ON THE LOCALLY BRANCHED EUCLIDEAN METRIC GAUGE

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To Professor Yuriĭ Grigor'evich Reshetnyak on his seventieth birthday

Abstract
A metric gauge on a set is a maximal collection of metrics on the set such that the identity map between any two metrics from the collection is locally bi-Lipschitz. We characterize metric gauges that are locally branched Euclidean and discuss an obstruction to removing the branching. Our characterization is a mixture of analysis, geometry, and topology with an argument of Yu. Reshetnyak to produce the branched coordinates for the gauge.

1. Introduction

A *metric gauge* on a set is a maximal collection of metrics on the set such that the identity map between any two metrics from the collection is locally bi-Lipschitz; that is, locally the ratio $d(x, y)/d'(x, y)$ of two metrics is bounded from above and below by positive constants independent of the points x and y. In this paper, we present a characterization for metric gauges that are locally "branched Euclidean" and discuss an obstruction to removing the branching. We consider n-dimensional gauges that are embeddable in a finite-dimensional Euclidean space and whose local cohomology groups in dimensions $(n-1)$ and higher are similar to those of an n-manifold. Our approach is to stipulate enough structure so that one can consider differential Whitney 1-forms on the gauge together with an orientation on the measurable cotangent bundle that is compatible with a chosen local topological orientation. We call an n-tuple $\rho = (\rho_1, \dots, \rho_n)$ of locally defined 1-forms on an n-dimensional gauge a (local) *Cartan-Whitney presentation* of the gauge if

$$\operatorname{ess\,inf} *(\rho_1 \wedge \cdots \wedge \rho_n) > 0. \tag{1.1}$$

We prove that if the gauge supports, in addition, a Poincaré inequality, then each (local) Cartan-Whitney presentation ρ determines a positive integer-valued func-

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tion $\operatorname{Res}(\rho, \cdot)$, the *residue* of the presentation, such that the metric gauge is locally Euclidean at a point p if and only if the residue (of some presentation) satisfies $\operatorname{Res}(\rho, p) = 1$. Moreover, for each presentation ρ, the residue function $\operatorname{Res}(\rho, \cdot)$ assumes the value 1 on a dense open set of full measure with complement at most $(n-2)$-dimensional. In particular, the existence of local Cartan-Whitney presentations implies that the gauge is locally Euclidean almost everywhere.

The main ingredient of the proof is a general form of a theorem of Reshetnyak [Re1]. We show that the map

$$x \mapsto f(x) = \int_{[p,x]} (\rho_1, \ldots, \rho_n), \tag{1.2}$$

defined through integration of the 1-forms $\rho_1, \ldots, \rho_n$ as in (1.1), defines a Lipschitz branched cover into $\mathbf{R}^n$, with the property that

$$\liminf_{y \to x,\ y \neq x} \frac{|f(x) - f(y)|}{d(x, y)} \geq c > 0 \tag{1.3}$$

for all x and for some $c > 0$ independent of x. The residue $\operatorname{Res}(\rho, p)$ is the local index of the map (1.2) at p. All this is made more precise in our main theorem, Theorem 4.2. To prove the theorem, we make use of the recent advances in differential analysis and nonlinear potential theory on metric measure spaces with Poincaré inequality.

The metric gauges that admit local Cartan-Whitney presentations need not be manifolds in general, and even if they are manifolds they need not be locally Euclidean (see Examples 2.4). But they are always branched Euclidean. Indeed, our study leads to a characterization of a locally branched Euclidean metric gauge.

Definition 1.4
A metric gauge is said to be *locally branched Euclidean* if it is n-dimensional, satisfies the local cohomology condition as in Axiom I, and admits local BLD-maps into $\mathbf{R}^n$.

To describe the terminology in Definition 1.4, let (X, d) be a locally compact, n-dimensional metric space, $n \geq 2$, with integral cohomology groups in degrees $(n-1)$ and higher locally equivalent to those of an n-manifold (as in Axiom I). We call X *locally BLD-Euclidean* if every point in X has an open neighborhood U and a finite-to-one, open and sense-preserving Lipschitz map $f : U \to \mathbf{R}^n$ such that

$$\frac{1}{L} \operatorname{length} \alpha \leq \operatorname{length} f \circ \alpha \leq L \operatorname{length} \alpha \tag{1.5}$$

for each path α in U, where the constant $L \geq 1$ is independent of α. Such maps are called *maps of bounded length distortion* or *BLD-maps*. Note that the local cohomology condition allows us to speak about sense-preserving maps. Finally, the BLD-condition is bi-Lipschitz invariant, and so it makes sense to speak about (local) BLD-maps of a metric gauge into $\mathbf{R}^n$.

In Euclidean spaces, BLD-maps form a subclass of more general *quasi-regular mappings* or *mappings of bounded distortion*, introduced by Reshetnyak in the 1960s (see [Re1], [Re2], [Ri], [MV], [Su]). In general spaces, BLD-maps are examples of *regular maps* in the terminology of [DS] (see [HR2, Theorem 4.5]). The local degree function for the map in (1.2) was studied in [Su] in the context of Lipschitz manifolds; in particular, condition (1.3) was proved in [Su] in this case. Note that condition (1.3) easily follows from the BLD-condition by the path-lifting property for discrete and open maps (see [HR2, Section 3.3]).

We show that a locally branched Euclidean metric gauge is characterized by four axioms, Axioms I – IV presented in Section 2, provided that we also make the a priori assumption that the gauge is locally embeddable in a finite-dimensional Euclidean space with a metric orientation on its measurable tangent bundle (see Section 3.4 for the terminology). The axioms are a mixture of analysis, geometry, and topology. They stipulate local cohomological and measure-theoretic properties of the gauge, the existence of a Poincaré inequality, and the existence of local Cartan-Whitney presentations.

It remains an interesting open problem to find an additional axiom that would remove the branching in the gauge. In our approach this amounts to an analytic characterization of local Cartan-Whitney presentations whose residue is everywhere 1 (see Remark 2.5 for a conjecture). Only a few nontrivial sufficient conditions for a locally Euclidean metric gauge are known: L. Siebenmann and D. Sullivan [SS] characterized the polyhedra in high dimensions that are Lipschitz manifolds, and T. Toro [T1], [T2] found positive answers in two other special cases. For related studies and examples, see [HR1], [HR2], [L], [Se2], and [Se3].

2. The axioms

Let $\mathscr{M}$ be a metric gauge on a set X, and let $n \geq 2$ be an integer. We describe four axioms that are shown to be necessary and sufficient for $X = (X, \mathscr{M})$ to be locally branched Euclidean. The axioms are explained and analyzed more carefully in Section 3.

Axiom I. X is locally compact and has integral cohomology modules in degrees $(n - 1)$ and higher locally equivalent to those of an n-manifold.

Axiom II. X is metrically n-dimensional, locally bi-Lipschitz embeddable in some Euclidean space, and locally metrically orientable.

Axiom III. X supports a Poincaré inequality.

As is explained in Sections 3.21 and 3.23, Axioms I and II allow us to define gauge Whitney 1-forms on X. These are bounded measurable 1-forms on X with bounded exterior derivatives.

