p-universal spaces and rational homotopy types

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1 Introduction

Let p be a prime or 0. A map $f : X \longrightarrow Y$ between topological spaces is called a *p-equivalence* if the induced homomorphism

$$f^* : H^*(Y; \mathbb{Z}/p) \longrightarrow H^*(X; \mathbb{Z}/p)$$

is an isomorphism. A p-equivalence, however, is not an equivalence relation; in particular, the symmetricity does not hold in general. In [MT] Mimura and Toda introduced a class of spaces in which a p-equivalence is an equivalence relation. They called such spaces *p-universal*. About 20 years ago the first and the fourth authors observed in the unpublished draft [BS] that the p-universality does not depend on a particular prime p but on its rational homotopy type, although they gave only the outline of the proof. The purpose of the present note is to give a detailed proof of it and to show that the class of p-universal spaces coincides exactly with that of spaces whose rational homotopy type has "positive weights" in the sense of Morgan and Sullivan. That is, our main theorem is stated as follows.

Theorem A. *Let X be a simply connected finite* CW*-complex. Then the following statements are equivalent:*

(1) *X is p-universal for a prime p or* 0;

(2) *the rational homotopy type of X has positive weights;*

(3) *X is p-universal for any prime p and* 0.

We also prove

Theorem B. *Let X be a simply connected* CW*-complex such that*

$$\sum_{i=2}^{\infty} \dim_{\mathbb{Q}} \pi_i(X) \otimes \mathbb{Q} < \infty.$$

Then there is a p-universal space K for any prime p having the same rational homotopy type as X if and only if X has $\mathbb{Q}$-positive weights.

Theorem A does not hold for infinite complexes (see Remark 3.6). In §2 we study a space whose rational homotopy type has positive weights. In Theorem 2.7 we give a various characterization of it and show that it is independent of the ground field. In fact, the characterization (1) in Theorem 2.7 is stated in [BS]. The method there is to show that the closure of $\mathbb{Q}$-split torus of the group of automorphisms of minimal model in the space of endomorphisms contains a zero homomorphism. The detailed proof, however, was not given in [BS]. We give here its proof by using the Galois group action on one parameter subgroups.

In §3 we prove (1) $\Longrightarrow$ (2) in Theorem A by using (1) in Theorem 2.7. Then following the idea of [BS], we realize the one parameter subgroup $\lambda(q)$, where q is a positive integer, by a self map of K which has the same rational homotopy type as a given complex. From this, we prove Theorem B as well as (2) $\Longrightarrow$ (3) of Theorem A. Finally we show in Proposition 3.7 that homogeneous spaces of compact Lie groups are p-universal for any prime p and 0.

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2 Positive weights

Let $V = \underset{n\geq 2}{\oplus} V^n$ be a graded vector space and denote by $m = \Lambda(V)$ a minimal differential graded-commutative algebra (minimal DGA for short) over $\mathbb{Q}$ ([H] and [Su]). Let $\mathbb{K}$ be a field such that $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$. We take a basis $\{x_1^{(n)}, \dots, x_{k_n}^{(n)}\}$ for $V^n \otimes \mathbb{K}$ and assign a positive integer $w(x_j^{(n)})$ to each $x_j^{(n)}$. The integer $w(x_j^{(n)})$ is called the *weight* of $x_j^{(n)}$. Let U_s^n be a subspace of $V^n \otimes \mathbb{K}$ spanned by the elements with weight s. We extend the definition of the weight by

$$w(x_i^{(n)} \cdot x_j^{(m)}) = w(x_i^{(n)}) + w(x_j^{(m)}).$$

Then for m^+ the ideal of positive elements, we have the weight decomposition

$$m^+ = \underset{s\geq 1}{\oplus} U_s, \quad \text{where} \quad U_s = \underset{n\geq 2}{\oplus} U_s^n.$$

Let X be a CW-complex and denote by $m(X) = \Lambda(\bigoplus_{n\geq 2} V^n)$ its minimal model with a differential operator d.

Definition 2.1. The $\mathbb{K}$-homotopy type of a CW-complex X, $m(X)\otimes\mathbb{K}$, is said to *have $\mathbb{K}$-positive weights* if we can choose a basis $\{x_1^{(n)},\dots,x_{k_n}^{(n)}\}$ of $V^n\otimes\mathbb{K}$ for $n\geq 2$ and give weight $w(x_i^{(n)})$ for each $x_i^{(n)}$ such that it satisfies

$$w(dx_i^{(n)}) = w(x_i^{(n)}) \quad (i = 1,\dots,k_n; n\geq 2). \tag{1}$$

In this case we simply say that X *has $\mathbb{K}$-positive weights.*

We denote by $m(X)(n)$ the sub DGA of $m(X)$ generated by the elements of degree $\leq n$ and by $m(X)(n)^i$ the subspace spanned by the elements of degree i. We also denote by $G_n(\mathbb{Q})$ the group of $\mathbb{Q}$-DGA automorphisms of $m(X)(n)$. If we fix a $\mathbb{Q}$-basis of $m(X)(n)^i$ for $i = 2,\dots,2n$, then $G_n(\mathbb{Q})$ is represented by the subgroup of $\mathrm{GL}(N,\mathbb{Q})$ defined by polynomial equations with coefficients in $\mathbb{Q}$, where $N = \sum_{i=2}^{2n}\dim_{\mathbb{Q}} m(X)(n)^i$. Let G_n be the subgroup of $\mathrm{GL}(N,\mathbb{C})$ defined by the same equations. Then G_n is an algebraic group defined over $\mathbb{Q}$ and $G_n(\mathbb{Q})$ is the set of $\mathbb{Q}$-rational points of G_n.

