abstract: There is an interpretation of open string field theory in algebraic topology. An interpretation of closed string field theory can be deduced from this open string theory to obtain as well the interpretation of open and closed string field theory combined. The algebraic structures derived from the first string interactions are related to algebraic models discussed in work of (Atiyah-Segal), (Moore-Segal) and (Getzler and Segal). For example the Corollary 1 of §1 says that the homology of the space of paths in any manifold beginning and ending on any submanifold has the structure of an associative dialgebra satisfying the module or Frobenius compatibility (see appendix). Corollary 2 gives another structure.

§1 *Open string states in M :* The open string theory interpretation in topology includes a collection of linear categories ϑM one for each ambient space M . The objects of ϑM are smooth oriented submanifolds L_a, L_b, L_c, \dots of M . The set of morphisms ϑ_{ab} between two objects L_a and L_b are graded chain complexes, linearly generated by smooth oriented families of paths from L_a to L_b . An element in ϑ_{ab} is called an open string state. A path is a piecewise smooth map $[0,1] \rightarrow M$.

The first open string interactions are

i) two endpoint restrictions: $\vartheta_{ab} \xrightarrow{r} \vartheta_{a'b}$ and $\vartheta_{ab} \xrightarrow{r} \vartheta_{ab'}$ where $L_{a'}$ is a submanifold of L_a and $L_{b'}$ is a submanifold of L_b . Degree $r = -\text{cod}$ of submanifold.

ii) joining or composition $\vartheta_{ab} \otimes \vartheta_{bc} \xrightarrow{\wedge} \vartheta_{ac}$, degree $\wedge = -\dim L_b$

iii) cutting or cocomposition $\vartheta_{ac} \xrightarrow{\vee} \vartheta_{ab} \otimes \vartheta_{bc}$, degree $\vee = -\text{cod } L_b + 1$

Namely,

i) (restriction) for an open string state in ϑ_{ab} (ie. a chain in ϑ_{ab}) one can intersect transversally in L_a the chain of beginning points in L_a with $L_{a'}$ to obtain a chain in $\vartheta_{a'b}$. The same idea works in L_b for the endpoints of paths to construct $\vartheta_{ab} \xrightarrow{r} \vartheta_{ab'}$.

ii) (joining) the transversal intersection in L_b of the chain of endpoints for an open string state in ϑ_{ab} with the chain of beginning points for an open string state in ϑ_{bc} is a chain labelling composable paths which after composing defines an open string state in ϑ_{ac} , and the composition $\vartheta_{ab} \otimes \vartheta_{bc} \xrightarrow{\wedge} \vartheta_{ac}$.

iii) (cutting) Now it is required that L_a, L_b, L_c, \dots have oriented normal bundles. For example, this is true if the ambient space M is a smooth manifold. Then given an L_b and any open string state in ϑ_{ac} we may transversally intersect in M the paths with L_b . The intersection chain labels cuttings of the path at L_b defining $\vartheta_{ac} \xrightarrow{\vee} \vartheta_{ab} \otimes \vartheta_{bc}$. (We use Eilenberg-Zilber.)

The operation \vee refers to cutting at any time along the path whenever it crosses L_b . We can also consider the operation \vee_t of cutting at a specific

time $t \in [0, 1]$. All these \vee_t are chain homotopic. In fact \vee is the chain homotopy between \vee_0 cutting at time zero and \vee_1 cutting at time one.

Remark: Actually the above operations are directly defined by the above descriptions only for states satisfying transversality conditions. To go from such a typical definition to a complete definition perturbations of the identity creating transversality must be introduced. The combinatorics of these perturbations fits neatly into Stasheff's strong homotopy formalism [S]. An elegant treatment can be read in Fukaya et al [1], for the classical case of intersecting chains in a manifold.

Theorem: For each ambient oriented smooth manifold M there is an open string category whose objects are smooth submanifolds L_a, L_b, L_c, \dots and whose morphisms are chains $\vartheta_{\alpha\beta}$ on paths between objects L_α and L_β . Only the objects L_a which are compact (without boundary) have identity maps (which commute with the boundary operator). For transversal open string states in $\vartheta_{\alpha\beta, \dots}$ composition \wedge is associative, cocomposition \vee is coassociative, and the derivation compatibility holds between \vee and $\wedge(x, y) = x \cdot y$, $\vee(x \cdot y) = x \cdot \vee y + \vee x \cdot y$ (see appendix). \wedge and \vee_t commute with ∂ but $[\vee, \partial] = \vee_1 - \vee_0$.