A locally defined n-tuple $\rho = (\rho_1, \ldots, \rho_n)$ of gauge Whitney 1-forms on X is called a (local) Cartan-Whitney presentation of the gauge if the associated volume form has constant sign and is uniformly bounded away from zero (in the almost everywhere sense).

Axiom IV. Cartan-Whitney presentations exist locally on X.

It is implicitly assumed that each of the above axioms includes the preceding ones to the extent its definition so requires.

We have the following theorem.

THEOREM 2.1
Let X be a metric gauge that satisfies Axioms I and II. Then X is locally branched Euclidean if and only if it satisfies Axioms III and IV.

The sufficiency part in Theorem 2.1 follows from our main theorem, Theorem 4.2, which is formulated in Section 4. For a further discussion of the axioms and for a proof of the necessity part in Theorem 2.1, see Section 5.

The following theorem sums up some of the consequences of Theorem 4.2.

THEOREM 2.2
If a metric gauge satisfies Axioms I–III, then each (local) Cartan-Whitney presentation ρ of the gauge determines (locally) a positive upper semicontinuous integer valued function, $p \mapsto \operatorname{Res}(\rho, p)$, called the residue of the presentation, which for a fixed p is continuous in ρ in the L^∞-topology, with the property that the gauge is locally Euclidean at a point $p \in X$ if and only if a Cartan-Whitney presentation ρ can be found near p such that the residue satisfies $\operatorname{Res}(\rho, p) = 1$.

Moreover, for each (local) Cartan-Whitney presentation ρ, there is a closed set of zero measure and of topological dimension at most $(n-2)$ such that $\operatorname{Res}(\rho, p) = 1$ for each p outside the closed set.

COROLLARY 2.3
If a metric gauge satisfies Axioms I–IV, then it is locally Euclidean outside a closed set of zero measure and of topological dimension at most $(n-2)$.

Examples 2.4
If X is a compact polyhedron in some Euclidean space such that every point in X has a cone neighborhood with a link that is a homology $(n-1)$-sphere, then X satisfies Axioms I–IV. The first two axioms are straightforward to verify, and the Poincaré inequality follows, for example, from [HeKo2, Section 6]. To establish Axiom IV, we observe that the method of J. Alexander [Al] (see also [BE]) can be used to map X locally to the standard (polyhedral) n-sphere by a sense-preserving piecewise linear branched cover. The pullback of the standard coframe on the sphere provides gauge Whitney 1-forms with property (1.1).

It is known that the n-sphere $\mathbf{S}^n$ has polyhedral metrics as above that are not locally Euclidean if $n \geq 5$ (see [Ca2], [Ca1], [E], [SS]). Also, on the lower-dimensional spheres $\mathbf{S}^3$ and $\mathbf{S}^4$, there are metric gauges that are not locally Euclidean even though they satisfy Axioms I–IV (see [Se2], [Se3], [HR1], [HR2]). It is not known whether a 2-dimensional gauge is locally Euclidean if it satisfies Axioms I–III. Recently, T. Laakso [L] proved that a 2-dimensional gauge need not be locally Euclidean if it satisfies Axioms I and III, and is metrically 2-dimensional.

Remark 2.5
It remains an interesting open problem to find a sharp analytic condition on a Cartan-Whitney presentation ρ that would assure that the residue of the presentation satisfies $\operatorname{Res}(\rho, p) = 1$ for all p. We conjecture that this is the case for each presentation ρ in the Sobolev class $H^{1,2}$ (see Section 3.12 for the definition of Sobolev classes). This condition would be sharp, as shown by the pullback presentation in $\mathbf{R}^2$ under the map $(r, \theta) \mapsto (r, 2\theta)$ in polar coordinates. It was proved in [HeKi] that $\operatorname{Res}(\rho, \cdot) \equiv 1$ for presentations $\rho \in H^{1,2}$ in $\mathbf{R}^n$ which are closed (i.e., $d\rho = 0$).

One can also ask if the membership of ρ in the space VMO (or perhaps in BMO with small norm) leads to the residue value 1. Here VMO and BMO stand for the spaces of vanishing mean oscillation and bounded mean oscillation, respectively.

3. Description of the axioms

In this section, we describe the content of our axioms more carefully.

3.1. Cohomology manifolds
Axiom I concerns only the local homology of the gauge. The second requirement means that for each $x \in X$ there are arbitrarily small open neighborhoods U of x such that $H_c^n(U) = \mathbf{Z}$, that $H_c^p(U) = 0$ for $p = n-1$ and for $p > n$, and that the standard homomorphism $H_c^n(V) \to H_c^n(U)$ is a surjection if $V \subset U$ is an open neighborhood of x. (Here H_c^* denotes the Alexander-Spanier cohomology with compact supports.)

If X satisfies Axiom I and is finite-dimensional, then $\dim X = n$ (see [HW, p. 151]).

3.2. Metrically n-dimensional sets

We call a metric space *metrically n-dimensional* if it can be expressed as a countable union of Lipschitz images of subsets of $\mathbf{R}^n$ plus a set of Hausdorff n-measure zero, and if for each compact set K in the space, there is a constant $C_K \geq 1$ such that

$$C_K^{-1} r^n \leq \mathscr{H}_n\big(B(x,r)\big) \leq C_K r^n \tag{3.3}$$

for all balls $B(x,r)$ of radius $r < C_K^{-1}$ centered at $x \in K$. Here and hereafter, $\mathscr{H}_n$ denotes the Hausdorff n-measure in a metric space.

Thus, a space is metrically n-dimensional if it is (*countably*) *n-rectifiable* in the sense of geometric measure theory and if it satisfies an appropriate local version of the condition known as *Ahlfors n-regularity*. In particular, a metrically n-dimensional space has Hausdorff dimension n, and the Hausdorff n-measure is locally finite and positive. Being metrically n-dimensional is a bi-Lipschitz invariant condition, and it is now clear what the first requirement in Axiom II means.

The second requirement, that X be locally embeddable in some Euclidean space, means that every point in X has a neighborhood that can be bi-Lipschitz embedded in some $\mathbf{R}^N$.

3.4. Metric orientation

A gauge as in Axiom I is locally orientable in the sense that every point in it has a connected neighborhood U with $H_c^n(U) = \mathbf{Z}$. A choice of a generator g_U in $H_c^n(U)$ is an *orientation* of U; it canonically determines an orientation of each connected open subset V of U, for the canonical homomorphism $H_c^n(V) \to H_c^n(U)$ is an isomorphism.

Assume now that U is an oriented open subset of X and that $f : U \to E$ is a continuous map into an oriented n-dimensional real vector space E. Then, for each open connected set D with compact closure in U and for each component A of $E \setminus f(\partial D)$, the map

$$f|f^{-1}(A) \cap D : f^{-1}(A) \cap D \to A$$

is proper, and the *local degree* $\mu(A, D, f)$ is the integer that satisfies

$$\xi_E \mapsto \mu(A, D, f)\, g_U \tag{3.5}$$

under the map

$$H_c^n(E) \xleftarrow{\approx} H_c^n(A) \to H_c^n\big(f^{-1}(A) \cap D\big) \to H_c^n(U), \tag{3.6}$$

where ξ_E and g_U denote the fixed orientations of E and U, respectively.

