On the foundation of geometry, analysis,

and the differentiable structure for manifolds

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There are levels of structure on a set beneath that of the differentiable structure on a smooth manifold and above that of the topological structure where one has more than enough to define an adequate algebra of differential forms, exterior d, and their non-linear versions connections and curvatures on vector bundles over the space.

One example described in Whitney's book [1] is the structure of any metric gauge which is locally equivalent to that of a polyhedron. By a metric gauge we mean a maximal class of metrics on the set where for any two metrics in the class all the respective local distances d(x,y) are in bounded ratio. The idea of Whitney's construction is to start with lipschitz chains and put a new norm which makes two chains close if they are up to a small mass error homologous by a chain of small mass. The continuous cochains for this Whitney norm on chains form a graded commutative differential algebra – the Whitney forms for the metric gauge. One obtains in this way forms w so that w and dw have bounded measurable coefficients. Exterior d is a bounded operator for the Whitney norm. There are also forms with square integrable coefficients with exterior d a closable unbounded operator with dense domain containing the Whitney forms.

If the metric gauge is locally equivalent to that of Euclidean space Teleman developed Hodge theory using these Whitney and L^2 forms. [2] He found that exterior d plus the adjoint of exterior d relative to any L^2 inner product where multiplication by functions is self adjoint is itself essentially self adjoint.

Such a Hilbert space norm on forms is determined by a bounded measurable Hodge * operator and it is not clear a priori but true that d and d* have a common domain. A key point is that d has closed image which follows as usual from deRham's theorem. The self adjoint signature operator d + d* was used by Teleman to develop a version of the

Atiyah–Singer Index theorem for these metric gauges. Other corollaries [3], [4] were a construction of characteristic classes and the K–homology orientation [17] from the metric gauge.

One knows the following result [5].

Theorem 1: Locally Euclidean metric gauges exist and are unique up to small isotopy on every locally Euclidean topological space if the dimension is not equal to four.

This uses [5] together with work of Bing and Moise below dimension four and the annulus work of Kirby based on Novikov in dimensions above four. (see [5] for further references).

The existence and uniqueness result of Theorem 1 is also true for locally Euclidean conformal gauges [5]. By a conformal gauge we mean a maximal class of metrics on a set where for any two metrics in the class the bounded local relative distances $^{d(x,y)}/d(x,y')$ in one are also bounded in the other. Now differential forms, wedge product, and exterior d can also be constructed given a locally Euclidean conformal gauge on a set. This is described in [6] and can be based on either work of the Morrey school or that of the Helsinki school. Here one obtains forms that have p^{th} power integrable measurable coefficients where (degree of form) · (power of integrability) equals (ambient dimension). Exterior d will be an unbounded operator and it is interesting to note that the composition of a local Poincaré lemma singular integral operator with a Sobolev embedding defines a local inverse of exterior d, $\{^n/k^{th}$ power integrable k–forms$\}$ goes by Poincaré lemma transform to $\{1^{st}$ derivative $^n/k^{th}$ power integrable (k–1) – forms$\}$ which goes by Sobolev embedding to $\{(^n/k-1)^{th}$ power integrable (k–1) – forms$\}$.

Modifications of this picture in dimension zero and the top domension n are sometimes required e.g.., one can replace bounded measurable functions (L^∞) by functions of bounded mean ascillation (BMO) and replace integral n–forms (L^1) by the Hardy space n forms (H^1). This all makes good sense in the locally Euclidean conformal gauge.

In [6] Teleman's work was recapitulated for the locally Euclidean conformal gauge. The phase of the signature operator (but not its absolute value) could be defined in the conformal gauge via an algebraic device that circumvented Teleman's delicate issue of a common domain of d and d*. This device worked in even dimensions and developing the picture in odd dimensions presents interesting new features. The Atiyah–Singer theorem was developed also in the conformal gauge and eventually in dimension four the entire Yang–Mills–Donaldson theory, [6].

Then using work of Michael Freedman [7] and Simon Donaldson [16] one can prove the following theorem [6].

Theorem 2: The conformal gauges of Kaehler complex surfaces in one topological type can form an infinite number of isomorphism classes. Some locally Euclidean topological spaces in dimension 4 do not admit locally Euclidean conformal gauges.

Using calculations of Donaldson invariants by Friedman and Morgan [8] for Kaehler complex surfaces and the obvious but beautiful fact that two generic algebraic surfaces in a connected algebraic family are diffeomorphic one can deduce the following result.

Theorem 3: Fixing the conformal gauge of a Kaehler complex surface determines the diffemorphism type up to finitely many possibilities.