On the full space of open string states, associativity for \wedge and coassociativity for \vee_t hold up to strong homotopy in the sense of Stasheff. There are conjecturally similar strong homotopy statements for coassociativity of \vee and the derivation or infinitesimal bialgebra compatibility between \wedge and \vee . (see appendix).

Corollary 1: For each object L_a the homology of ϑ_{aa} is an associative algebra via the composition operation \wedge (with identity if L_a is compact without boundary). The operation \vee_t is a coassociative coalgebra (which if non zero implies L_a cannot be deformed off of itself). The \wedge, \vee_t dialgebra satisfies the module or Frobenius compatibility (see appendix).

Proof of corollary: i) The algebra statement follows from a) \wedge commutes with ∂ operator on open string states and so passes to homology b) homotopy associativity at the chain level implies associativity at the homology level.

ii) a) The fixed time cutting operation \vee_t also commutes with the ∂ operator and passes to homology. b) because different times are chain homotopic we can choose them conveniently to prove the module or Frobenius compatibility. To calculate $\vee_t(x \cdot y)$ we can choose t in x 's time to see that we get $\vee_t(x) \cdot y$ or in y 's time to see that we get $x \cdot \vee_t(y)$. See the remark 2) for the rest.

Sketch proof of theorem: 1) One sees the indicated identities hold for transversal chains by looking at the picture. For example, when cutting a joining of paths, the cut can happen in the first part or the second part. This yields the derivation compatibility.

2) The strong homotopy properties follow using i) manifolds are locally contractible

ii) transversality can be created in manifolds by arbitrarily small perturbations.

Remarks: 1) The coalgebra \vee_t is chain homotopic to \vee_0 which may be written as a composition involving the restriction and the diagonal mapping. Let $L_{a'}$ be the transversal intersection of L_a with itself. Then \vee_0 is the composition of, *first the restriction of the beginning point to $L_{a'}$, next the inclusion of $\vartheta_{a'a}$ into ϑ_{aa} , next the diagonal map on generating chains of ϑ_{aa} , next the cartesian product on chains of the beginning point operator (thought of as a constant path) with the identity and finally Eilenberg-Zilber.* A similar composition and statement hold for \vee_1 .

2) We can use remark 1) to define a new coalgebra structure on homology when L_a is deformable off itself, say to L_b . Then define $\vee : \vartheta_{aa} \rightarrow \vartheta_{ab} \otimes \vartheta_{ba}$ cutting at variable time and note that \vee_0 and \vee_1 are zero on the chain level. Thus \vee commutes with ∂ and passes to homology. We use the obvious equivalences $\vartheta_{aa} \sim \vartheta_{ba} \sim \vartheta_{ab}$ to obtain:

Corollary 2: If L_a is deformable off of itself, the homology of open string states on L_a has the structure of an associative dialgebra satisfying the derivation or infinitesimal bialgebra compatibility (see appendix).

Examples: *i)* (manifolds) $L_a = M$ the ambient space. Then ϑ_{aa} is equivalent to the ordinary chains on M since paths in M is homotopy equivalent to M . Then the strong homotopy associativity algebra structure on ϑ_{aa} is equivalent to the intersection algebra of chains on M . The operation $\vee_0 \sim \vee_t \sim \vee_1$ is chain equivalent to the diagonal mapping on chains. One recovers the known fact that on passing to homology one obtains a graded commutative algebra structure $C \otimes C \xrightarrow{\wedge} C$ and a graded cocommutative coalgebra structure $C \xrightarrow{\vee} C \otimes C$ satisfying the module or Frobenius

compatibility $\vee(x \cdot y) = x \cdot (\vee y) = (\vee x) \cdot y$ where the notation refers to multiplication on the left and right factors of the tensor product respectively (see appendix).