For any field $\mathbb{K}\supseteq\mathbb{Q}$, there is a maximal torus $T^{\mathbb{K}}$ of the connected component of G_n defined over $\mathbb{K}$ by Theorem 18.2 of [B]. Then by Proposition of [B;p.121] we have a decomposition over $\mathbb{K}$

$$T^{\mathbb{K}} = T_a^{\mathbb{K}}\cdot T_d^{\mathbb{K}},\quad T_a^{\mathbb{K}}\cap T_d^{\mathbb{K}} = \text{ finite}, \tag{2}$$

where $T_a^{\mathbb{K}}$ is the largest anisotropic subtorus defined over $\mathbb{K}$ and $T_d^{\mathbb{K}}$ is the largest split (i.e., diagonalizable over $\mathbb{K}$) subtorus of $T^{\mathbb{K}}$.

Let $\mathbb{C}^*$ be the multiplicative group of $\mathbb{C}$.

Definition 2.2. A group homomorphism $\lambda : \mathbb{C}^* \longrightarrow G_n$ is called *a one parameter subgroup* of G_n defined over $\mathbb{K}$ if it is represented by

$$\lambda(t) = \left\{ \left(\begin{array}{ccc} t^{a_1} & & 0 \\ & \ddots & \\ 0 & & t^{a_N} \end{array} \right) \middle| \, t \in \mathbb{C}^* \right\}$$

with respect to some $\mathbb{K}$-basis of $\bigoplus_{i=2}^{2n} m(X)(n)^i \otimes \mathbb{K}$, where $a_1, \dots, a_N$ are integers.

Proposition 2.3. *Let $\mathbb{K}$ be a field such that $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$. A* CW*-complex X has $\mathbb{K}$-positive weights if and only if, for each n, there is a one parameter subgroup $\lambda(t)$ of $T_d^{\mathbb{K}}$ defined over $\mathbb{K}$ such that $\lim_{t \to 0} \lambda(t) = 0$, where the topology of G_n is the metric one induced from $\mathbb{C}^{N^2}$.*

Proof. If X has $\mathbb{K}$-positive weights, then the correspondence

$$\lambda(t) : x \longmapsto t^s x \quad (x \in U_s, t \in \mathbb{K}^*)$$

defines a one parameter subgroup of $T_d^{\mathbb{K}}$ satisfying the required property. In fact, one can take s positive by the assumption that X has $\mathbb{K}$-positive weights. Conversely, if there is a one parameter subgroup $\lambda(t)$ of $T_d^{\mathbb{K}}$ defined over $\mathbb{K}$ such that $\lim_{t \to 0} \lambda(t) = 0$, one can choose a basis for $\bigoplus_{i=2}^{2n} m(X)(n)^i \otimes \mathbb{K}$ and positive integers $a_1, \dots, a_N$ so that $\lambda(t)$ is represented by

$$\lambda(t) = \left\{ \begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_N} \end{pmatrix} \middle| \, t \in \mathbb{K}^* \right\}. \tag{3}$$

Then one obtains a weight decomposition of $m(X)(n) \otimes \mathbb{K}$ by putting

$$U_{a_i} = \{x \in m(X)(n) \otimes \mathbb{K} \mid \lambda(t)x = t^{a_i}x\} \quad \text{for} \quad i = 1, \ldots, N. \qquad \text{QED}$$

Let E_n and G_n be the set of $\mathbb{C}$-DGA endomorphisms and the set of $\mathbb{C}$-DGA automorphisms of $m(X)(n)\otimes\mathbb{C}$ respectively. Then E_n is an algebraic set defined over $\mathbb{Q}$ and G_n is a Zariski open set of E_n. Recall that the Zariski closure of G_n in E_n coincides with the metric closure of G_n in E_n (see for example [M]), which we denote by $\overline{G_n}^E$.

Lemma 2.4. *Let B be a Borel subgroup of G_n. If the zero homomorphism is contained in $\overline{G_n}^E$, then so is in the metric closure of B .*

Proof. Let M be a compact maximal subgroup of G_n. Then M acts on the complete variety G_n/B transitively, and hence we have

$$G_n = M \cdot B.$$

Let $\{x_n\}$ be a sequence of the points in $M \cdot B$ such that $\lim_{n\to\infty} x_n = 0$. Then each x_n can be expressed as

$$x_n = u_n \cdot b_n,$$

where $u_n \in M, b_n \in B$. There is an accumulation point α of $\{u_n\}$ such that

$\alpha \in M$, since M is compact. Then

$$\lim_{n\to\infty} \alpha \cdot b_n = 0.$$

Hence by multiplying α^{-1} we have

$$\lim_{n\to\infty} b_n = 0\,. \qquad \text{QED}$$

By (4) of Theorem 10.6 of [B] we have a semi-direct product decomposition

$$B = T_1 \cdot U,$$

where T_1 is a maximal torus of G_n and U is a unipotent subgroup of G_n by Corollary 11.3 of [B], since B is solvable by definiton.

