

**CLOSED STRING OPERATORS IN TOPOLOGY
LEADING TO LIE BIALGEBRAS AND HIGHER STRING
ALGEBRA**

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Imagine a collection of closed oriented curves depending on parameters in a smooth d -manifold M . Along a certain locus of configurations strands of the curve may intersect at certain sites in M . At these sites in M the curves may be cut and reconnected in some way. One obtains operators on the set of parametrized collections of closed curves in M . By making the coincidences transversal and compactifying, the operators can be made to act in the algebraic topology of the free loop space of M when M is oriented. The process reveals collapsing sub graph combinatorics like that for removing infinities from Feynman graphs.

Let \mathbb{H} denote the equivariant homology with rational coefficients of the pair (Maps (S^1, M) , constant maps (S^1, M)) relative to the S^1 action of rotating the source. \mathbb{H} is called the reduced equivariant homology of the free loop space of M .

Associated to the diagrams I and II of the figure there are operators $c_n : \mathbb{H}^{\otimes n} \rightarrow \mathbb{H}$ and $s_n : \mathbb{H} \rightarrow \mathbb{H}^{\otimes n}$, $n = 2, 3, 4, \dots$. These operators satisfy various relations e.g. c_2 and s_2 satisfy the relations-Jacobi, coJacobi, and Drinfeld compatibility which utilize diagrams III, IV, V, and VI.

Diagram I (the operator c_n , $n = 5$):

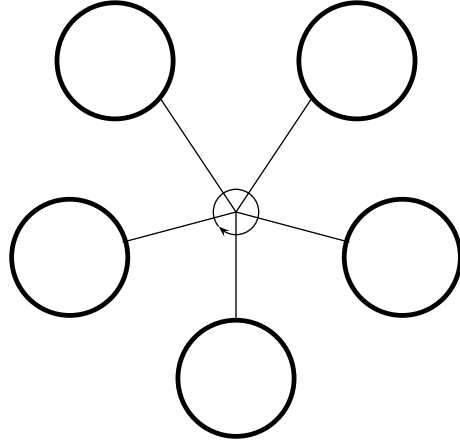


Diagram II (the operator s_n , $n = 6$):

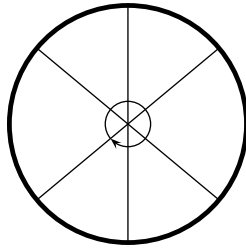


Diagram III (Jacobi identity for c_2):

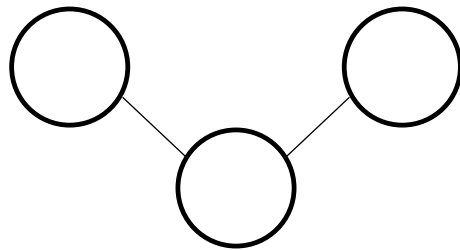


Diagram IV (Cojacobi identity for s_2):

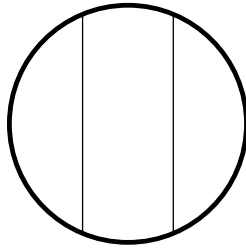


Diagram V (Drinfeld compatibility for c_2 and s_2):

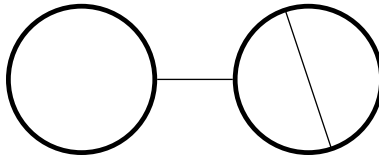
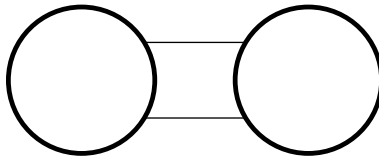


Diagram VI (Drinfeld compatibility for c_2 and s_2):



Thus we prove the

Theorem: The reduced equivariant homology \mathbb{H} of the free loop space of a smooth oriented d -manifold M has the structure of a Lie bialgebra generated by string operators $c_2 : \mathbb{H} \otimes \mathbb{H} \rightarrow \mathbb{H}$ and $s_2 : \mathbb{H} \rightarrow \mathbb{H} \otimes \mathbb{H}$ of degree $2 - d$. The string operators $c_n : \mathbb{H}^{\otimes n} \rightarrow \mathbb{H}$ and $s_n : \mathbb{H} \rightarrow \mathbb{H}^{\otimes n}$ are also defined for $n > 2$ and have degree $n + (1 - n)d$. The relations conjecturally satisfied by c_n and s_n are described below.