Next, assume that $U \subset \mathbf{R}^N$ is a metrically n-dimensional embedded neighborhood of a point in X of finite Hausdorff n-measure. Then U has a unique (approximate) tangent n-plane T_xU at $\mathscr{H}_n$-a.e. point $x \in U$ (see [F, Theorem 3.2.19]). We view the collection of these planes as a *measurable tangent bundle* TU over U. The bundle TU is a measurable subbundle of $T\mathbf{R}^N$ (with respect to the measure $\mathscr{H}_n \lfloor U$) and inherits a metric from $\mathbf{R}^N$. An *orientation* $\xi = \{\xi_x\}$ of TU is a measurable choice of orientations on the approximate tangent planes:

$$\xi = \big\{\xi_x : \xi_x \in \wedge_n T_xU \text{ is a simple (unit) } n\text{-vector}\big\}. \tag{3.7}$$

To say that U is metrically oriented is to say that an orientation ξ on TU can be chosen so as to be compatible with a given local orientation g_U on U; such compatibility allows us to use a degree theory for Lipschitz mappings as in the case of a smooth manifold.

To give a precise definition, let $x \in U$ be a point such that the tangent space $T_x = T_xU$ exists. Because U satisfies (3.3) locally, the set

$$\big\{y \in \mathbf{R}^N : \operatorname{dist}(y - x, T_x) > \epsilon|y - x|\big\} \tag{3.8}$$

does not meet U near the point x for each $\epsilon > 0$. (Indeed, otherwise the Ahlfors regularity condition (3.3) would imply that U has positive n-density at x along a set as in (3.8), contradicting the definition for approximate tangent planes; see [F, Theorem 3.2.19].) Thus, if π_x denotes the projection

$$\pi_x : \mathbf{R}^N \to x + T_x$$

to the affine n-plane $x + T_x$, the preimage $\pi_x^{-1}(x)$ does not meet $U \setminus \{x\}$ near x. In particular, π_x induces a map

$$H_c^n(T_x) \to H_c^n(U) \tag{3.9}$$

as in (3.6). It is easy to see that this map does not depend on the choice of the domain D in (3.6) for D sufficiently small. Then we say that U is *metrically orientable* if U is orientable and if there is an orientation g_U of U and an orientation $\xi = \{\xi_x\}$ of TU such that

$$\xi_x \mapsto g_U$$

under the map in (3.9) for $\mathscr{H}_n$-a.e. point $x \in U$. The pair (g_U, ξ) is called a *metric orientation* of U. Finally, we say that X is *locally metrically orientable* if every point in X has a neighborhood that is metrically orientable.

Example 3.10
A metric space is *locally linearly contractible* if for each compact set K in the space

there is a constant $C_K > 0$ such that for points $x \in K$ and radii $r < C_K^{-1}$, the metric balls $B(x, r)$ are contractible in concentric balls $B(x, C_K r)$. It clearly makes sense to speak about a locally linearly contractible metric gauge.

It is not hard to see that if X satisfies Axiom I, is metrically n-dimensional and locally embeddable in some Euclidean space, and is locally linearly contractible, then X is locally metrically orientable. Indeed, if U is as above, the local linear contractibility guarantees that there is a neighborhood G of U in $\mathbf{R}^N$ and a retraction $\psi : G \to U$ such that, locally,

$$|\psi(y) - y| \leq C \operatorname{dist}(y, U) \tag{3.11}$$

with $C \geq 1$ independent of y. It is then easy to see, with the above notation, that the map

$$f = \psi \circ \pi_x : \overline{D} \to U$$

is homotopic to the identity through maps $\overline{D} \setminus \{x\} \to U \setminus \{x\}$ if D is a small enough connected open neighborhood of x in U. By using this and (3.11), one checks that π_x induces an isomorphism in (3.9), and the metric orientation can be defined via this isomorphism. (See [Se1] for more discussion on local linear contractibility and related issues.)

3.12. Sobolev classes
Assuming that X satisfies Axiom II, we define Sobolev spaces $H^{1,p}(U)$ in each embedded metrically n-dimensional neighborhood U in $\mathbf{R}^N$. (The metric orientation is not needed here.) Although the spaces to be defined depend on the chosen embedding, the membership in a space of a particular degree of integrability does not. Therefore, it makes sense to speak about (local) *Sobolev classes* of functions on X. Because of the rectifiability properties of U, the definition of the Sobolev space $H^{1,p}(U)$ is rather straightforward. In particular, we do not need the recent and more sophisticated (albeit equivalent) Sobolev space theories as in, for example, [Cr], [FHK], or [Sh].

Thus, let U be a metrically n-dimensional set in $\mathbf{R}^N$ of finite Hausdorff n-measure. There is a bounded linear operator d from Lipschitz functions defined on U to bounded measurable sections of T^*U which vanishes on (locally) constant functions and satisfies

$$|du| \leq \operatorname{Lip}(u), \tag{3.13}$$

where $\operatorname{Lip}(u)$ is the Lipschitz constant of u, as well as

$$d(uv) = v\,du + u\,dv \tag{3.14}$$

and

$$d(f \circ u) = f'(u)\,du \tag{3.15}$$

for all $f \in C^1(\mathbf{R})$, where it is understood that (3.13) – (3.15) hold almost everywhere with respect to the Hausdorff measure $\mathcal{H}_n$ on U. Indeed, du is the *approximate differential* of u, as in [F, Theorem 3.2.19].

For $1 < p < \infty$ we define the *Sobolev space* $H^{1,p}(U)$ as the closure of all Lipschitz functions on U in the norm

$$\|u\|_{1,p} = \left(\int_U |u|^p \, d\mathcal{H}_n\right)^{1/p} + \left(\int_U |du|^p \, d\mathcal{H}_n\right)^{1/p}. \tag{3.16}$$

Thus, $u \in H^{1,p}(U)$ if and only if $u \in L^p(U)$ and there are a measurable L^p-integrable section α of T^*U and a sequence (u_j) of Lipschitz functions on U such that $u_j \to u$ in $L^p(U)$ and $du_j \to \alpha$ in $L^p(U)$.

3.17. Poincaré inequality

We say that X supports a *Poincaré inequality* if X is locally pathwise connected and if every point in X has a metrically n-dimensional embedded neighborhood U, together with constants $C \geq 1$ and $\tau \geq 1$, such that

$$\int_B |u - u_B|^2 \, d\mathcal{H}_n \leq C \, (\operatorname{diam} B)^2 \int_{\tau B} |du|^2 \, d\mathcal{H}_n \tag{3.18}$$

for each metric ball B satisfying $\tau B \subset U$ and for each Lipschitz function u in τB, where τB denotes the ball that is concentric with B, but with radius τ times the radius of B, and u_B denotes the integral average of u in B.

The validity of a uniform Poincaré inequality of the type (3.18), together with mild assumptions on the Hausdorff measure, implies that the space possesses many strong geometric and analytic properties (see, e.g., [Cr], [HaKo], [HeKo2], [Se5], [Sh]). In particular, we require the following fact, proved in [FHK, Theorem 10].

PROPOSITION 3.19

If U is a metrically n-dimensional embedded neighborhood that supports a Poincaré inequality as in (3.18), then the operator d from Lipschitz functions on U to L^2-sections of T^*U is closable.