Lemma 2.5. *If the closure (metric) of B contains* 0, *so does the closure of* T_1.

Proof. Let $\{x_n\}$ be a sequence of B such that $\lim_{n\to\infty} x_n = 0$. We can express x_n by an upper triangular matrix

$$x_n = \begin{pmatrix} \beta_{11} & \cdots & \beta_{1N} \\ & \ddots & \vdots \\ 0 & & \beta_{NN} \end{pmatrix}.$$

Since $\lim_{n\to\infty} \beta_{ii} = 0$ for $i = 1, 2, \ldots, m$ and since

$$\begin{pmatrix} \beta_{11} & & 0 \\ & \ddots & \\ 0 & & \beta_{NN} \end{pmatrix} \in T_1,$$

we have the lemma. QED

There exists a maximal torus $T^{\mathbb{Q}}$ of G_n defined over $\mathbb{Q}$. Then, if $\overline{G_n}^E$ contains 0, so does the metric closure of $T^{\mathbb{Q}}$ by Lemmas 2.4 and 2.5, since maximal tori of G_n are conjugate. By Corollary 18.8 of [B], $T^{\mathbb{Q}}$ splits over a finite normal extension $\mathbb{K}$ of $\mathbb{Q}$ so that the elements of the $\mathbb{K}$-rational points $T^{\mathbb{Q}}(\mathbb{K})$ are diagonalizable over $\mathbb{K}$. That is, with respect to some $\mathbb{K}$-basis for $m(X)(n) \otimes \mathbb{K}$, $T^{\mathbb{Q}}$ can be represented as

$$\left\{ \left(\begin{array}{ccc} t_1^{a_1^1} \cdot \cdots \cdot t_m^{a_m^1} & & 0 \\ & \ddots & \\ 0 & & t_1^{a_1^N} \cdot \cdots \cdot t_m^{a_m^N} \end{array} \right) \middle| \, t_1, \ldots, t_m \in \mathbb{K}^* \right\}, \tag{4}$$

where $\mathbb{K}^*$ is the multiplicative group of $\mathbb{K}$, m is the dimension of $T^{\mathbb{Q}}$ and a_i^j are integers for $1 \leq i \leq m, 1 \leq j \leq N$.

Lemma 2.6. *If the closure (metric) of $T^{\mathbb{Q}}$ contains* 0, *there is a one parameter subgroup $\lambda(t)$ of $T^{\mathbb{Q}}$ defined over $\mathbb{K}$ such that*

$$\lim_{t \to 0} \lambda(t) = 0.$$

In particular the metric closure of the $\mathbb{K}$-rational points $T^{\mathbb{Q}}(\mathbb{K})$ contains 0.

Proof. We denote a matrix in (3) by $M(t_1, \ldots, t_m)$. By the assumption there is a sequence $\{M(x_1^{(k)}, \ldots, x_m^{(k)})\}_{k=1}^{\infty}$ such that

$$\lim_{k \to \infty} |(x_1^{(k)})^{a_1^j} \cdots (x_m^{(k)})^{a_m^j}| = 0 \quad \text{for} \quad j = 1, \ldots, m. \tag{5}$$

We choose a positive number $t < 1$ such that $x_\ell^{(k)} = t^{\alpha_\ell^k} e^{i\theta_\ell^k}$, where α_ℓ^k and θ_ℓ^k

are real numbers. Then

$$|(x_1^{(k)})^{a_1^j} \cdots (x_m^{(k)})^{a_m^j}| = t^{\sum_{\ell=1}^m \alpha_\ell^k a_\ell^j}.$$

By (4) for large k, the numbers $\sum_{\ell=1}^{m} \alpha_\ell^k \cdot a_\ell^j$ for $j = 1, \ldots, N$ are simultaneously positive. Then from the density of $\mathbb{Q}$ in $\mathbb{R}$, we can choose rational numbers $\beta_1, \ldots, \beta_m$ so that

$$\sum_{\ell=1}^{m} \beta_\ell a_\ell^j > 0 \quad \text{for} \quad j = 1, \ldots, N.$$

Hence we have integers $P_1, \ldots, P_m$ such that

$$\sum_{\ell=1}^{m} P_\ell a_\ell^j > 0 \quad \text{for} \quad j = 1, \ldots, N.$$

Then the one parameter subgroup defined by

$$\lambda(t) = \{M(t^{P_1}, \ldots, t^{P_m}) \mid t \in \mathbb{C}^*\}$$

satisfies

$$\lim_{t \to 0} \lambda(t) = 0. \qquad \text{QED}$$

Thus we have proved, by virtue of Proposition 2.3 together with Lemmas 2.4, 2.5 and 2.6, that $m(X)(n)$ has $\mathbb{K}$-positive weights, if $\overline{G_n}^E$ contains 0, where $\mathbb{K}$ is a finite normal extension of $\mathbb{Q}$.