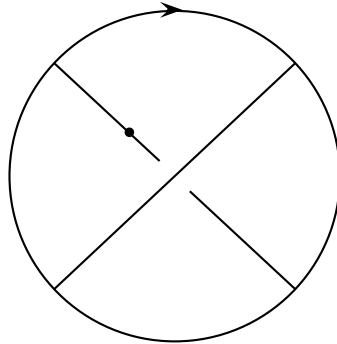
Problem and Conjecture: There is evidence (see c_n and s_n Identities paragraph below) that one may prove the c_n and s_n generate an algebraic

structure on \mathbb{H} that is Koszul dual in the sense of [Gan] to the positive boundary version (the number of inputs and the number of outputs are both positive) of the algebraic structure (genus zero) of symplectic topology [M].

Remark: This Lie bialgebra is non trivial. For example, when $d = 2$ it is isomorphic to the Lie bialgebra discovered by Goldman and Turaev[T].

General Program: The theorem is part of a more elaborate structure of closed string operators including higher genus acting at the chain level of the spaces of maps of families of closed curves into M . We describe the elements of this general theory and illustrate with the examples indicated by diagrams I, II, III, IV, V, and VI which are required to treat the theorem and diagram VII used to prove the genus one involutive identity.

Diagram VII



(Involutive identity $c_2 \cdot s_2 = 0$)

Generalized Chord Diagram: First of all there are operators of order zero, order one, etc. The operators of order zero are associated to diagrams as above. The most general *order zero diagram* D (generalized chord diagram) is specified by the data- a finite subset F of a union C of directed circles, a partition of the subset F into parts of cardinality at least two, and a cyclic order on each part. Higher order operations

correspond to general chord diagrams with additional combinatorial data (see paragraphs Diagrams V, Diagram VI). To the order zero diagram one can associate a graph $\Gamma(D)$ by attaching an n -prong respecting the cyclic order (as in diagram I or II) to each part of the partition. There is also a surface $\Sigma'(D)$ obtained by crossing the collection of circles C with the unit interval $[0,1]$ and attaching along n segments a thickened (n -prong), a disk with n tabs, to $C \times \{1\}$. So $\Sigma'(D)$ equals $C \times [0,1]$ union (thickened n -prongs) along $C \times \{1\}$.

Ribbon surface and cyclic graph of a diagram: We denote by $\Sigma(D)$ the surface obtained from $\Sigma'(D)$ by collapsing to points the n -prongs slightly extended into the collar to level $C \times \{1/2\}$. The quotient of $C \times \{1/2\}$ by this collapse is called the cyclically ordered graph $\Gamma(D)$ of the diagram D . $\Sigma(D)$ is a *ribbon surface* associated to the *cyclically ordered graph* $\Gamma(D)$, associated to the *generalized chord diagram* D .

Closed String Operator: Let us try to define the chain operator associated to D . The input is a bundle η with fibre C and a map of the total space of η into M smooth along the C direction. (These objects can be used to compute tensor products of the equivariant homology reduced or not). We form the associated bundle to η with fibre all isotopic configurations of F embedded in C . We restrict attention to the D -locus in the total space of the associated bundle, defined as the locus where the map identifies in M all the points in each part of the partition of F (now considered as a subset of $C \times \{1/2\}$). Along this locus there are induced

maps of the quotient graph $\Gamma(D)$ of C into M and thus also of the ribbon surface $\Sigma(D)$ which retracts to $\Gamma(D)$. The restriction of this map on $\Sigma(D)$ to C' , the rest of the boundary of $\Sigma(D)$ besides C (actually $C \times \{0\}$) in each fibre is a map of the total space of a C' fibration over the D locus into M . This is the output of the D operation at the level of set theory or topological spaces.

Chain Operator: To have a good object in algebraic topology there are several issues compactness, transversality, and orientation. We want the output to be a relative chain representing an element in a chain complex computing $\mathbb{H}^{\otimes i}$, $i =$ number of components of C , if the input is. We now discuss these issues in the order mentioned.

Diagram I: Now the compactness property of chains is not a problem for diagram I because its configuration space is already compact being an n -torus. Similarly, compactness is easy to arrange for any cactus diagram (diagram *III'*) generalizing diagram III. (see paragraph Diagram III below). The homological content of this part of theory was discussed in [CS] where the non reduced equivariant homology, the Lie bracket c_2 and the higher analogues c_n were considered.

Diagram II: For diagram II the configuration space is non compact and there is an obstacle to overcome. Note however that when two prongs come close enough together (relative to the input C-family) the output C' family has for each parameter at least one component which is small in M . The role of the reduced equivariant homology is to take advantage of this fact. Because families with constant (or small) map components are

considered to be null or zero for the reduced discussion we obtain that the configuration space of diagram II is (relatively) compact for the purpose of producing a (relative) chain for the computation of $\mathbb{H}^{\otimes i}$, the reduced equivariant homology.

Relative Compactness: To treat the chain operator for a general diagram we need to complete its configuration space enough so that this relative compactness is achieved. Namely, any strata omitted from a true compactification must correspond to output families which for each parameter have a positive number of tiny components.