Thus, under the presence of a Poincaré inequality, the section α above is independent of the sequence (u_j); it is denoted by du and called the *weak differential* of u.

Besides being used in Proposition 3.19, inequality (3.18) is used in Proposition 4.22, which in turn is crucial in the proof of our main theorem, Theorem 4.2.

We note that there are variants of condition (3.18) that could equally well be used in Axiom II; we have chosen (3.18) for its relative simplicity (see Section 5.2 for a further discussion).

Example 3.20
If X satisfies Axioms I – II and in addition is locally linear contractible (as defined in Example 3.10), then X supports a Poincaré inequality as described above. This follows from work of S. Semmes [Se1]. As pointed out in Section 5.4, in Theorem 2.1 we cannot replace Axiom III by the requirement that X be linearly locally contractible.

3.21. Whitney forms
Let U be an n-rectifiable subset of $\mathbf{R}^N$ of finite Hausdorff n-measure, and let $\xi = \{\xi_x\}$ be an orientation on TU as in (3.7). Then the pair (U, ξ) defines an n-dimensional current by integration: for each smooth n-form ω in $\mathbf{R}^N$, the action

$$\langle (U, \xi), \omega \rangle = \int_U \langle \omega(x), \xi_x \rangle \, d\mathscr{H}_n(x) \tag{3.22}$$

is defined in the usual way by using the chosen a.e. defined orientation and the Hausdorff measure.

The action (3.22) can be extended to a larger class of differential forms in $\mathbf{R}^N$, which we call *Whitney forms*. By definition, these are forms ω of bounded measurable coefficients whose distributional exterior differential $d\omega$ also has bounded measurable coefficients. For Whitney forms we have $dd\omega = 0$ in the sense of distributions. One can also pull back Whitney forms by Lipschitz maps, and $dF^*(\omega) = F^*(d\omega)$ if F is Lipschitz and ω is a Whitney form.

To briefly explain why (3.22) extends to Whitney forms, we recall that the space of Whitney m-forms as defined above can be identified as the dual of *flat m-chains* in $\mathbf{R}^N$ in the *flat norm* (see [W2, Section IX.7], [F, Section 4.1.19]). Now every m-dimensional rectifiable current in $\mathbf{R}^N$ is a flat m-chain by [F, Section 4.1.24], and every oriented m-dimensional rectifiable set is an m-dimensional rectifiable current through formula (3.22) by [F, Section 4.1.28] (cf. Section 3.26). Although Whitney m-forms are a priori only a.e. defined with respect to Lebesgue measure of the ambient space $\mathbf{R}^N$, they have representatives such that the action on m-dimensional rectifiable currents makes sense by integration. This is a theorem of H. Whitney [W2, Theorem 9A, p. 303]. (Compare this with the special and better-known case $m = 0$ when the Whitney forms are nothing but Lipschitz functions; the differential of a Lipschitz function has a well-defined restriction to each rectifiable curve.) Whenever we are dealing with Whitney forms in this paper, we tacitly assume that the good representatives have been picked.

Finally, Whitney forms can be defined and studied in any open set in $\mathbf{R}^N$.

3.23. Gauge Whitney 1-forms
If X satisfies Axioms I and II, then we can define Whitney 1-forms locally on X via their action on rectifiable curves by using the local embeddings in Euclidean space

and the bi-Lipschitz invariance of line integrals. We call these invariantly defined forms *gauge Whitney 1-forms*. To be more precise, we let U be open in X; we abuse notation and understand that U is already embedded in some $\mathbf{R}^N$. Then every Whitney 1-form ω defined in an open neighborhood G of U in $\mathbf{R}^N$ determines a gauge Whitney 1-form: if U' is a different embedding of U in $\mathbf{R}^M$, then there is a bi-Lipschitz homeomorphism h from U' onto U; the map h can be extended to a Lipschitz map $H : \mathbf{R}^M \to \mathbf{R}^N$ by Kirszbraun's theorem (see [F, Theorem 2.10.43]), and the Whitney 1-form $H^*(\omega)$ is defined in the open neighborhood $G' = H^{-1}(G)$ of U' and satisfies

$$\int_{H^{-1}(\gamma)} H^*(\omega) = \int_\gamma \omega$$

for all rectifiable curves γ in U.

Axiom IV means that for each point in X there are a metrically oriented neighborhood U and an n-tuple of gauge Whitney 1-forms $\rho_1, \dots, \rho_n$ defined in U such that

$$\operatorname{ess\,inf} * (\rho_1 \wedge \cdots \wedge \rho_n) > 0, \tag{3.24}$$

where the Hodge star operator $* : \wedge^n TU \to \mathbf{R}$ is determined by the given metric orientation. Note that $\rho_1 \wedge \cdots \wedge \rho_n$ is a Whitney n-form by [W2, p. 277] or by [Re2, Lemma 4.4, p. 133]. Condition (3.24) is independent of the chosen embedding of U in Euclidean space.

Remark 3.25

We could have defined gauge Whitney 1-forms in Section 3.23 more intrinsically (say, in the spirit of [W1, p. 4]) without the requirement of local extension in the ambient Euclidean space. However, such an extension is necessary for our proof of Theorem 4.2. If we strengthened Axiom II by requiring that X be locally embeddable in some Euclidean space as a (local) Lipschitz retract, then local extensions of intrinsically defined forms would exist. This new axiom would also imply the current Axiom III, via local linear contractibility as in Remark 3.20 but would not be necessary for the gauge to be locally branched Euclidean (see Section 5.4). It is not clear whether intrinsically defined gauge Whitney 1-forms can be extended (locally) to the ambient Euclidean space under the present axioms, nor whether our proof could be made to work without such extension.

3.26. Stokes cycles

We now address the precise technical sense in which our objects define "abstract cycles" locally; namely, for each local embedding of the gauge in $\mathbf{R}^N$, we have the expected integration by parts formula. In more detail, recall that an *n-dimensional rectifiable current* in $\mathbf{R}^N$ is a current with compact support which is representable by integration over an oriented n-rectifiable set with integer multiplicities (see [F, Theorem 4.1.28]). Thus, each n-dimensional rectifiable current is associated with a triple (W, ξ, μ), where W is an n-rectifiable set, $\xi = \{\xi_x\}$ is a measurable choice of unit n-vectors on $\wedge_n TW$, and μ is an integer-valued $\mathscr{H}_n$-integrable (multiplicity) function on W. We call a current T in $\mathbf{R}^N$ an *n-dimensional Stokes current* if it is an n-dimensional rectifiable current with compact support and if an associated triple can be chosen so that the set W is locally compact and satisfies

$$W \cap \operatorname{spt} \partial T = \emptyset. \tag{3.27}$$

In other words, each point in W should have a neighborhood such that

$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle = 0$$

for each smooth, and hence Whitney, $(n-1)$-form ω with support in the neighborhood.

If we start with an n-rectifiable, locally compact bounded set W, together with a choice of orientation ξ on TW, and if (3.27) holds for the current $T = (W, \xi)$, then we say that *W represents an n-dimensional Stokes current in the orientation ξ*.

Stokes currents allow for localization: if (W, ξ) is an n-dimensional Stokes current and $W' \subset W$ is open in W, then (W', ξ') is an n-dimensional Stokes current, where $\xi' = \xi | W'$ is the restriction of the orientation ξ to W'.