The one parameter subgroup $\lambda(t)$ defined over $\mathbb{K}$ in Lemma 2.6 is represented by matrices $S(t)$ with respect to some ($\mathbb{Q}$-) basis of $\bigoplus_{i=2}^{2n} m(X)(n)^i$ such that each entry b_{ij} of $S(t) = (b_{ij})$ is in $\mathbb{K}$ if $t \in \mathbb{K}^*$.For an element σ of the Galois group $G(\mathbb{K}/\mathbb{Q})$, we set

$$S(t)^\sigma = (b_{ij}^\sigma).$$

Then the entries of the matrix

$$A(t) = \prod_{\sigma \in G(\mathbb{K}/\mathbb{Q})} S(t)^\sigma$$

are in $\mathbb{Q}$ if $t \in \mathbb{K}^*$. Hence $A(t)$ defines elements of $T^{\mathbb{Q}}(\mathbb{Q})$, the $\mathbb{Q}$-raotional points of $T^{\mathbb{Q}}$. For $t \in \mathbb{Q}^*$ we decompose $\bigoplus_{i=2}^{2n} m(X)(n)^i$ into $A(t)$-invariant, irreducible $\mathbb{Q}$-subspaces

$$\bigoplus_{i=2}^{2n} m(X)(n)^i = \bigoplus_{j=1}^{\ell} V_j.$$

The restriction of $\lambda(t)$ for $t \in \mathbb{Q}^*$ on V_j is represented by a matrix $A_j(t)$ whose entries are in $\mathbb{Q}$. The matrix $A_j(t)$ is diagonalizable over $\mathbb{K}$; there is an invertible matrix P_j with entries in $\mathbb{K}$ such that

$$B_j(t) = P_j^{-1} A_j(t) P_j = \begin{pmatrix} k_1(t) & & 0 \\ & \ddots & \\ 0 & & k_{n_j}(t) \end{pmatrix},$$

where $k_1(t), \ldots, k_{n_j}(t)$ are eigenvalues of $A_j(t)$ which are conjugate over $\mathbb{Q}$ if $t \in \mathbb{Q}^*$. For an element σ of the Galois group $G(\mathbb{K}/\mathbb{Q})$ we set

$$B_j^\sigma(t) = \begin{pmatrix} k_1(t)^\sigma & & \\ & \ddots & \\ & & k_{n_j}(t)^\sigma \end{pmatrix}.$$

Then we have

$$C_j(t) = \prod_{\sigma \in G(\mathbb{K}/\mathbb{Q})} B_j^\sigma(t) = \begin{pmatrix} r_j(t) & & \\ & \ddots & \\ & & r_j(t) \end{pmatrix},$$

where $r_j(t)$ is in $\mathbb{Q}^*$ if $t \in \mathbb{Q}^*$. Hence if we set

$$D(t) = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_\ell \end{pmatrix} \begin{pmatrix} C_1(t) & & \\ & \ddots & \\ & & C_\ell(t) \end{pmatrix} \begin{pmatrix} P_1^{-1} & & \\ & \ddots & \\ & & P_\ell^{-1} \end{pmatrix},$$

then it is of the form

$$D(t) = \begin{pmatrix} r_1(t) & & & & & & \\ & \ddots & & & & & \\ & & r_1(t) & & & & \\ & & & \ddots & & & \\ & & & & r_\ell(t) & & \\ & & & & & \ddots & \\ & & & & & & r_\ell(t) \end{pmatrix}.$$

Then the matrix $D(t)$ defines a one parameter subgroup $\mu(t)$ of $T^{\mathbb{Q}}$ defined over $\mathbb{Q}$ such that $\lim_{t \to 0} \mu(t) = 0$.

Then we have the following:

Theorem 2.7. *The following conditions are equivalent:*

(1) *The Zariski closure of G_n in E_n contains the zero homomorphism for each n,*

(2) *X has $\mathbb{C}$-positive weights,*

(3) *X has $\mathbb{Q}$-positive weights.*

Proof. [(1) $\Rightarrow$ (2)] The metric closure of G_n in E_n contains 0 for $n \geq 2$. Then by Lemmas 2.4, 2.5 and 2.6 there is a one parameter subgroup $\lambda(t)$ of $T^{\mathbb{Q}}$ defined over $\mathbb{K}$ such that $\lim_{t \to 0} \lambda(t) = 0$. Hence by Proposition 2.3 X has $\mathbb{K}$-positive weights, where $[\mathbb{K} : \mathbb{Q}] < \infty$. In particular, we have (2).

[(2) $\Rightarrow$ (3)] If X has $\mathbb{K}$-positive weights for such a that $[\mathbb{K} : \mathbb{Q}] < \infty$, then from the above argument we have a one parameter subgroup $\mu(t)$ of $T^{\mathbb{Q}}$ defined over $\mathbb{Q}$ such that $\lim_{t \to 0} \mu(t) = 0$. Hence by Proposition 2.3 we have (3).

[(3) $\Rightarrow$ (1)] This is obvious by Proposition 2.3. QED

If X is a formal space, then one can see that it has $\mathbb{Q}$-positive weights by grading automorphisms (see [Su] and [Shi]). Thus, the property "having positive weights" does not depend on the ground field, as does in the case of formal spaces ([Su]).

3 p-universal spaces

In this section, we will prove that the rational homotopy type of a p-universal space has positive weights. Among the various definitions of the p-universality ([MOT]), we adopt the following for the sake of our convenience:

Definition 3.1. A simply connected CW-complex X is called *p-universal* if for any prime q different from p, there exists a map $f : X \longrightarrow X$ such that

(1) $f_* : H_*(X; \mathbb{Z}/p\mathbb{Z}) \longrightarrow H_*(X; \mathbb{Z}/p\mathbb{Z})$ is isomorphic,

(2) $f_\sharp \otimes 1 : \pi_*(X) \otimes \mathbb{Z}/q\mathbb{Z} \longrightarrow \pi_*(X) \otimes \mathbb{Z}/q\mathbb{Z}$ is trivial.