Note when M is a closed oriented manifold \wedge and \vee are related by the non degenerate intersection pairing, Poincare duality.

ii) (based loop space) M is any space and L_a is a point in M . Then ϑ_{aa} is the chains on the based loop space of M and the algebra structure on ϑ_{aa} is the Pontryagin algebra of chains on the based loop space (the original setting of Stasheff's work). No transversality is needed here because all paths are composable. Here one has Hopf's celebrated compatibility with the diagonal map \vee' on chains that \vee' is a map of algebras. The connection of the latter with the open string theory here is a mystery (but compare [2] and remark 1) above).

If M is a manifold of dim M near L_a and L_a is a point, the cocomposition \vee_t is defined but is zero in homology. The operation \vee can then be refined to a chain mapping and passes to homology (remark 2)). ϑ_{aa} obtains a coassociative coalgebra structure on homology of degree $(-\dim M) + 1$ satisfying the derivation or infinitesimal bialgebra compatibility (of the theorem) with the Pontryagin product. Here one is splitting a based loop where it passes again through a (nearby) base point.

iii) (free loop space) Let $M = L \times L$ and $L_a \subset M$ be the diagonal. Then paths in M beginning and ending on L_a is homeomorphic to the free loop space of $L = \text{Maps}(\text{circle}, L)$. Then the algebra structure on ϑ_{aa} is chain homotopic to the loop product of "String Topology" [2]. This is

a graded commutative algebra structure on the homology of the free loop space of the manifold L . The degree is zero if we grade by the negative codimension $(k - \dim M)$.

The product interacts with the circle action differential Δ of degree $+1$. The deviation of Δ from being a derivation of the loop product $\Delta(x \cdot y) - (\Delta x) \cdot y - x \cdot (\Delta y)$ is a Lie bracket of degree $+1$ which is compatible via the Leibniz identity with the loop product (all on homology). This Lie bracket is a geometric version [2] of Gerstenhaber's bracket in the (Hochschild) deformation complex of an associative algebra. For simply connected closed manifolds L the Hochschild complex $\bigoplus_k \text{Hom}(A^{\otimes k}, A)$ applied to the intersection algebra A of chains on L is a model of the free loop space of L (Cohen-Jones, Tradler) which realizes the above comparison (Tradler).

The Lie product on the free loop space of degree $+1$ is compatible via the connecting morphism M between equivariant homology and ordinary homology with a Lie bracket on the equivariant free loop space homology [2]. The latter Lie bracket generalizes to all manifolds the Goldman bracket (related to the Poisson structure on flat bundles over a surface) on the vector space generated by conjugacy classes in the fundamental group of a surface [Goldman] (see closed strings §2 below).

If the coalgebra part \vee_t of the Frobenius dialgebra on homology of the free loop space of L is non zero, then L is a closed manifold with non-zero Euler characteristic. Otherwise a homotopy class of non-zero vector fields on L allows a refining of the operation \vee cutting at variable time to an

operation commuting with ∂ and we obtain in this case an infinitesimal bialgebra structure (appendix) on the homology of the free loop space.

§2 *Closed string states in M (now called L):* For closed string states in L we take the chains for the equivariant free loop space of L relative to the circle action rotating the domain. There are maps

$$\begin{array}{l} \dots \xrightarrow{C} \text{closed string states in } L \xrightarrow{M} \text{open string states on the diagonal in } L \times L \\ \xrightarrow{E} \text{closed string states in } L \xrightarrow{C} \dots \end{array}$$

leading to the long exact sequence relating ordinary homology and equivariant circle homology. Here we are thinking of the free loop space of L as paths in $L \times L$ beginning and ending on the diagonal.

The connecting chain map C has degree -2 and intersects with a representative of the 1st chern class of the line bundle associated to the S^1 action made free by crossing with a contractible space on which S^1 acts freely. The chain map M has degree +1 and is associated to adding a mark to a closed string in all ways to get a circle of free loops. The chain map E has degree zero and is associated to forgetting the mark on a loop to get a closed string. The composition $EM = 0$ and the composition ME is Δ the differential associated to the circle action.