We call an n-dimensional metric gauge a *(local) Stokes cycle* if every point in the gauge has an embedded neighborhood in some Euclidean space which represents an n-dimensional Stokes current in a metric orientation; whether or not an embedded neighborhood has this property is independent of the choices.

We learned the proof of the following proposition from Stephen Semmes, whose participation we thus gratefully acknowledge.

PROPOSITION 3.28
If X satisfies Axioms I and II, then X is a local Stokes cycle.

Proof
Let U be an embedded (in $\mathbf{R}^N$) metrically oriented neighborhood of a point p in X, and denote by $T = (U, \xi)$ the corresponding n-current. We have to show that there is $\delta > 0$ such that

$$\langle T, d\omega \rangle = \int_U \langle d\omega(x), \xi_x \rangle \, d\mathscr{H}_n(x) = 0 \tag{3.29}$$

for each smooth $(n-1)$-form ω with support in the N-ball $B(p, \delta)$. We first show that (3.29) is true for forms ω of the form

$$\omega = F^*(\alpha), \tag{3.30}$$

where α is a smooth $(n-1)$-form on $\mathbf{S}^n$ and $F : \mathbf{R}^N \to \mathbf{S}^n$ is a smooth map, homotopic to a constant through maps $F_t : \mathbf{R}^N \to \mathbf{S}^n$ such that $F_t | U \setminus B(p, \delta) \equiv \sigma$, where $\sigma \in \mathbf{S}^n$ is independent of t. To this end, we use the pushforward current $F_{\#}(T)$, which satisfies

$$\langle F_{\#}(T), d\alpha \rangle = \langle T, dF^*(\alpha) \rangle = \langle T, d\omega \rangle.$$

We have the integral expression

$$\langle F_{\#}(T), d\alpha \rangle = \int_{\mathbf{S}^n} \langle d\alpha(y), \eta(y) \rangle \, dy,$$

where $dy = d\mathscr{H}_n(y)$ on $\mathbf{S}^n$, and $\eta(y) = 0$ or

$$\eta(y) = \sum_{x \in F^{-1}(y) \cap U} a(x) \mathscr{O}_y = \mathscr{O}_y \sum_{x \in F^{-1}(y) \cap U} a(x).$$

Here $\mathscr{O} = \{\mathscr{O}_y\}$ denotes the standard orientation on $\mathbf{S}^n$ and $a(x) = 1$ if the approximate differential $dF(x) : (T_x, \xi_x) \to (T_y, \mathscr{O}_y)$ is sense preserving, and $a(x) = -1$ in the opposite case (see [F, Section 4.1.30] for these facts). Now the sum on the right is the sum of the signs of the Jacobians of F, which equals the degree of F, thus zero, almost everywhere (cf. (4.29)). (The assumption on metric orientation is used here.) Therefore, (3.29) follows for forms as in (3.30).

To prove the general case, we observe first that by linearity and by change of coordinates we may assume that ω is of the form $\omega(x) = u(x)\, d\lambda_{n-1}$, where $d\lambda_{n-1} = dx_1 \cdots dx_{n-1}$ and u is a smooth function with compact support. If $b(x)$ is any smooth bump function with support on $B = B(p, \delta)$ and total integral 1, then the convolution $\omega_\epsilon = b_\epsilon * \omega$ satisfies $\langle \partial T, \omega_\epsilon \rangle \to \langle \partial T, \omega \rangle$ as $\epsilon \to 0$ (see [W2, (12), p. 176]). On the other hand, one easily computes that

$$\begin{aligned} \langle \partial T, \omega_\epsilon \rangle &= \langle T, d\omega_\epsilon \rangle = \langle T, db_\epsilon * \omega \rangle \\ &= \int_{\operatorname{spt}\omega} u(y) \int_U \langle db_\epsilon(x-y) d\lambda_{n-1}, \xi_x \rangle \, dx \, dy, \end{aligned}$$

which is zero, provided

$$\int_U \langle db_\epsilon(x-y) d\lambda_{n-1}, \xi_x \rangle \, dx = 0$$

for all $y \in \mathbf{R}^N$. This reduces the problem to the case where

$$\omega(x) = b(x)\, d\lambda_{n-1} \tag{3.31}$$

and $b(x)$ is a bump function of our choice.

It remains to find a form ω that is both of the form (3.30) and the form (3.31). To this end, let α be an $(n-1)$-form on $\mathbf{S}^{n-1}$ which is a volume form multiplied by a nonnegative (but nonzero) function on $\mathbf{S}^{n-1}$ so that $\alpha \equiv 0$ in a small neighborhood of a point $\sigma \in \mathbf{S}^{n-1}$. Next, extend α to be an $(n-1)$-form on $\mathbf{S}^n \supset \mathbf{S}^{n-1}$ as follows: first, pull back by the map $\pi_1 \circ \pi_2$, where $\pi_2 : \mathbf{R}^{n+1} \setminus \{x_{n+1} - \text{axis}\} \to \mathbf{R}^n \setminus \{0\}$ and $\pi_1 : \mathbf{R}^n \setminus \{0\} \to \mathbf{S}^{n-1}$ are projections; then multiply by a nonnegative function of $|x_{n+1}|$ which vanishes if $|x_{n+1}| \geq 1/4$ and is equal to 1 in a neighborhood of 0; and finally, restrict (and extend) to $\mathbf{S}^n \subset \mathbf{R}^{n+1}$. Note that this extension α vanishes on definite neighborhoods of the south and the north poles of $\mathbf{S}^n$, as well as on a neighborhood of a great circle that connects the poles through the point σ.

We now describe a mapping $F : \mathbf{R}^N \to \mathbf{S}^n$ as required in (3.30). First, let $F_1 : \mathbf{R}^{n-1} \to \mathbf{S}^{n-1}$ be a map of degree 1 that assumes the value σ outside $B(0, \delta)$. (We may assume that $p = 0 \in \mathbf{R}^N$ and that $\delta > 0$ is small.) Then, using coordinates (λ, x') in $\mathbf{R}^N = \mathbf{R}^{n-1} \times \mathbf{R}^{N-n+1}$, we define $F = \varphi \circ F_2$, where $F_2(\lambda, x') = (F_1(\lambda), (1/\pi)\arctan(|x'|/\delta))$, $F_2 : \mathbf{R}^N \to \mathbf{S}^{n-1} \times [0, 1/2)$, and φ is the projection of $\mathbf{S}^{n-1} \times [0, 1/2)$ onto an open subset of $\mathbf{S}^n$. It is easy to see that $\omega = F^*(\alpha)$ depends only on $\lambda = (x_1, \ldots, x_{n-1})$ and $|x'| = |(x_n, \ldots, x_N)|$; indeed, it is easy to see from the definitions that ω is of the form $\omega(\lambda, x') = f(\lambda)h(|x'|)\, dx_1 \cdots dx_{n-1}$ for some nonnegative (but nonzero) functions f and h. □

This completes the proof of Proposition 3.28.