Constraint Normal Bundle: Besides this compactness consideration the other "sine qua non" issue is the normal bundle to the constraint locus. Imposing the conditions defining the locus of the diagram in the cases above amounts to taking the preimage of a diagonal with a normal bundle. Then the transversal preimage of a chain will be a chain (the Thom map at the chain level) and we can work in the context of algebraic topology.

We refer to this as the constraint normal bundle issue and we must keep this *constraint normal bundle* as we add pieces to the configuration spaces to obtain (relative) compactness.

Diagram III: Let us consider diagram III. When two different chords on the middle circle coalesce we converge to constraints as in diagram I ($n=3$). The number of constraints is the same (two) and the normal bundles fit together perfectly. This works for the normal bundle consideration to compactify all the configuration spaces of cactus like diagrams, diagram *III'* (planar trees with circles inserted at some of the vertices)

leading to genus zero n to one operations. The point is that compactness is achieved by allowing *different* parts of the partitions to coalesce where the constraint normal bundle persists. Because of the tree like property it is impossible to let parts self collide. Except for orientations the discussion is conceptually complete in these cases- we have actual compactness and a normal bundle to define a transversal Thom chain map, in the equivariant context. (compare [CS] and [V]).

Diagram *III'* :

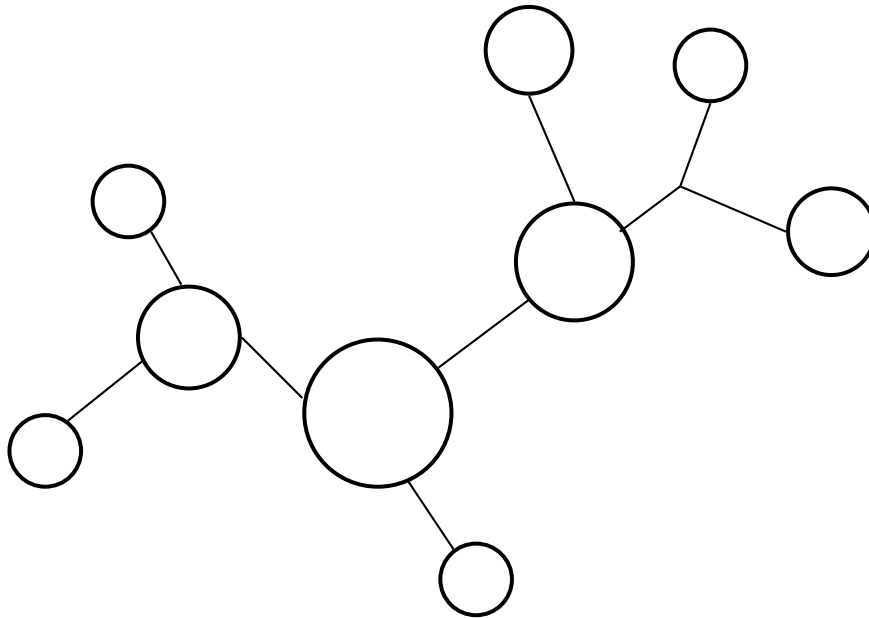
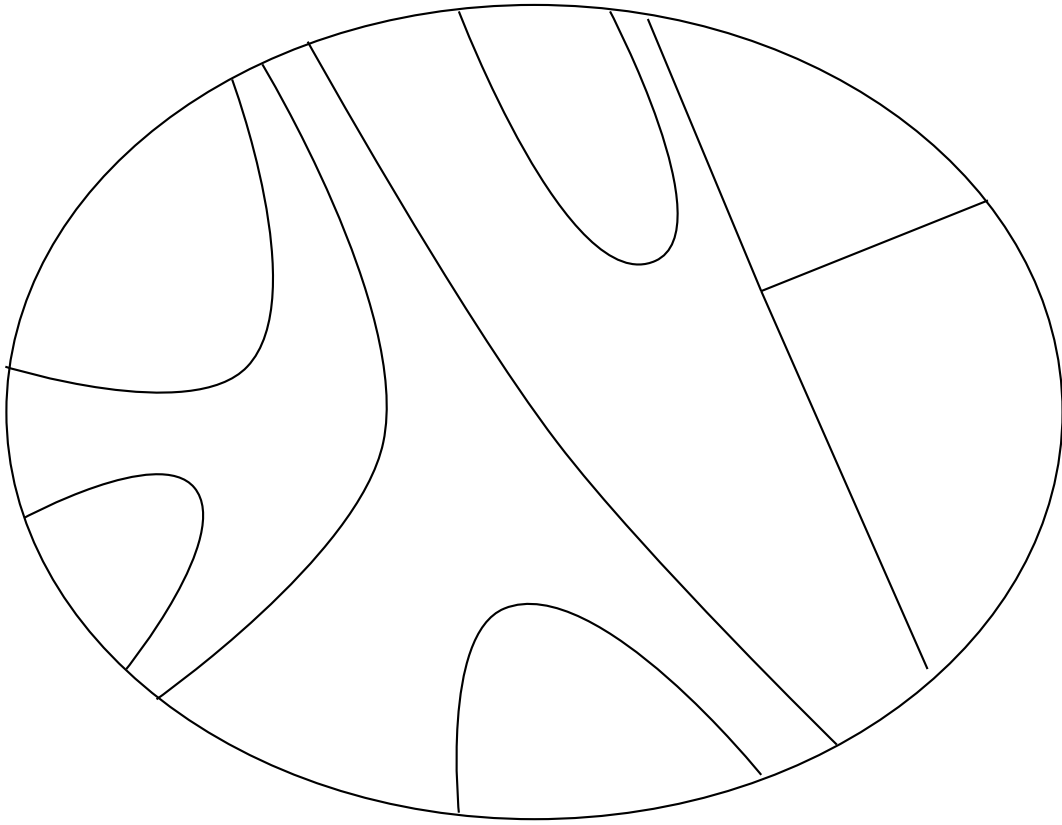


