REMINISCENCES OF MICHEL HERMAN'S FIRST GREAT THEOREM

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In the middle seventies, Henri Epstein and I would walk over to Orsay from the IHES to hear Michel Herman's lectures on circle diffeomorphisms. We marveled at how much structure and elegance evolved from Michel's study of the iteration of $x \mapsto x + y(x)$ where y is any smooth function of periodicity one.

A couple of years earlier a new edition of Denjoy's work was published by the CNRS and Michel was involved. This provided Michel the opportunity to reconsider Denjoy's arguments showing a twice differentiable circle diffeomorphism either has a periodic orbit or only dense orbits. Basically, Denjoy was controlling the nonlinearity of the first return iterates q_1, q_2, q_3, ... rising the differentiability hypothesis of f and the disjointness of an orbit of intervals up to first return. This involved calculating the first and second derivatives of long iterates of f. The first derivative is just the usual chain rule while second derivative involves in modern terms the chain rule for the non linearity $\dfrac{f''}{f'} = (\ln f')'$.

Michel wanted to control the higher derivative of the iterates in order to attack Arnold's conjecture that if the q_1, q_2, ... did not grow rapidly the Denjoy continuous conjugation Df of f to a rotation would actually be smooth. From the Kolmogoroff-Arnold-Moser theory, Michel already knew that if he could introduce coordinates to make the non linearity small enough for a given growth condition on the first returns $q_1, q_2, \ldots$, then he would win.

A prodigious calculation of third and higher derivatives with a "miraculous cancellation" led to Michel Herman's initial big breakthrough. First he showed that for any set of first returns, $q_1, q_2, \ldots$ the transformation f was ergodic for the Lebesque measure class even though there need not be a smooth invariant measure —the smoothness of the unique invariant measure being controlled by the smoothness of the Denjoy conjugacy Df.

Then a logically complex argument evolved showing Arnold's conjecture if the $q_1, q_2, \ldots$ were of bounded type. Michel presented this argument at my urging in a marathon seminar at the IHES —approximately

30 hours over three or four days. For the last half I was lost, but the other two surviving participants, Pierre Deligne and Adrien Douady were still checking and absorbing the proof. Adrien made a wonderful flow chart of the logic which I transcribed in colored inks on the back of a large Colette poster and Pierre undertook the task of presenting Michel's argument at the next Bourbaki.

I distinctly remember Pierre hunched over his desk in the evenings at IHES struggling with the enormous proof. One evening he seemed particularly worried and it turned out he had doubts. He had completely understood everything concrete and constructive in the proof and one estimate going from "big O" to "little o" was missing. Pierre was right and Michel was also right. The missing estimate came from an application of the ergodic theorem —a-non constructive passage from "big O" to "little o."

Soon the Bourbaki event took place. Just before Deligne's lecture on Michel Herman's achievement, I met Michel in the hall outside the lecture room at the Institute Henri Poincaré. He was so overcome with emotion that he couldn't attend the lecture and he went for a walk instead. I did attend and it was really something. Pierre's first sentence developed the essential aspects of the theory of the rotation number of homeomorphisms of the circle without period points, "Given f there is a unique irrational rotation $R\alpha$ so that any of its orbits and any of f's orbits $1, 2, 3, \dots$ have the same circular order type." Of course this α is called the rotation number of f.

While on this walk or perhaps a little later, Michel began to see the solution to the entire question when the first returns $q_1, q_2, \dots$ were not of bounded type. He soon had the full theorem that Denjoy's conjugacy Df was smooth precisely under the hypothesis of the perturbation result given by the KAM theory for diffeomorphisms close to rigid rotations.

A very special case of this general statement allows some appreciation of its depth. Take the golden number as rotation number so that the first returns of an orbit near its start 1 are the Fibonacci iterates $3, 5, 8, \dots$ These appear on alternate sides of 1 and converge down to 1. Michel Herman's theorem implies this convergence is geometric with precise ratio the golden number $.62\dots$ itself. In other words, the asymptotic geometry of the return is rigid by Michel's theorem —while the earlier result of Denjoy only implies there is some convergence. During this period I decided that this rigidity result of Michel Herman's was something really worth understanding deeply.

As great as Michel Herman's bounded type result was, the new progress was worth another Bourbaki report at a meeting the next year. This time Harold Rosenberg took up the task of presenting the result as well as its relationship to smooth geometric questions about codimension one foliations.