4. Locally branched Euclidean gauge

We assume in this section that X satisfies Axioms I–IV. Pick a point $p \in X$ and an embedded (open, connected) neighborhood U of p in some $\mathbf{R}^N$. We assume that U is metrically oriented by (g_U, ξ), that the current (U, ξ) is an n-dimensional Stokes current, and that a Cartan-Whitney presentation $\rho = (\rho_1, \ldots, \rho_n)$ is given on U. Thus, the Whitney 1-forms $\rho_1, \ldots, \rho_n$ are defined in a neighborhood G of U in $\mathbf{R}^N$, and they satisfy

$$\operatorname{ess\,inf} * (\rho_1 \wedge \cdots \wedge \rho_n) \geq \delta > 0 \tag{4.1}$$

in U. By shrinking U if necessary, we may also assume that each 2-simplex $[x, y, z]$, generated by points x, y, z in U, lies in G. In particular, each line segment $[p, x]$ for $x \in U$ lies in G.

Under these assumptions, we prove the following theorem, which should be regarded as the main result of this paper.

THEOREM 4.2
The neighborhood U can be chosen small enough so that the mapping

$$f(x) = \int_{[p,x]} (\rho_1, \ldots, \rho_n), \quad x \in U, \tag{4.3}$$

is a sense-preserving, discrete, and open Lipschitz mapping from U to $\mathbf{R}^n$ which sat-

isfies (1.5). In particular, we have

$$\liminf_{y\to x,\, y\neq x} \frac{|f(x)-f(y)|}{|x-y|} \geq c > 0 \tag{4.4}$$

for all $x \in U$ and for some $c > 0$ independent of x. The mapping f is locally bi-Lipschitz outside the branch set B_f of measure zero and of topological dimension at most $n-2$.

A mapping $f : U \to \mathbf{R}^n$ is *sense-preserving* if the local degree $\mu(A, D, f)$ defined in (3.5) is positive for each domain D compactly contained in U and for each component A of $\mathbf{R}^n \setminus f(\partial D)$ that meets $f(D)$; we assume that $\mathbf{R}^n$ is equipped with its standard orientation. A map is *discrete* if the preimage of each point is a discrete set, and the *branch set* B_f is the closed point set in the domain of f where f does not define a local homeomorphism. For a discrete and open mapping f, the branch set B_f always has topological dimension at most $n-2$ by [Ch1], [Ch2], and [V2]. (The hypotheses in [Ch1], [Ch2], [V2] are somewhat different from what is required by Axiom I. However, the proof in [V2] in particular is valid in the present context.)

What we show here is that the mapping f given in (4.3) is a discrete, open, and sense-preserving map with volume derivative uniformly bounded away from zero. The BLD-property (1.5) for such maps follows from [HR2, Theorem 6.18]. To be precise, our axioms are slightly weaker than the assumptions on the source space X in [HR2]. The axioms are sufficient, however, to run the proof in [HR2, Theorem 6.18], for the required analysis there needs only the validity of a Poincaré inequality. This is clear from the references used in the proof of [HR2, Theorem 6.18]. As an additional technical point, one needs to know here that a Poincaré inequality as in (3.18) implies quasiconvexity of the space (for this, see [Cr, Appendix] or [HaKo, Proposition 4.4]).

Finally, observe that property (1.3) follows from (1.5) via a simple path-lifting argument (cf. [HR2, Section 3.3]) and that the sufficiency part of both Theorem 2.1 and Corollary 2.3 follows from Theorem 4.2.

Next, we define the *residue* of ρ by

$$\mathrm{Res}(\rho, p) = \text{the local degree at } p \text{ of the map } f \text{ given in (4.3).} \tag{4.5}$$

Thus, $\mathrm{Res}(\rho, p) = 1$ if and only if p lies outside the branch set of f, and we conclude that Theorem 2.2 follows from Theorem 4.2, except the claim about continuity in ρ, which is clear from the proof.

Proof of Theorem 4.2

The proof is presented in several subsections.

4.6. f is Lipschitz with uniformly positive volume derivative

One should compare the argument here to that given in [Su] in the context of Lipschitz manifolds. We have

$$\left|f(x)-f(p)\right| = \left|f(x)\right| \leq \|\rho\|_\infty |x-p|,$$

where $\rho = (\rho_1, \dots, \rho_n)$. If $x \in U$, $x \neq p$, and y is near x, then

$$\begin{aligned}\left|f(x)-f(y)\right| &= \left|\int_{[p,x]} \rho - \int_{[p,y]} \rho + \int_{[x,y]} \rho - \int_{[x,y]} \rho\right| \\ &\leq \left|\int_{[p,x,y]} d\rho\right| + \left|\int_{[x,y]} \rho\right| \\ &\leq \|d\rho\|_\infty \left|[p,x,y]\right| + \|\rho\|_\infty |x-y|.\end{aligned}$$

The area $|[p,x,y]|$ of the 2-simplex $[p,x,y]$ is at most a constant times $|x-y|$, and we conclude that f is uniformly locally Lipschitz in U. In fact, f is uniformly locally Lipschitz in a small neighborhood of p in the ambient space $\mathbf{R}^N$ by the same argument. We may therefore assume that f is Lipschitz in U.

Because f is Lipschitz, its (approximate) differential $df = (df_1, \dots, df_n)$ determines Whitney 1-forms in a neighborhood of U in $\mathbf{R}^N$. (See [F, Theorem 3.2.19] and [W2, Chapter X], and recall that we can extend f to a Lipschitz map $\mathbf{R}^N \to \mathbf{R}^n$ by Kirszbraun's theorem.) In particular, $df \in L^\infty$, and for $x, y \in U$ we have

$$\begin{aligned}\int_{[x,y]} (\rho - df) &= \int_{[x,y]} \rho - \big(f(y)-f(x)\big) \\ &= \int_{[x,y]} \rho - \int_{[p,y]} \rho + \int_{[p,x]} \rho = \int_{[p,x,y]} d\rho,\end{aligned}$$

which gives

$$\begin{aligned}\left|\int_{[x,y]} (\rho - df)\right| &\leq \|d\rho\|_\infty \left|[p,x,y]\right| \\ &\leq \|d\rho\|_\infty |p-x||x-y|\end{aligned}$$

if $|x-y| \ll |x-p|$. This implies that the L^∞-norm of $\rho - df$ satisfies

$$\|\rho - df\|_\infty \leq \|d\rho\|_\infty |p-x|.$$

By further shrinking U if necessary, we thus find that

$$df_1 \wedge \cdots \wedge df_n \geq \delta' > 0 \tag{4.7}$$

almost everywhere in U (cf. [W2, Theorem 7C, p. 265]).

4.8. Potential theory

Here we follow the fundamental ideas of Reshetnyak [Re1]. For $b \in \mathbf{R}^n$, the function

$$u(y) = u_b(y) = -\log|y - b| \tag{4.9}$$

solves the quasi-linear elliptic equation

$$-*d*|du|^{n-2}\,du = 0$$

in $\mathbf{R}^n \setminus \{b\}$, where $*$ is the Hodge star operator in $\mathbf{R}^n$. In particular, the $(n-1)$-form

$$\alpha = *|du|^{n-2}\,du$$

is closed in $\mathbf{R}^n \setminus \{b\}$. Because α is smooth and f is Lipschitz, we have

$$df^*(\alpha) = f^*(d\alpha) = 0 \tag{4.10}$$

in the nonempty (relatively) open set $U \setminus f^{-1}(b)$. Note that f is not constant in U by (4.7).