Diagram IV' :



Duality Diagram:

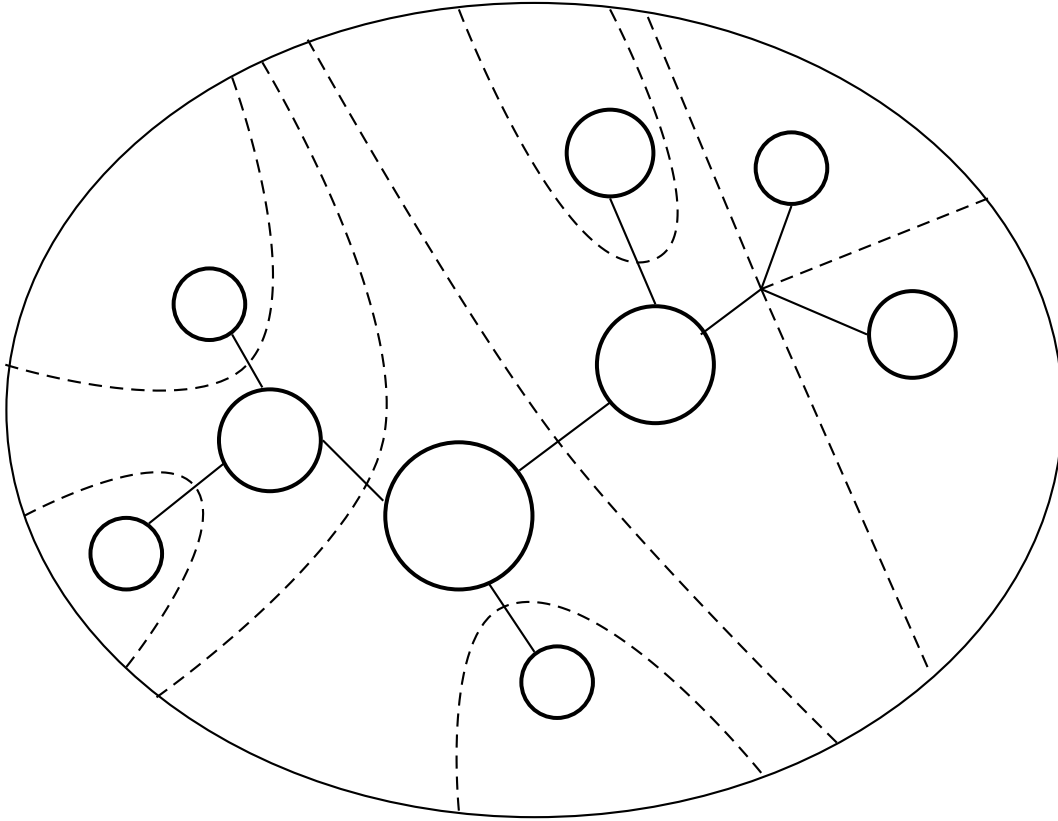


Diagram IV: Now consider diagram IV. When different chords come together *at one point* the constraints converge to those of a diagram like II ($n=3$). The normal bundles fit together perfectly as before. By adding the configuration spaces of diagram II to that of IV we obtain a *good* constraint normal bundle and *relative compactness* for Diagram IV. This also applies to more general internal chord or n -prong diagrams on one circle (disjoint n -prongs in one circle Diagram *IV'*) which are dual to planar cactii with roles of input and output interchanged (see Duality Diagram). We can add strata preserving the constraint normal bundle and achieve relative compactness. Thus dual to all the n to one genus zero

operations Diagram III' from the Diagram III paragraph we have one to n genus zero operations Diagram IV' in the reduced theory (however see next paragraph).