Let $m(X) = \Lambda(\bigoplus_{n \geq 2} V^n)$ be a minimal model of X. Then we have a diagram

$$\begin{array}{ccc} V^n & \underset{\varphi}{\xrightarrow{\sim}} & \mathrm{Hom}(\pi_n(X), \mathbb{Q}) \\ & & \cup \\ & & \mathrm{Hom}(\pi_n(X), \mathbb{Z}), \end{array}$$

where φ is a linear isomorphism of $\mathbb{Q}$-vector spaces ([Su], [H]). Let L^n be the free abelian subgroup of V^n which is mapped isomorphically by φ onto $\mathrm{Hom}(\pi_n(X), \mathbb{Z})$. We form a multiplicative lattice:

$$\hat{L}(X) = \Lambda(\bigoplus_{n\geq 2} L^n).$$

Then $\hat{L}(X)$ is a free graded commutative algebra over $\mathbb{Z}$. Denote by $\hat{L}(X)\langle n\rangle$ the sub $\mathbb{Z}$-module of the elements of degrees $\leq n$.

Suppose that X is p-universal. Then the map $f : X \longrightarrow X$ in Definition 3.1 induces an automorphism $\hat{f} : m(X) \longrightarrow m(X)$ such that $\hat{f}$ preserves $\hat{L}(X)$. Let $\{e_1, \ldots, e_s, h_1, \ldots, h_t\}$ be a basis for $\hat{L}(X)\langle n\rangle$ such that

$$e_i \in \bigoplus_{j=2}^{n} L^i \quad (i = 1, \ldots, s)$$

and

$$h_j \in \hat{L}(X)^+ \cdot \hat{L}(X)^+ \quad (j = 1, \ldots, t),$$

where $\hat{L}(X)^+$ is the set of the elements of positive degrees. Then by (2) of Definition 3.1 with respect to this basis, the restriction $\hat{f}|\hat{L}(X)\langle n\rangle$ is represented by a matrix with integer entries

$$C = \left(\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right)$$

such that each entry of A and B is divisible by q. Let $\mathbb{K}$ be a finite field extension of $\mathbb{Q}$ containing all the eigenvalues of C. Let ν_q be a normal valuation of $\mathbb{Q}$ defined by

$$\nu_q\left(\frac{aq^s}{b}\right) = q^{-s},$$

where a, b are integers prime to q. We extend ν_q to $\mathbb{K}$, which is also denoted ν_q by abuse of notation. All the coefficients except the highest degree of the equation

$$\det(tI - C) = t^\ell + a_{\ell-1}t^{\ell-1} + \cdots + a_0 = 0 \tag{6}$$

are divisible by q, where $\ell = s + t$ is the dimension of $\hat{L}(X)\langle n\rangle$. Let λ be one of eigenvalues of C. Then we have

$$\nu_q(\lambda^\ell) = \nu_q(-(a_{\ell-1}\lambda^{\ell-1} + \cdots + a_0)).$$

Since λ is an algebraic integer, we have

$$\nu_q(\lambda) \leq 1.$$

Hence we have

$$\nu_q(-(a_{\ell-1}\lambda^{\ell-1} + \cdots + a_0)) \leq \max(\nu_q(a_{\ell-1}), \ldots, \nu_q(a_0)) \leq q^{-1},$$

from which we have

$$\nu_q(\lambda^\ell) \leq q^{-1}. \tag{7}$$

Let G_n be the set of $\mathbb{K}$-DGA automorphisms of $\hat{L}(X)\langle n\rangle \otimes \mathbb{K}$. We choose a basis of $\hat{L}(X)\langle n\rangle \otimes \mathbb{K}$ so that C is represented by an upper triangular matrix

$$C' = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_\ell \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_\ell \end{pmatrix} \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_\ell$ are the eigenvalues of C. Then the matrix

$$C_s = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_\ell \end{pmatrix}$$

is the semi-simple part of C'. By the Jordan decomposition, one can see that C_s is also an element of $G_n(\mathbb{K})$, the set of $\mathbb{K}$-rational points of G_n.

Let $\alpha(X_{11}, X_{12}, \ldots, X_{nn})$ be a polynomial with coefficients in $\mathbb{Q}$ such that

$$\alpha(g) = 0 \quad \text{for all} \quad g \in \mathrm{Aut}_{\mathbb{C}}(m(X)(n) \otimes \mathbb{C}).$$

Then after multiplying some integer, the equation

$$\alpha(C_s^k) = 0 \quad (k = 1, 2, \ldots)$$

will become

$$\sum_{t \geq 1} \sum_{i_1 + \cdots + i_\ell = t} a_{i_1 \cdots i_\ell} \lambda_1^{ki_1} \cdots \lambda_\ell^{ki_\ell} + d = 0,$$

where $a_{i_1 \ldots i_\ell}$ and d, the constant term of α, are all integers. Then by using (7) we have

$$\nu_q(\sum_{t \geq 1} \sum_{i_1 + \cdots + i_\ell = t} a_{i_1 \ldots i_\ell} \lambda_1^{ki_1} \cdots \lambda_\ell^{ki_\ell}) \leq q^{-\psi(k)},$$

where $\psi(k)$ is an integer such that

$$\lim_{k \to \infty} \psi(k) = \infty.$$

As k can be arbitrarily large, the constant term d must be zero. This implies that the Zariski closure of G_n in E_n contains 0.