The string product on closed string states satisfying Jacobi (at the transversal chain level) may be defined by the formula $[\alpha, \beta] = E(M\alpha \wedge M\beta)$ where \wedge is the open string product (the procedure in example 3 above only satisfies Jacobi up to a non trivial chain homotopy). Other independent closed string operations c_n can be defined by $c_n(\alpha_1, \alpha_2, \dots, \alpha_n) = E(M\alpha_1 \wedge$

$M\alpha_2 \wedge \dots \wedge M\alpha_n$) (cf. [2] and [G]). These all commute with the ∂ operator and satisfy other identities transversally [2].

The collision operators c_n pass to the reduced equivariant complex or *reduced closed string* states which is defined to be the equivariant chain complex for the S^1 pair, (free loop space, constant loops).

We can define a closed string cobracket s_2 by the formula $s_2(\alpha) = (E \otimes E)(\vee(M\alpha))$. In the reduced complex s_2 commutes with ∂ and passes to homology (but not so in the unreduced complex).

Theorem: The closed string bracket $c_2(\alpha, \beta) = E(M\alpha \wedge M\beta)$ where $x \wedge y = \wedge(x \otimes y)$ and the closed string cobracket $s_2(\alpha) = (E \otimes E)(\vee M\alpha)$ satisfy respectively Jacobi, coJacobi, and Drinfeld compatibility (appendix). The term satisfy means either on the level of integral homology, for transversal chains on the chain level, or conjecturally at the Stasheff level of strong homotopy.

Proof: These formulae in terms of open strings are reinterpretations as in [2] of the definitions given in "Closed string operators in topology leading to Lie bialgebras and higher string algebra" [3]. There the identities at the transversal chain level were considered.

Corollary: Homology of reduced closed string states forms a Lie bialgebra, [3].

Remark: Independent splitting operations s_3, s_4, \dots can be defined similarly by iterations of \vee , $s_n(\alpha) = E \otimes \dots \otimes E(\dots \vee \otimes 1 \cdot \vee(M\alpha))$. These also commute with ∂ and pass to homology in the reduced equivariant theory. A conjecture about $c_2, c_3, \dots, s_2, s_3, \dots$ generating genus zero closed string

operators and the algebraic form of this structure was proposed in [3] and is mentioned below in the summary. Also, compare [Chas] for the original questions motivating this work.

Interplay between open and closed string states: Let \mathcal{C} denote the closed string states in M , a manifold of dimension d , and let ϑ denote any of the complexes of open string states. Transversality yields an action of closed strings on open strings,

$$\mathcal{C} \otimes \vartheta \rightarrow \vartheta \quad \text{degree} = (-d + 2)$$

and a coaction of closed strings on open strings

$$\vartheta \rightarrow \mathcal{C} \otimes \vartheta \quad \text{degree} = (-d + 2)$$

In the coaction we let the open string hit itself at any two times and split the event into a closed string and an open string. In the action we let a closed string combine with an open string to yield an open string.

The action is a Lie action of the Lie algebra of closed strings by derivations at the transversal chain level. Both the action and the coaction have a non trivial commutator with the boundary operator on chains.

§3 *Connection to work of (Atiyah-Segal), (Moore-Segal) and (Getzler and Segal):* Dialgebras satisfying the module or Frobenius compatibility give examples of 1+1 TQFT's in the positive boundary sense. In the commutative case we associate the underlying vector space to a directed circle, its tensor products to a disjoint union of directed circles and to a connected $2D$ oriented bordism between two non empty collections the morphism obtained by decomposing the bordism into pants and composing accordingly the algebra or coalgebra map. The module or Frobenius compatibility is

just what is required for the result to be independent of the choice of pants decomposition.

N.B. this description differs from the usual one because we don't have disks to close up either end of the bordism. One knows these discs at both ends would force the algebra to be finite dimensional and the algebra and coalgebra to be related by a non degenerate inner product. We refer to these generalizations of the Atiyah-Segal concepts as the positive boundary version of TQFT (a name due to Ralph Cohen).

An exactly similar discussion with associative dialgebras satisfying the module or Frobenius compatibility leads to a positive boundary version of a relative TQFT using open intervals. Now the algebra and coalgebra are associated to $1/2$ pants (a disc with ∂ divided into six intervals-three ($1/2$ seams) alternating with two ($1/2$ cuffs) and one ($1/2$ waist)). Any planar connected bordism between two nonempty collections of intervals determines a mapping between inputs and outputs.