Null chains and Degenerate chains: There is one additional caveat about working in the reduced theory even for the diagrams considered up to now. We need to know the null subcomplex consisting of maps where at least one component is constant is invariant by the operations. At first glance this seems problematic but it works out in the end. A constant loop component may be cut up by an operation and mixed into other components. If so we no longer have a constant component. However, the situation is saved because we obtain a degenerate chain-one whose geometric dimension is too low. For as the parameters of the configurations vary in the component which is mapped to a constant loop the image chain is not varying. A transversal pull back will not have the full homological dimension and can be ignored. In fact we can mod out by degenerate chains from the beginning. (This point has to be considered carefully when extraordinary homology theories are studied here). On the other hand if the operation doesn't touch the null component we still have a null component in the output. In summary definable string chain operators act in the relative complex defining the reduced equivariant theory. For example we have all the ingredients now to define the operations c_n and s_n , $n = 2, 3, 4, \dots$ in the reduced equivariant theory.

One need only add that diagrams I and II correspond to *cycles* since no strata were added to create (relative) compactness. Thus the chain operators corresponding to c_n and s_n commute with the ∂ operators on chains and pass to homology. (Orientations will be discussed in the paragraph below).

To prove the relations of a Lie bialgebra among the compositions of c_2 and s_2 we have to consider diagrams V and VI which bring forth two further considerations.

Diagram V: To achieve relative compactness for the configurations space of diagram V we have to let the two chords touch at one point which is a case already considered above for Diagram III (and IV). We also have to allow the internal chord of Diagram V to collapse to the endpoint of the connecting chord (from opposite sides only- because a one sided approach leads to a tiny output circle and a null chain). This creates in the limit a diagram of type I for $n = 2$ and if we do nothing else the number of constraints goes down and we lose the normal bundle property. However, the collapsing internal chord and the constraint that values at the endpoints of this chord coincide say that in the limit the derivative of the map in the C direction is null at the limit point.

Thus we are led to an order one diagram, a diagram of order zero of type I for $n = 2$ with the additional data that one of the attaching points of the chord is a point of multiplicity two. This means that when the locus of this diagram of order one is defined the condition coincidence of values at the endpoints of the chord is augmented by the condition that

the 1st derivative in the C direction is zero at the point of multiplicity two. (In general at a point of multiplicity k the first $(k - 1)$ derivatives would be required to be zero). Then the constraint normal bundle extends continuously over the added stratum for the relative compactness. This treats diagram V.

Diagram VI: One more feature appears in treating diagram VI. To the generic $4D$ configuration space of diagram VI we add three $3D$ strata and two $2D$ strata for relative compactness. One of the $3D$ strata will involve a new kind of consideration similar to that in the Fulton MacPherson compactification of configuration spaces [FM]. The other strata will be of the type already considered. Namely, two of the $3D$ strata allow the two chords to touch on one circle or the other. This has already been considered. The new case appears when the two chords approach each other (on opposite sides again) at both endpoints at commensurable distances. We add a $3D$ stratum to our space which records the limiting single chord (two parameters) and a third parameter which can record the signed ratio of the small distances in the approach. We call this an FM-stratum.

We also add two strata for the chords touching first at one endpoint and then at the second endpoint which-like the discussion of diagram V-produces in each case a multiple point. These strata account for the approach of chords at both endpoints with incommensurable distances at the endpoints.

When defining the locus for this completion (to a relatively compact configuration space) we treat the FM stratum in the following way. We

have a chord diagram of type I ($n = 2$) with a third ratio parameter λ . We ask first that the map agrees at endpoints of the chord as before and then ask further that derivatives in the C direction at these points be proportional with ratio $1/\lambda$, (the factor $1/\lambda$ because distances appear in the denominator when calculating derivatives).

If the other strata are treated as described above the constraint normal bundle extends continuously over this entire (relative) compactification of the configuration space of diagram VI. This treats diagram VI.

Now we have all the ingredients to define the chain operators of diagrams I through VI up to a question of orientation.

Orientations: It is possible to avoid a nightmare of sign difficulties using a categorical approach to orientations motivated by [D]. First there is the "graded line" functor from finite dimensional real vector spaces to $Z/2$ graded vector spaces. It assigns to V the top exterior power placed in even degree if dimension V is even and in odd degree if dimension V is odd.