To justify equality (4.10) and the other upcoming differential calculus on Lipschitz forms, we refer the reader to the discussion in [W2, Section X.9] (see also [Re2, Sections II.4.3 and 4.4]). Recall that we can think of f as being defined in all of $\mathbf{R}^N$.

Next, if φ is a compactly supported Lipschitz function in $U \setminus f^{-1}(b)$, we calculate

$$d\big(f^*(\alpha)\varphi\big) = df^*(\alpha)\varphi + (-1)^{n-1} f^*(\alpha) \wedge d\varphi = (-1)^{n-1} f^*(\alpha) \wedge d\varphi, \tag{4.11}$$

and recalling Proposition 3.28, we thus obtain

$$\begin{aligned} 0 = \big(\partial U, f^*(\alpha)\varphi\big) &= \int_U d\big(f^*(\alpha)\varphi\big) \\ &= (-1)^{n-1} \int_U f^*(\alpha) \wedge d\varphi. \end{aligned} \tag{4.12}$$

(We suppress the fixed orientation ξ from the notation here and below.) In conclusion,

$$0 = \int_U f^*(\alpha) \wedge d\varphi = \int_U \big\langle *f^*(\alpha), d\varphi \big\rangle \, d\mathscr{H}_n, \tag{4.13}$$

where the $*$-operator is determined by the fixed inner product and orientation on T^*U. Equality (4.13) means that the 1-form $*f^**|du|^{n-2}\,du$ is coclosed in $U \setminus f^{-1}(b)$ in a weak sense; that is,

$$\int_U \big\langle *f^* * |du|^{n-2}\,du, d\varphi \big\rangle \, d\mathscr{H}_n = 0 \tag{4.14}$$

for all compactly supported Lipschitz functions φ in $U \setminus f^{-1}(b)$.

Let us reformulate (4.14) as follows. By (4.7), for almost every x in U we can define a linear map

$$G(x) : \Lambda^1 T_x U \to \Lambda^1 T_x U$$

by the formula

$$G(x) = \det df(x)^{2/n} [df(x)]^{-1} [df(x)]^{-1^T}. \tag{4.15}$$

Here T denotes transpose, determined by the fixed inner products, and we use the natural fiberwise identification of the tangent and cotangent spaces. Now (4.14) states that

$$\int_U \langle \mathscr{A}_x(dh(x)), d\varphi(x) \rangle \, d\mathscr{H}_n(x) = 0 \tag{4.16}$$

for all compactly supported Lipschitz functions φ in $U \setminus f^{-1}(b)$, where $h = u \circ f$ is a (locally) Lipschitz function in $U \setminus f^{-1}(b)$ and

$$\mathscr{A}_x(\eta) = \langle G(x)\eta, \eta \rangle^{(n-2)/2} G(x)\eta.$$

To see this, we observe first that

$$*f^* * = (-1)^{n-1} \det df \, df^{-1} \tag{4.17}$$

on 1-forms by Laplace's formula (see, e.g., [Ri, Chapter I.1]). A pointwise calculation now shows that

$$dh = df^T du$$

and hence that

$$\begin{aligned}
\mathscr{A}_x(dh) &= \det df^{(n-2)/n} \langle df^{-1} df^{-1^T} dh, dh \rangle^{(n-2)/2} \det df^{2/n} df^{-1} df^{-1^T} dh \\
&= \det df \, |df^{-1^T} dh|^{n-2} df^{-1} df^{-1^T} dh \\
&= \det df \, |du|^{n-2} df^{-1} du \\
&= (-1)^{n-1} * f^* * |du|^{n-2} du
\end{aligned}$$

almost everywhere, as required. It follows that (4.14) and (4.16) are indeed reformulations of each other.

Finally, we observe that because df is bounded and because condition (4.7) holds, we have

$$F_x(\eta) = \langle G(x)\eta, \eta \rangle^{n/2} = \langle \mathscr{A}_x(\eta), \eta \rangle \approx |\eta|^n \tag{4.18}$$

for all measurable sections η of T^*U. The constants in (4.18) are independent of η.

4.19. Quasi-continuous Sobolev functions

Recall the definition for Sobolev space $H^{1,p}(U)$ from Section 3.12. In what follows, we only need the case where $p = n$. (An analogous discussion is valid for all $1 < p < \infty$.) For a set $E \subset U$, we define its *n-capacity* to be the number

$$C_n(E) = \inf \int_U \left(|u|^n + |du|^n\right) d\mathscr{H}_n, \tag{4.20}$$

where the infimum is taken over all $u \in H^{1,n}(U)$ such that $u \geq 1$ almost everywhere in an open neighborhood of E. We also need the following variational counterpart of C_n. Assume that E is a compact subset of an open set $V \subset U$. Then the *variational n-capacity of E in V* is the number

$$\operatorname{cap}_n(E, V) = \inf \int_V |du|^n \, d\mathscr{H}_n, \tag{4.21}$$

where the infimum is taken over all compactly supported Lipschitz functions u in V such that $u \geq 1$ on E.

Note that (4.21) was defined by using Lipschitz test functions, whereas (4.20) used arbitrary Sobolev functions. These are the most natural ways to define the two capacities, although it is true (and important) that in both cases the pool of test functions can be altered without altering the value of the capacity. Also, note that the definition of $\operatorname{cap}_n(E, V)$ can be extended to arbitrary subsets of V in a standard manner (see [HKM, p. 27]).

A set E in U is said to be of *n-capacity zero* if $C_n(E) = 0$. One can show by using the Poincaré inequality (3.18) that for a compact set E, we have $C_n(E) = 0$ if and only if $\operatorname{cap}_n(E, V) = 0$ for every (equivalently, some) relatively compact open set V containing E (see the arguments in [HKM, pp. 49, 34]).

The next result follows from [HeKo2, Theorem 5.9]; the Poincaré inequality (3.18) is crucial here.

PROPOSITION 4.22

A compact set of zero n-capacity in U has Hausdorff dimension zero.

A real-valued function u defined on a set $E \subset U$ is said to be *n-quasi-continuous* if for each $\epsilon > 0$ there is an open set G with $C_n(G) < \epsilon$ such that $u|E \setminus G$ is continuous. A sequence (u_j) of functions on E is said to converge *n-quasi-uniformly* to a function u on E if for each $\epsilon > 0$ there is an open set G with $C_n(G) < \epsilon$ such that $u_j \to u$ uniformly in $E \setminus G$. If a property holds except on a set of zero n-capacity, we say it holds *n-quasi-everywhere*.

PROPOSITION 4.23

A function in the Sobolev space $H^{1,n}(U)$ has an n-quasi-continuous representative, and two such representatives agree n-quasi-everywhere. Every convergent sequence of n-quasi-continuous functions in $H^{1,n}(U)$ subconverges n-quasi-uniformly to an n-quasi-continuous function.

The proof for the existence part in Proposition 4.23 is standard. We leave its detailed verification to the reader following the presentation of [HKM, Chapters 2, 4] and using properties (3.13) – (3.15). The same holds true for the last assertion in Proposition 4.23. The proof of the uniqueness up to a set of zero capacity of the quasi-continuous representative in [HKM, Theorem 4.12] is somewhat complicated, relying on nontrivial results from the theory of quasilinear variational inequalities in $\mathbf{R}^n$, and it is not clear if the argument can be used in the present setting. However, T. Kilpeläinen [K] has recently given a short, elementary proof for the uniqueness that applies very generally; in particular, it applies in our case, and Proposition 4.23 follows.