Thus by Theorem 2.7, we have proved the following:

Propositon 3.2. *If X is p-universal for a prime p, the rational homotopy type of X has $\mathbb{Q}$-positive weights.*

Let X be a CW-complex such that $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i \geq n_0$, where n_0 is some positive integer.

Proposition 3.3. *If there is a one parameter subgroup $\lambda(t)$ of* Aut $(m(X))$ *such that $\lim_{t \to 0} \lambda(t) = 0$, then there is a* CW*-complex K satisfying the following conditions:*

(a) *there is a 0-equivalence $g : X \longrightarrow K$,*

(b) *for any two distinct primes p, q, there is a p-equivalence $f_q : K \longrightarrow K$ inducing $f_{q\sharp} \otimes 1 = 0 : \pi_*(K) \otimes \mathbb{Z}/q\mathbb{Z} \longrightarrow \pi_*(K) \otimes \mathbb{Z}/q\mathbb{Z}$.*

Proposition 3.3 follows from Lemma 3.4 below.

We can represent $\lambda(t)$ by

$$\lambda(t) = \left\{ \left(\begin{array}{ccc} t^{a_1} & & \\ & \ddots & \\ & & t^{a_m} \end{array} \right) \middle| \, t \in \mathbb{Q}^* \right\}$$

with respect to some $\mathbb{Z}$-basis for $L(X)\langle n \rangle$, where $a_1, \ldots, a_m$ are positive integers. In particular, $\lambda(s)$ preserves $\hat{L}(X)$ for a positive integer s.

Lemma 3.4. *For each positive integer n, there is a complex K_n such that the following conditions are satisfied:* (a) there is a D.G.A. isomorphism

$$\rho_n : m(X)(n) \longrightarrow m(K_n)$$

(b)

$$\pi_i(K_n) = \begin{cases} \textit{torsion free} & \textit{for} \quad i \leq n \\ 0 & \textit{for} \quad i > n \end{cases};$$

(c) *for any distinct primes p and q, there is a p-equivalence*

$$f_q : K_n \longrightarrow K_n$$

satisfying the following conditions:

(1) *the induced homomorphism*

$$f_{q\sharp} \otimes 1 : \pi_*(K_n) \otimes \mathbb{Z}/q\mathbb{Z} \longrightarrow \pi_*(K_n) \otimes \mathbb{Z}/q\mathbb{Z}$$

is trivial;

(2) *the following diagram is commutative:*

$$\begin{array}{ccc} m(X)(n) & \xrightarrow{\lambda(q)_n} & m(X)(n) \\ \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho_n} \\ m(K_n) & \xrightarrow{\hat{f}_q} & m(K_n), \end{array}$$

where $\hat{f}_q$ is a map induced by f_q and $\lambda(q)_n$ is the restriction of $\lambda(q)$ on $m(X)(n)$.

Proof. We will prove the lemma by induction on n. As an inductive hypothesis we assume that there is a complex K_n satisfying the conditions (a) $\sim$ (c). We can choose a basis $\{e_1, \ldots, e_s\}$ for L^{n+1} ($\simeq \mathrm{Hom}_{\mathbb{Z}}(\pi_{n+1}(X), \mathbb{Z})$) such that each e_i is an eigenvector of $\lambda(q)_{n+1}$. Then $de_i \in m(X)(n)$ is an either an eigenvector or 0. Let N be a positive integer such that Nde_i ($i = 1, \ldots, s$) represents an element of $H^{n+2}(K_n; \mathbb{Z})$ via ρ_n. Then $(de_1, \ldots, de_s)$ represents an element

$$\chi \in [K_n, K(\mathbb{Z}^s/N, n+2)],$$

where [,] denotes the set of homotopy classes. Since $L^{n+1} \simeq \mathbb{Z}^s/N$ as $\mathbb{Z}$-modules, λ_q induces a map

$$\lambda_q : K(\mathbb{Z}^s/N, n+2) \longrightarrow K(\mathbb{Z}^s/N, n+2)$$

so that the diagram

$$\begin{array}{ccc} K_n & \xrightarrow{f_q} & K_n \\ \downarrow \chi & & \downarrow \chi \\ K(\mathbb{Z}^s/N, n+2) & \xrightarrow{\lambda_q} & K(\mathbb{Z}^s/N, n+2) \end{array}$$

is homotopy commutative. Let

$$\Omega K(\mathbb{Z}^s/N, n+2) \longrightarrow P \xrightarrow{\pi} K(\mathbb{Z}^s/N, n+2) \tag{7}$$

be the path fibration. Let $\hat{\lambda}_q : P \longrightarrow P$ be a map defined by $\hat{\lambda}_q(\ell)(t) = \lambda_q(\ell(t))$, where $\ell \in P$ and $t \in [0,1]$. Set

$$\begin{aligned} K_{n+1} &= \{(x,\ell) \in K_n \times P \mid \chi(x) = \pi(\ell)\}, \\ C_{n+1} &= \{(x,\ell) \in K_n \times P \mid \lambda_q\chi(x) = \pi(\ell)\}, \\ E_{n+1} &= \{(x,\ell) \in K_n \times P \mid \chi f_q(x) = \pi(\ell)\}. \end{aligned}$$