Proof: Friedman and Morgan show that two Kaehler surfaces with the same Donaldson invariants up to deformation lie in one algebraic family. [8]

Recently, Friedman and Morgan have arrived at a much faster proof of the statement that two Kaehler surfaces with the same Seiberg Witten invariants up to deformation lie in one algebraic family. This swifter calculation does not literally yield a proof of Theorem 3 even if certain conjectures by Mrowka, Kronheimer, and Witten are verified [9]. The point is that the Seiberg Witten theory depends on the existence of the Dirac operator on spinors ("square roots" of differential forms). We conjecture [10] that an appropriate Dirac package does not exist for a locally Euclidean conformal gauge or a locally Euclidean metric gauge unless the gauge contains a smooth structure. To summarize all the above consider the table suggestive of several conjectural statements some of which are theorems.

Operator	Structure
phase of signature operator	locally euclidean conformal gauge
signature operator	locally euclidean metric gauge
Dirac operator	locally euclidean differeniable structure

This table was also discovered by Alain Connes [11] in the context of "non commutative geometry" where the operator on a Hilbert space h plays the primary role in extracting the non commutative geometry and analysis from the non commutative topology (a C* algebra) and its non commutative measure theory (a self adjoint representation of the C* algebra in the Hilbert space h). The link is provided by Atiyah's seminal idea relating K–homology and abstract elliptic operators [12] that the operator and the representation commute modulo lower order terms.

When Connes developed Chern Weil formalism in the non–commutative context, he discovered cyclic cohomology [11]. These non commutative ideas come together with [6] in [13] to develop, using the phase of the signature operator, a local formula for the characteristic classes of an even dimensional manifold with a locally Euclidean conformal gauge and a choice of bounded measurable * in the middle dimension. The discussion is closely related to the measurable Riemann mapping theorem ((Morrey–Ahlfors–Bers) in 2D and the Donaldson Yang Mills theory in 4D. In a related paper [14] a locally Euclidean metric gauge and a full choice of a bounded measurable * are used to develop local formula for characteristic classes which converges to the classical Chern–Weil formula when a small parameter tends to zero, at least in regions where the * is smooth. Both these discussions [13], [14] can be viewed as an operator theoretic or quantum version of curvature and Chern–Weil formalism.

Let us come now to the idea of the differential or smooth structure itself. In [10] one assumes a locally Euclidean metric gauge and makes use of the Whitney forms mentioned above. One can define a "vector bundle of one forms" – a pair (E,γ) consisting of a Lipschitz vector bundle E and a positive bounded embedding γ of its Lipschitz sections into Whitney one forms [10]. It makes sense to define the torsion of a connection on a "vector bundle of one–forms". A cotangent structure for a metric gauge is by definition a "vector bundle of one forms" (E,γ) which admits a torsion free connection. The main result of [10] asserts that a cotangent structure on X determines an index function $X \longrightarrow \{1,2,3 \ldots\}$ which depends continuously on (E,γ) so that (E,γ) is isomorphic (up to $\epsilon > 0$) to the (E',γ') of a smooth structure iff the index function is identically one, and more generally there are branched covering "charts" whose local degrees agree with the value of index function.

Thus the notion of cotangent structure provides a way to characterize and generalize the smooth structures inside a locally Euclidean metric gauge.

The connection with Dirac comes from calculations of Cheeger and his former student Chou [15] for polyhedra. Imagine the branching charts of some cotangent structures are equivalent to polyhedral branched covers, then [15] shows the geometric Dirac operator defined away from the branching set is not essentially self adjoint if one takes as domain the Lipschitz sections of the pulled back spinor bundle by the branched cover. In other words the Dirac operator "sees" the branching singularities of a cotangent structure by failing to be essentially self adjoint for the natural domain of Lipschitz sections of the relevant spinor bundle.

In setting up the Seiberg Witten theory [9] one needs abstractly – 1) differential forms and exterior d, 2) a Hilbert space structure on forms so that multiplication by functions is self adjoint, 3) a vector bundle of abstract spinors with the expected algebraic relation to forms and a torsion free orthogonal connection with its correct algebraic relation to Clifford multiplication, and finally 4) the essential self adjointness of the associated Dirac operator with domain the regular sections of the spinor bundle.

It seems that 1), 2), and 3) are generally possible in the locally Euclidean metric gauge context but that 1), 2), 3) and 4) are only possible in the smooth context [10].

The discussion in the lectures will concentrate on the above remarks related to references [1], [2], [6], [10], and [13].

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