Michel Herman's career blossomed after all this —marked by very successful research, very conscientious contributions to mathematical scholarship, and very fruitful and constructive interactions with graduate students. For example, his student Yoccoz completed the equivalence of the Segal condition on the $q_1, q_2, \ldots$ and the C^∞ nature of the Denjoy conjugacy for C^∞ diffeomorphisms and together with Michel related the "miraculous cancellation" of the third derivative calculation with the chain rule for the Schwarzian derivative.

Now I would like to mention a development in the history of ideas and in Michel's attitude which, while being completely understandable, was not always constructive. At about this time (1975) numerical work was being done on first return geometry in another area of one dimensional mappings by Feigenbaum in the U.S. and by Collett and Tresser in France. Feigenbaum found that the 2^n-th iterates of the critical point for the limit of period doubling mappings converges geometrically quickly to itself with a certain universal ratio (.39...). Analogous numerical studies of critical circle mappings by mathematical physicists revealed more universal geometry in the asymptotics of first return mappings. This work purported to describe self similar structure on the boundary of the KAM region and the papers used an assertive tone in the statements —as in mathematical theorems— but did not provide proofs beyond heuristics and numerics. Such papers offended Michel, who had very laboriously proved estimates about the size of the good KAM regions getting rigorous numbers like .001 while the "chaos papers" were claiming numbers like .8 based on handwaving and numerics.

One unfortunate consequence was that for some years Michel was suspicious when the catch word of all these numerical papers "renormalization" was invoked. The physicists used this phrase in dynamics to refer to the process of replacing one dynamical system f on the circle or the unit interval by the first return to a tiny interval (and in the former case by gluing neighborhoods of the end points together by an iterate of f) to get a new dynamical systems Rf on a new tiny circle or interval which would then be rescaled (or renormalized) up to unit size.

In the intervening years, the renormalization viewpoint was used by Khanin and Sinai to redo the circle KAM theory more geometrically and conceptually. They also treat Michel Herman's theorem in this way. I am not familiar with their exact work, but in a final tribute to the fundamental nature of Michel's theorem, I would like to close with a little mathematical story about how the original complex edifice of Michel Herman's proof becomes understandable all at once if enough of its methodology is absorbed into the foundations. For me this is the hallmark of a deep and complex result which is also a great result.

For a circle transformation f we can consider the sequence of first returns $q_1, q_2, \ldots$ to appropriate intervals glued up by appropriate powers of f to obtain a sequence of renormalizations Rf, R^2f, ... which are diffeomorphisms of new "abstract circles" S_1, S_2, ... The rotation numbers of Rf, R^2f, ... are just the points of the orbit of the rotation number of f iterated by the ergodic continued fraction mapping $(a_0, a_1, a_2, \ldots) \mapsto (a_1, a_2, \ldots)$, where the a_i are from the continued fraction. Now three things are happening at once which reduces Michel Herman's theorem to the KAM perturbation result.

1. The renormalizations f, Rf, R^2f, ... being first return mappings, all have the same set of orbits so that a smooth invariant measure for one gives such for all. Thus we don't lose the question of the smoothness of the Denjoy conjugacy by replacing f by any of its renormalizations.
2. The first return iterates used to define the renormalized circles and the renormalized diffeomorphisms have two important properties.
   a) Their non-linearity is bounded by the sum of the lengths of disjoint intervals on the circle (the Denjoy argument),
   b) their deviation from being Moebius or projective as measured by the Schwarzian derivative tends to zero as the renormalization index tends to infinity. This follows from a) and the chain rule for the Schwarzian derivative. Namely, a) controls the non-linearity so that the chain rule estimates the Schwarzian of the first returns by the initial C^3 constant of f and the sum of the squares of the lengths of the disjoint intervals (whose lengths are all going to zero by Denjoy's original theorem).
3. Thus relative to coordinate systems of bounded non-linearity from the initial one the abstract circles are becoming closer and closer to projectively flat and the renormalizations are becoming closer and closer to projective transformations with bounded non-linearity. This means that up to bounded non-linearity smooth coordinate changes the renormalizations are converging to rotations. Now KAM says that a positive measure cantor set of good rotation numbers in such a neighborhood implies smooth invariant measures. The renormalizations are eventually inside such a KAM neighborhood and by the ergodicity of the continued fraction

transformation, their rotation numbers are infinitely often in the cantor set of good rotation numbers for this neighborhood.

This is the view at a glance of Michel's theorem —1) renormalization puts any diffeomorphism eventually near rotations and 2) starting with any set of good numbers the renormalized rotation number visits the KAM set of positive measure (specific constants) infinitely often.

Michel Herman was a great analyst and dynamicist whose mathematics was an inspiration.