The structures we have found (including ∂ labels L_a, L_b, \dots) for open strings using the composition \wedge and fixed time cutting \vee_t satisfies this Frobenius compatibility up to chain homotopy and we can apply it at the homology level in the relative TQFT scheme just mentioned. This fits with the work of Moore-Segal [M].

As we begin to look at the chain homotopy coproduct \vee the derivation or infinitesimal bialgebra compatibility appears. According to [Gan] the derivation or infinitesimal bialgebra compatibility is related to the notion of module or Frobenius compatibility via Koszul duality (see appendix).

Now we are entering into a third stage—the proposal of Segal (and independently Getzler) enriching the earlier notion of TQFT by chain complexes and chain homotopies.

Recall the free loop space above gives on the ordinary (chain) homology level a (strong homotopy) commutative associative product and a cocommutative coassociative coproduct (cutting at a fixed time) satisfying the module or Frobenius compatibility. This together with the associative Frobenius category above for open strings fits with the model [M]. In that model ordinary and equivariant levels are not distinguished.

We saw that passing to the equivariant setting the product and the cutting at variable time gave a Lie bialgebra in the reduced theory. According to [Gan] Lie dialgebras with Drinfeld compatibility are related to commutative dialgebras with Frobenius compatibility by Koszul duality (see appendix).

§4 *Summary*: We have described the part of the interpretation of open and closed string field theory in topology associated to the basic product and coproduct (and in the equivariant setting certain implied n -variable splitting and collision operators as in [3]). The coproduct discussion has two levels involving a coproduct \vee_t and an associated chain homotopy coproduct \vee .

We found the open string product and the coproduct \vee_t satisfied the module or Frobenius compatibility on the level of homology. In a setting where \vee_0 and \vee_1 were zero or even deformable to zero, \vee emerges as or can be deformed to a coproduct commuting with ∂ and thus a coproduct

\vee on homology of one higher degree. Then a new compatibility with the product is observed- the derivation or infinitesimal bialgebra compatibility (true transversally).

Similarly for the closed string one has to consider the free loop space in both the ordinary and equivariant versions. For the open string with diagonal boundary conditions the relevant ordinary (chains) homology of the free loop space becomes a (strong homotopy) commutative dialgebra with the module or Frobenius compatibility. Passing to the equivariant theory required for the closed string interpretation and reducing to kill \vee_0 and \vee_1 which makes \vee commute with ∂ , the product coproduct pair becomes a Lie dialgebra with the derivation or Drinfeld compatibility (equals Lie bialgebra). According to [Gan] the associative and commutative dialgebras with the module or Frobenius compatibility are respectively Koszul dual to the associative and Lie dialgebras with the derivation or Drinfeld compatibility. This suggests that one of the structures will intervene in descriptions of strong homotopy versions (in the sense of Stasheff) of the dual structure (see appendix).

One can go further as discussed in [3] and visualize conjecturally all the above collision and splitting operations of the closed string theory $c_2, c_3, \dots, s_2, s_3, \dots$ defining on homology a structure Koszul dual to the positive boundary version of the Frobenius manifold structure described in [Manin].

The above is only a partial interpretation. The full interpretation of open closed string field theory in topology involves full families of arbitrary cutting and reconnecting operations of a string in an ambient space M . For closed curves some full families of these operators were labelled combinatorially by decorated even valence ribbon graphs obtained by collapsing chords in [3]. There is a serious compactness issue for the full families discussed there for realizing these in algebraic topology. The issue is a correct computation of the boundary. The problem has a parallel with renormalization in Feynman graphs. For the compactness algebraic topology issue one needs to associate operators to families of geometric graphs where various subgraphs are collapsing. When all the components of the collapsing subgraphs are trees there is no real problem as discussed in [3]. Similarly for Feynman graphs it is my understanding that if there were only tree collapses there is no problem of renormalization.

In both cases algebraic topology transversality normal bundle and Feynman graphs the loops in collapsing subgraphs cause the problems.