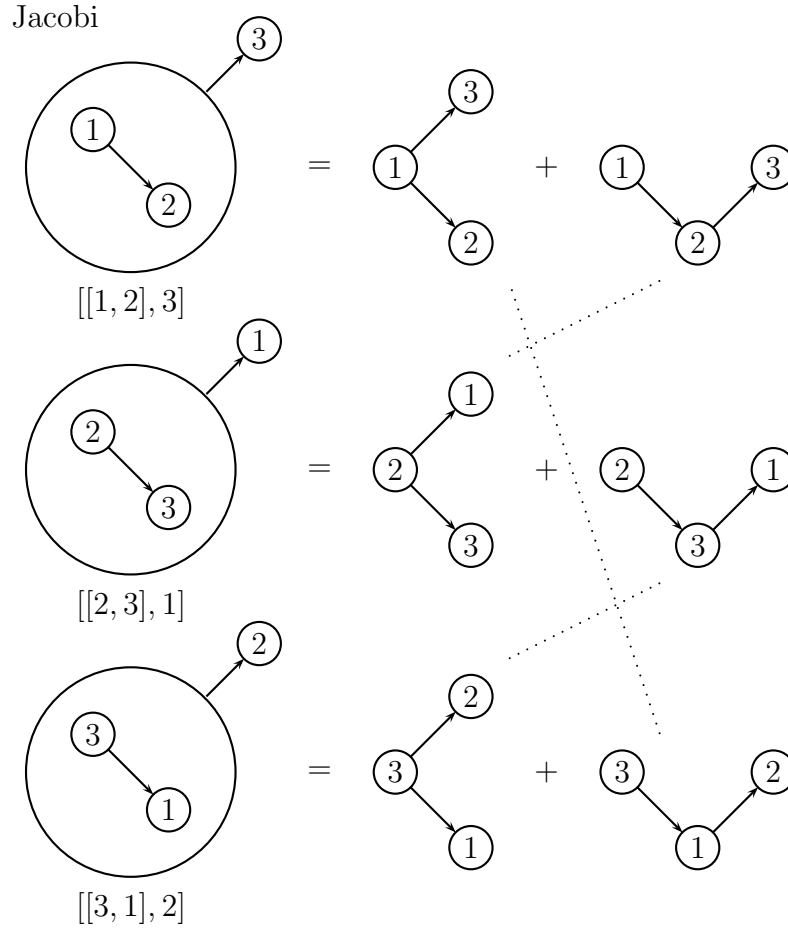
An orientation of V is by definition a generator of the graded line of V up to positive multiples. For any finite family of spaces V_f , $f \in F$ there is a canonical notion of direct sum $\bigoplus_{f \in F} V_f$ and if each V_f is oriented a canonical orientation of $\bigoplus_{f \in F} V_f$. Similarly in an exact sequence of spaces $0 \rightarrow V \rightarrow W \rightarrow V' \rightarrow 0$ an orientation of any two determines canonically an orientation of the third.

We apply this to the previous constructions as follows. Fix an orientation of M . Then any product $\prod_{f \in F} M$ indexed by a finite set has a canonical