Then we define a map

$$\tilde{\lambda}_q : K_{n+1} \longrightarrow C_{n+1}$$

by $\tilde{\lambda}_q(x,\ell) = (x, \hat{\lambda}_q(\ell))$. Since $\lambda_q \circ \chi$ is homotopic to $\chi \circ f_q$, there is a homotopy equivalence $h : C_{n+1} \longrightarrow E_{n+1}$ so that the diagram

$$\begin{array}{ccc} C_{n+1} & \xrightarrow[h]{} & E_{n+1} \\ \downarrow & & \downarrow \\ K_n & \xrightarrow[\mathrm{Id}]{} & K_n \end{array}$$

is commutative, where the vertical maps are the restrictions of the projection $K_n \times P \longrightarrow K_n$ respectively. We define a map

$$\widetilde{f}_q : E_{n+1} \longrightarrow K_{n+1}$$

by $\widetilde{f}_q(x, \ell) = (f_q(x), \ell)$. Then the diagram

$$\begin{array}{ccc} E_{n+1} & \xrightarrow[\widetilde{f}_q]{} & K_{n+1} \\ \downarrow & & \downarrow \\ K_n & \xrightarrow[f_q]{} & K_n \end{array}$$

is commutative. By setting $\overline{f}_q = \widetilde{f}_q \circ h \circ \widetilde{\lambda}_q$, we have a commutative diagram

$$\begin{array}{ccc} K_{n+1} & \xrightarrow{\bar{f}_q} & K_{n+1} \\ \downarrow & & \downarrow \\ K_n & \xrightarrow[f_q]{} & K_n \end{array}$$

We will show that the map $\bar{f}_q : K_{n+1} \longrightarrow K_{n+1}$ satisfies **(a)** $\sim$ **(c)**. First of all, **(a)** is easy to show from the construction. The other part is easily obtained from the following homotopy commutative diagram:

$$\begin{array}{ccccc} K(\mathbb{Z}^s/N, n+1) & \longrightarrow & K_{n+1} & \longrightarrow & K_n \\ \downarrow \overline{\lambda}_q & & \downarrow \bar{f}_q & & \downarrow f_q \\ K(\mathbb{Z}^s/N, n+1) & \longrightarrow & K_{n+1} & \longrightarrow & K_n, \end{array} \tag{8}$$

and the fact that $\overline{\lambda}_q$ is represented on $\pi_{n+1}(K(\mathbb{Z}^s/N, n+1))$ by a diagonal matrix whose entries are positive integer power of q. QED

Now we will complete the proof of Theorem A.

Let X be a finite complex whose rational homotopy type has $\mathbb{Q}$-positive weights. Then by Lemma 3.4 and Proposition 2.2 we obtain a complex K_n satisfying the following two conditions for any two distinct primes p and q:

(1) K_n has the same rational homotopy type as the n-th stage of the Postnikov tower of X;

(2) for any distinct primes p and q, there is a p-equivalence f_q such that

$$f_{q\sharp} \otimes 1 = 0 : \pi_*(K_n) \otimes \mathbb{Z}/q\mathbb{Z} \longrightarrow \pi_*(K_n) \otimes \mathbb{Z}/q\mathbb{Z}.$$

Let n be the dimension of X and K^i_{n+1} be the i-skeleton of K_{n+1}. Then K^i_{n+1} has the homotopy type of a finite CW-complex. The homomorphism $i^n_* : H_n(K^n_{n+1};\mathbb{Z}) \longrightarrow H_n(K_{n+1};\mathbb{Z})$ induced by the inclusion $i_n : K^n_{n+1} \longrightarrow K_{n+1}$ is surjective. From the homology sequence of the pair (K_{n+1}, K^n_{n+1}) we have an exact sequence

$$H_{n+1}(K_{n+1};\mathbb{Z}) \xrightarrow[j_*]{} H_{n+1}(K_{n+1}, K^n_{n+1};\mathbb{Z}) \xrightarrow[\partial_*]{} H_n(K^n_{n+1};\mathbb{Z}) \xrightarrow[i^*_n]{} H_n(K_{n+1};\mathbb{Z}).$$

Since $H_n(K^n_{n+1};\mathbb{Z})$ is free, we have a direct sum decomposition of a free $\mathbb{Z}$-module

$$H_{n+1}(K_{n+1}, K^n_{n+1};\mathbb{Z}) = \mathrm{Im}\ j^* \oplus A,$$

where A is isomorphic to $\ker i^*_n$ by ∂_*. We may assume that the map $f_q : K_{n+1} \longrightarrow K_{n+1}$ is cellular; let $f^n_q : K^n_{n+1} \longrightarrow K^n_{n+1}$be the restriction. Then A is f_{q*}-invariant. We may regard A as a free submodule of $\pi_{n+1}(K_{n+1}, K^n_{n+1})$.