In [3] we had to deal with some simple cases of one loop subgraph collapses to treat the identities defining the Lie bialgebra (in particular Drinfeld compatibility). This led to the idea of using the Fulton MacPherson compactification of configuration spaces to complete the discussion. There is a normal bundle issue related to transversality which requires more analysis to treat the general FM stratum. However for disjoint union of graphs with at most one loop per component this normal bundle for transversality can be easily described as in [3].

Now we expect a Riemann surface discussion to be sufficient to complete the string field theory transversality construction. This will complete the definition of the operations for this topological interpretation of open closed string field theory. The idea is that 1) general cutting and reconnecting operation on strings is isomorphic to the change in level that occurs when passing through a critical level of a harmonic function on a Riemann surface and 2) geometrical ideas due to Thurston and then Penner [P] allow an analysis of the combinatorial compactifications of spaces of Riemann surfaces in terms of ribbon graphs.

Thus if the transversality cutting and reconnecting operations of the string field theory interpretations are organized by ribbon graphs, then the compactness and transversality normal bundle issues discussed in [3] can be treated for open and closed strings. This is work in progress.

Appendix: (dialgebras and compatibilities) Let us call a linear space V with two maps $V \otimes V \xrightarrow{\wedge} V$ and $V \xrightarrow{\vee} V \otimes V$ a dialgebra. Associative dialgebra means \wedge is associative and \vee is coassociative. Commutative dialgebra means besides being associative \wedge and \vee are symmetric. Lie dialgebra means both maps are skew symmetric and that Jacobi and coJacobi hold.

In all these cases V and $V \otimes V$ have module structures over V and there are two kinds of compatibilities between \wedge and \vee relative to these. We get six kinds of structures (five appear in this paper, see table below) which are examples of definitions of algebras over dioperads [Gan]. These are structures whose generators and relations are described diagrammatically by trees.

The familiar example of a compatibility studied by Hopf that \vee is a map of algebras (associative or commutative case but not Lie) can only be described by a non tree diagram.

The compatibilities we consider here are

derivation compatibility $\vee(a \cdot b) = (\vee a) \cdot b + a \cdot \vee(b)$ and

module compatibility $\vee(a \cdot b) = \vee(a) \cdot b = a \cdot \vee(b)$

Table with names of compatibility and/or structure and/or examples.

	Module compatibility	Derivation compatibility
Associative dialgebra	Frobenius compatibility \Leftarrow Frobenius algebra= associative algebra with non degenerate invariant inner product	infinitesimal bialgebra compatibility= infinitesimal bialgebra (see Aguilar)
Commutative dialgebra	Frobenius compatibility \Leftarrow commutative Frobenius algebra	commutative cocommutative infinitesimal bialgebra
Lie dialgebra	Frobenius compatibility \Leftarrow Lie algebra with non degenerate invariant inner product	Drinfeld compatibility =Lie bialgebra

Where the \cdot refers to the algebra structure or the module structure (which means in the associative case $a \cdot (b \otimes c) = (a \cdot b) \otimes c$, $(a \otimes b) \cdot c = a \otimes (b \cdot c)$ and in the Lie case $a \cdot (b \otimes c) = -(b \otimes c) \cdot a = [a, b] \otimes c + b \otimes [a, c]$ where $[x, y] = \wedge(x \otimes y)$.)

In [Gan] Koszul dual pairs are defined and there it is proved that upper left and upper right are Koszul dual pairs and that middle left and lower

right are Koszul dual pairs. We suppose that the lower left and middle right are also Koszul dual pairs.

We note in passing a remark about derivation or Drinfeld compatibility and algebra or Hopf compatibility. A category of "power series" Hopf algebras \mathcal{U} was shown to be equivalent to the category of Lie bialgebras \mathcal{D} where $\mathcal{D} \rightarrow \mathcal{U}$ was a formal quantization and $\mathcal{U} \rightarrow \mathcal{D}$ was a semi classical limit (Etingof-Kahzdan).

We emphasize these Koszul relations because in several important situations a strong homotopy algebraic structure of one kind is very naturally expressed by freely generated diagrams decorated with tensors labeled by the Koszul dual structure. In the above discussion all the structures that are true transversally will almost certainly lead to strong homotopy versions on the entire space of states. So these might be expressed in this graphical Koszul dual way.

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