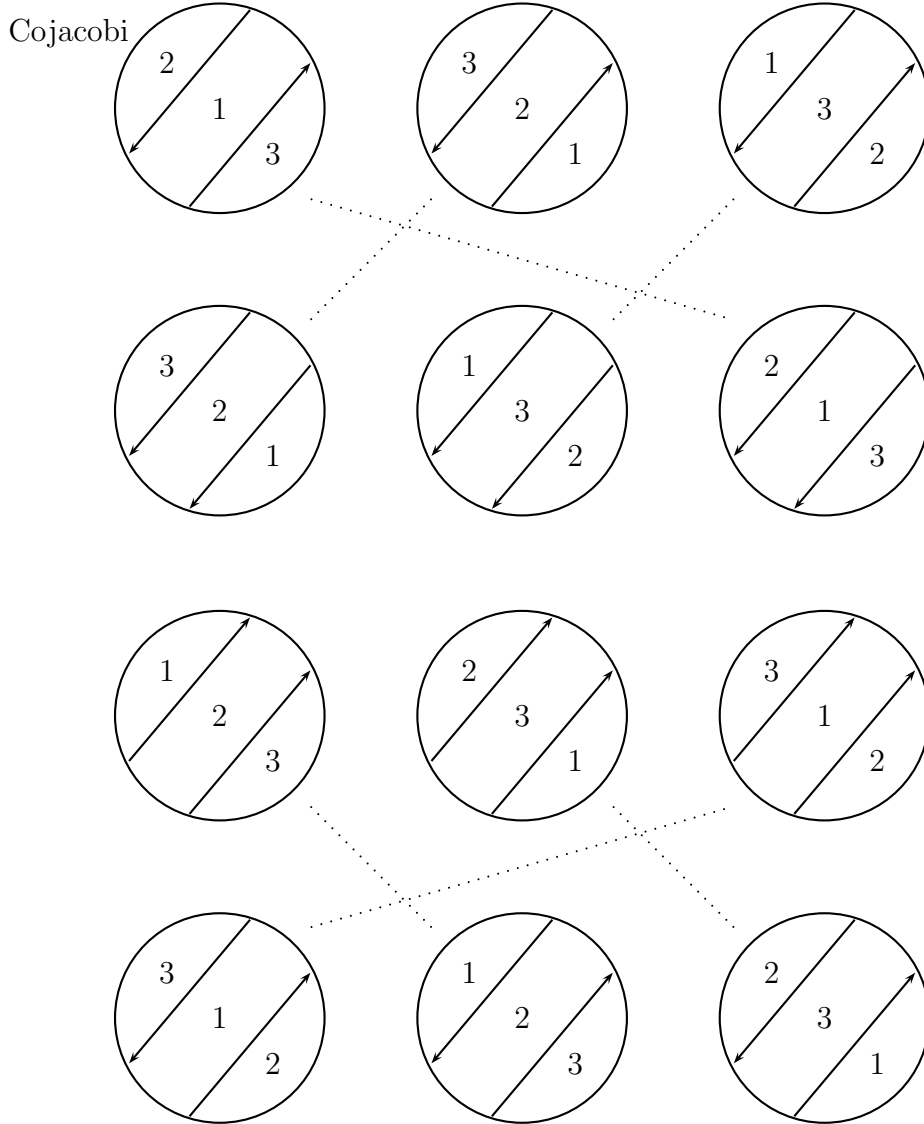
orientation by the first fact above. Using the second fact as well, any diagonal corresponding to a part of F and its normal bundle each has a canonical orientation. Further, if the base of a C bundle is oriented the total space η is canonically oriented (since C is oriented) and the total space of the associated bundle of F configurations in C is oriented (if F is ordered up to even permutations). In our examples Diagrams I, II, III, IV, V, VI this ordering on F comes from the ordering of input or output components. In example Diagram VII it comes from the ordering of the chords. Combining all this the transversal pull back is also canonically oriented. The actual definition of the operators c_n and s_n uses these canonical orientations.

Lie bialgebra identities, Jacobi:

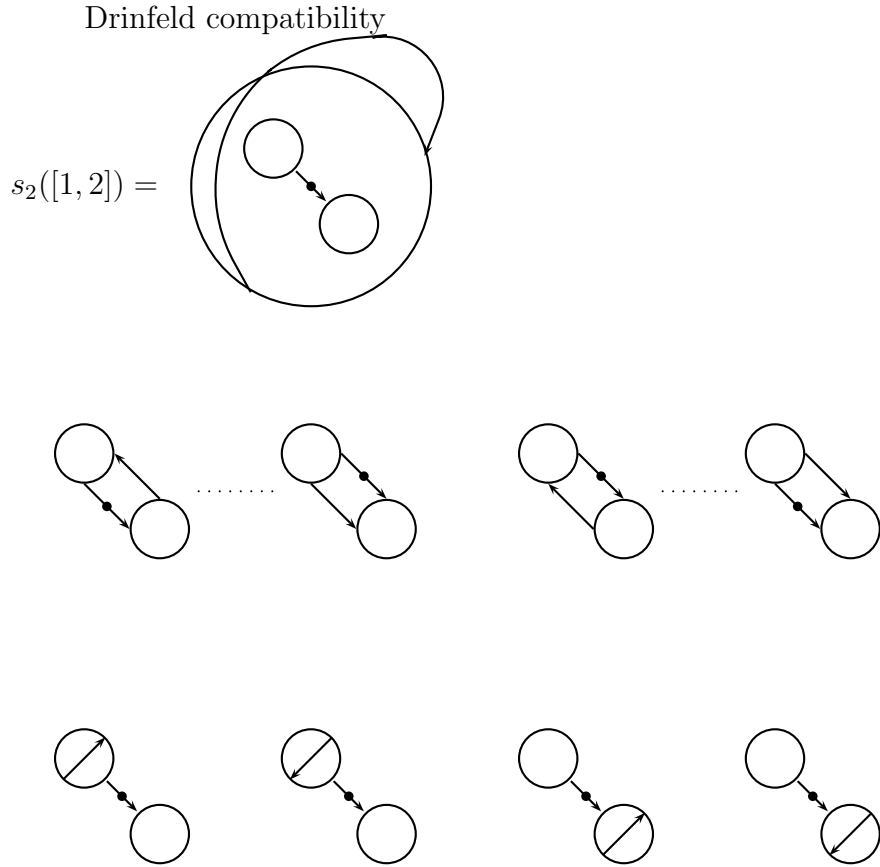
The proof of Jacobi for c_2 uses several versions of diagram III, to calculate $[[1,2],3]=c_2(c_2(1 \otimes 2) \otimes 3)$ and its cyclic permutations. The arrow on the chords is determined by the order of input arguments and determines, via an ordering in F up to even permutations, the orientation. The cancellation is indicated.