4.24. f is light

We show that the Lipschitz map $f : U \to \mathbf{R}^n$ given in (4.3) is *light*, that is, that the preimage of every point under f is a totally disconnected set. We show that the fiber $f^{-1}(b)$ has zero n-capacity in U for each $b \in \mathbf{R}^n$. This suffices by Proposition 4.22. Here we depart from Reshetnyak's original argument, which used Harnack's inequality for solutions to degenerate elliptic equations, and instead follow the proof in [HeKo1]. The idea in [HeKo1] (which avoids Harnack's inequality) was to construct an n-quasi-continuous function in a neighborhood B of each point in $f^{-1}(b)$ which takes only two values: 1 on $f^{-1}(b) \cap B$, and 0 elsewhere. Then necessarily $f^{-1}(b)$ has zero n-capacity.

To this end, pick $b \in \mathbf{R}^n$ and consider the function $u = u_b$ as defined in (4.9). Denote, for each positive integer $k \geq 1$,

$$E_k = \{h \geq k\} \cap \frac{1}{2}\overline{B},$$

where $h = u \circ f = -\log|f - b|$ as in Section 4.8, B is some fixed open ball (in U) centered at a point $x_0 \in \partial f^{-1}(b) \cap U$ such that the closed ball $\overline{B}$ lies in U, and $(1/2)\overline{B}$ denotes the closed ball with the same center as B but half the radius. Note that the required point x_0 exists because f is not constant and U is connected. It suffices to show that

$$E_\infty = f^{-1}(b) \cap \frac{1}{2}\overline{B}$$

has zero n-capacity.

With the discussion in Sections 4.19 and 4.8 understood, the argument is very similar to that in [HeKo1]. For convenience we repeat the main points. (In fact, the situation here is easier than in [HeKo1] because the degeneracy of the equation is less severe.)

First, we claim that the minimization problem

$$I_k = \inf_{\mathscr{F}_k} \int_B F_x\big(dv(x)\big)\, d\mathscr{H}_n(x), \tag{4.25}$$

where

$$\mathscr{F}_k = \big\{v \in H_0^{1,n}(B) : v \text{ is } n\text{-quasi-continuous and } v \geq 1 \; n\text{-quasi-everywhere on } E_k\big\}$$

and $F_x(\eta)$ is given in (4.18), is solved by a unique (up to a set of zero capacity) minimizer $v_k \in \mathscr{F}_k$. (Here the Sobolev space $H_0^{1,n}(B)$ is the closure of compactly supported Lipschitz functions in B with respect to the norm (3.16).) The proof of the claim follows the standard arguments of the calculus of variations (see, e.g., [HKM, Chapter 5] or [Re2, Chapter III.3]). We equip $H_0^{1,n}(B)$ with the uniformly convex norm

$$||v||_F = ||v||_n + \Big(\int_B F_x(dv)\, d\mathscr{H}_n\Big)^{1/n},$$

which by (4.18) is equivalent to the norm in (3.16) (with $p = n$), so that a minimizing sequence has a weakly convergent subsequence from which one can extract a sequence of convex combinations that converges strongly to a function v_k (by Mazur's lemma). By the lower semicontinuity of norms, v_k is a minimizer, and by Proposition 4.23, we may assume that v_k is in $\mathscr{F}_k$. Finally, the uniqueness follows from the strict convexity of F.

Next, analogously to [HeKo1, Lemma 4.8], one can show that

$$w_k = kv_k \leq h \tag{4.26}$$

quasi-everywhere in $B \setminus E_k$. The crucial fact in proving (4.26) is the validity of equation (4.16): because v_k uniquely minimizes (4.25), we easily obtain

$$\int_{\{w_k>h\}} F_x(dw_k)\, d\mathscr{H}_n < \int_{\{w_k>h\}} F_x(dh)\, d\mathscr{H}_n,$$

while on the other hand,

$$\begin{aligned}\int_{\{w_k>h\}} \big(F_x(dw_k) - F_x(dh)\big)\, d\mathscr{H}_n &\geq \int_{\{w_k>h\}} \big\langle \nabla_\eta F_x(dh), dw_k - dh\big\rangle\, d\mathscr{H}_n \\ &= n\int_{\{w_k>h\}} \big\langle \mathscr{A}_x(dh), dw_k - dh\big\rangle\, d\mathscr{H}_n = 0,\end{aligned}$$

where the last equality follows from (4.16) by (Lipschitz) approximation. It follows that $w_k \le h$ almost everywhere, hence quasi-everywhere by Proposition 4.23.

We have thus constructed a sequence (v_k) of quasi-continuous functions in $H_0^{1,n}(B)$ such that

(1) $v_k = 1$ n-quasi-everywhere on E_k;

(2) $v_k \le h/k$ n-quasi-everywhere on $B \setminus E_k$.

Moreover, the sequence (v_k) is bounded in $H_0^{1,n}(B)$, so that a sequence of convex combinations of v_k's converges strongly to a quasi-continuous function v_∞ in B. (This is, again, by Mazur's lemma and Proposition 4.23). By (1), $v_\infty = 1$ n-quasi-everywhere on E_∞, while by (2), $v_\infty = 0$ n-quasi-everywhere on $B \setminus E_\infty$ because h is finite outside E_∞. This implies that E_∞ has zero n-capacity, as was to be proved.

Therefore, $f : U \to \mathbf{R}^n$ is a light map.

4.27. f is sense-preserving

For almost every x in U, the approximate differential $df(x)$ exists (see [F, Section 3.2.16 and Theoreom 3.2.19]) and satisfies

$$\det df(x) = (df_1 \wedge \cdots \wedge df_n)(x) \ge \delta' > 0 \tag{4.28}$$

by condition (4.7). Let D be a relatively compact domain in U, and let A be a component of $\mathbf{R}^n \setminus f(\partial D)$ which meets $f(D)$. Because $f^{-1}(A) \cap D$ is open and nonempty, and because the fiber $f^{-1}(y)$ is finite for almost every $y \in \mathbf{R}^n$ by [F, Theorem 3.2.22], there is a point y in A whose preimage in D consists of finitely many points x such that $df(x)$ exists, (4.28) holds, and f is approximately differentiable at x. (We use here the fact that Lipschitz maps are absolutely continuous in measure.) It follows from an easy homotopy argument that

$$\mu(A, D, f) = \sum_{x \in f^{-1}(y) \cap D} \operatorname{sign} \det df(x) > 0, \tag{4.29}$$

as required. Thus, f is sense-preserving in U.

4.30. Conclusion

It is not hard to see that a sense-preserving light map $U \to \mathbf{R}^n$ is discrete and open (cf. [TY], [Re2, Section II.6.3], [Ri, Section VI.5]). We have thus shown that the mapping f given in (4.3) is a sense-preserving, discrete, and open Lipschitz mapping with a definite lower bound for the Jacobian determinant as in (4.28). As discussed right after the statement of Theorem 4.2, this suffices, and the proof of Theorem 4.2 is thereby complete. □