Cojacobi: The proof of Cojacobi for s_2 uses several versions of diagram IV. Again the direction on the chords is related to the order of arguments and subsequent orientations. Each column represents $(s_2 \otimes 1) \cdot s_2$ or its cyclic permutation. The numbers indicate the output arguments. The particular 4 element subset F pictured with its indicated coincidences could contribute in various ways to the chain operation. A chord could have either direction or be used as first chord or as second chord in the composed operation. Of the 8 possibilities in each case the 4 pictured are the only ones that occur. The cancellation is indicated.



Drinfeld compatibility: The proof of Drinfeld compatibility between c_2 and s_2 uses diagram V and diagram VI. The figure represents $s_2[1,2]$. The directions on chords indicate ordering of input variables for $s_2[1,2]$ and subsequent orientations. The dot on the chord indicates it is used first. Cancellation is indicated. The last four terms represent the right hand side of the compatibility equation, $s_2([1,2]) = [s_2(1),2] + [1,s_2(2)]$:



Identities for c_n and s_n : Using computations of Getzler [G] one can show the c_n taken together satisfy a generalized Jacobi identity. Some but not all of these were shown at the chain level in [CS]. The rest follow as in [G] from the BV homology structure of [CS]. The Getzler identities (defining what he calls a "gravity algebra") are Koszul dual [G] to the associative (or commutative) identities in the definition of a "Frobenius manifold" [M] describing the algebraic structure of genus zero Gromov Witten invariants of a closed symplectic manifold.

In the latter case there is a compatible non degenerate inner product—the Poincare duality of a closed symplectic manifold in its Floer homology

which is equal to its ordinary homology. One can always form the "positive boundary" version of an algebraic structure by trading in multiplications and inner product for multiplications and co multiplications satisfying the induced identities. (This idea we learned from David Kazhdan and the terminology "positive boundary" was suggested by Ralph Cohen.)

Do this for "Frobenius manifold", form the positive boundary version and then apply Koszul duality [G], [Gan], [M, p.87]. One obtains an algebraic structure which combines gravity algebra and gravity co algebra with Drinfeld type compatibilities. It contains the notion of Lie bialgebra [Gan]. At this point one knows the c_n satisfy the gravity algebra identities (generalized Jacobi [G]), the s_n by arrow reversal duality satisfy the gravity coalgebra identities (generalized Cojacobi) and c_2 and s_2 satisfy Drinfeld compatibility (this paper).

This is the evidence for the conjecture and problem mentioned above.

Involutive property of the Lie bialgebra: We may consider the operation $e = c_2 \circ s_2: \mathbb{H} \rightarrow \mathbb{H}$. If we think of s_2 and c_2 as associated to pairs of pants pointed in opposite directions then e is associated to a torus with one input circle and one output circle obtained by glueing these two pairs of pants. The diagram for e is one circle with (ordered) two chords whose endpoints are linked (see diagram VII). The (relative) compactification of this Diagram VII uses all of the considerations above- the different parts colliding of Diagrams III and IV, the multiple points of Diagram V, and the FM stratum of Diagram VI. Examining the chain operation shows that there is a complete cancellation because interchanging the ordering

of chords is orientation reversing for Diagram VII. Thus the genus one operator e is always zero in the Lie bialgebra.

Now e is the infinitesimal analogue for Lie bialgebras of the square of the canonical antiautomorphism of a Hopf algebra.

When this square is the identity one says the Hopf algebra is involutive so we say in analogy that since e is zero the *Lie bialgebra of the Theorem is involutive*. Note this is a genus one relation.

Higher genus operators: Forming the (relative) compactification of a general chord diagram D requires the addition of many strata. When different parts of F collide at one point the normal bundles fit together as indicated above. When the same part self collides at one point we introduce a multiplicity at that point. When several points collide at different rates we can reduce to the previous cases. When several points collide at commensurable rates we may have to introduce FM strata as in paragraph Diagram VI. The constraint bundle issue must be solved by adding relations among derivatives at the coincident points. These relations are associated to cycles in the subgraphs of $\Gamma(D)$ which are collapsing. For example the multiple point of Diagram V (and other examples) corresponds to collapsing a loop of $\Gamma(D)$. The FM stratum of Diagram VI corresponds to collapsing a cycle in $\Gamma(D)$ made out of two edges. More generally when intervals between points of F collapse this determines a collapsing subgraph of $\Gamma(D)$. Cycles on this graph give relations among derivatives. These are required to define the constraint normal bundle.

The theory begins to take on the structure of the collapsing graphs in the renormalization theory of Feynman diagrams appearing in the work of Kreimer et al [K]. This will be discussed elsewhere [S].

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