Let $\{\alpha_1, \ldots, \alpha_m\}$ be a basis of A. Let K be a complex obtained from K^n_{n+1} by attaching m cells of dimension $n+1$ via $\partial_\sharp \alpha_i \; (i = 1, \ldots, m)$, where

$$\partial_\sharp : \pi_{n+1}(K_{n+1}, K^n_{n+1}) \longrightarrow \pi_n(K^n_{n+1})$$

is the boundary operator. From the construction we may ragard K as a sub-complex of K_{n+1} such that

$$\begin{array}{lcll} H_i(K;\mathbb{Z}) & \simeq & H_i(K_{n+1};\mathbb{Z}) & \text{for } i \leq n, \\ H_i(K;\mathbb{Z}) & = & 0 & \text{for } i > n. \end{array}$$

The map f^n_q can be extended to $\widetilde{f}_q : K \longrightarrow K$ so that the diagram

$$\begin{array}{ccccc} K^n_{n+1} & \hookrightarrow & K & \hookrightarrow & K_{n+1} \\ \downarrow{\scriptstyle f^n_q} & & \downarrow{\scriptstyle \widetilde{f}_q} & & \downarrow{\scriptstyle f_q} \\ K^n_{n+1} & \hookrightarrow & K & \hookrightarrow & K_{n+1} \end{array}$$

is homotopy commutative. Then $\widetilde{f}_q$ is a p-equivalence such that the induced homomorphism

$$\widetilde{f}^*_q : H^*(K;\mathbb{Z}/q\mathbb{Z}) \longrightarrow H^*(K;\mathbb{Z}/q\mathbb{Z})$$

is trivial. Hence by (b) of Theorem 2.1 in [MOT], K is a p-universal for all p. Finally we construct a 0-equivalence $g : X \longrightarrow K$. Since D.G.A.s $m(X)(n+1)$ and $m(K_{n+1})$ are isomorphic, there is a homotopy equivalence between localised spaces at zero:

$$h : (X_{n+1})_{(0)} \longrightarrow (K_{n+1})_{(0)},$$

where X_{n+1} is the $(n+1)$-th stage of the Postnikov tower of X. Composing with the natural map and the localisation map, we obtain a map

$$\phi_{n+1} : X \longrightarrow (K_{n+1})_{(0)}.$$

Since $(K_{n+1})_{(0)}$ is obtained from $(K^n_{n+1})_{(0)}$ by attaching 'local cells' (cone over the local sphere) of dimension $\leq n+1$ ([Su 2]). By the cellular approximation theorem we obtain a map $X \longrightarrow (K^n_{n+1})_{(0)}$. By composing with the inclusion we have a map $\phi : X \longrightarrow K_{(0)}$ such that ϕ^* induces isomorphisms on rational cohomology. Since K is p-universal for every p and 0, the map ϕ factors as $X \xrightarrow{g} K \xrightarrow{\ell} K_{(0)}$. Then by Theorem 1.3 in [MT], X is also p-universal, and we have the desired result.

Remark 3.6. Theorem A does not hold for infinite complexes. Recall that the infinite quaternionic projective space $\mathbb{H}P^\infty$ has the same rational homotopy type as the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$, which is formal. As is well known, the degree of the induced map on $H^4(\mathbb{H}P^\infty; \mathbb{Z})$ of a self map is odd square. Hence $\mathbb{H}P^\infty$ is not p-universal ($p \neq 2$) in the sense of Definition 3.1. However $K(\mathbb{Z}, 4)$ is p-universal for any prime p.

As an application of Theorem A, we will show that homogeneous spaces of compact Lie groups are p-universal for any prime p.

Let G be a compact connected Lie group and H a closed connected subgroup of G. Let $S^*(G)$ be the ring of polynomial function with value in $\mathbb{R}$ on the Lie algebra $L(G)$. Then $S^*(G)$ is a symmetric algebra of $L(G)^* = \mathrm{Hom}_{\mathbb{R}}(L(G), \mathbb{R})$.

The degree of the elements of $L(G)^*$ is defined to be 2. Let $S^*(G)^G$ be the invariant subalgebra under the adjoint action of G. Then $S(G)^G$ is isomorphic to a graded polynomial algebra. Let $A(G)$ be the exterior algebra of $L(G)^*$, and $A(G)^G$ the invariant subalgebra under the adjoint action of G. Then $A(G)^G$ is the exterior algebra of the primitive space $P(G)$. We have the transgression $\tau : P(G) \longrightarrow S(G)^G$. Let $\gamma : S(G)^G \longrightarrow S(H)^H$ be the restriction of polynomial functions. Then we have a free DGA

$$A(G/H) = S(H)^H \otimes A(G)^G,$$

where the differential d_r is defined as follows:

$$\begin{aligned} d_r(x \otimes 1) &= 0, \quad \text{for } x \in S(H)^H, \\ d_r(1 \otimes y) &= \gamma r(y) \otimes 1, \quad \text{for } y \in P(G). \end{aligned}$$

The minimal model of G/H over $\mathbb{R}$ is isomorphic to that of $A(G/H)$. Let $m(G/H) \otimes \mathbb{R}$ be the minimal model of $A(G/H)$.

Proposition 3.7. Let G be a compact connected Lie group and H a closed connected subgroup. Then G/H is p-universal for any prime p.

Proof. By Theorem A, it is sufficient to show that the rational homotopy type of G/H has positive weights. For any $t \in \mathbb{R}^*$, there is a one parameter subgroup $\lambda(t)$ of DGA automorphisms of $A(G/H)$ defined by

$$\begin{aligned} \lambda(t)(x \otimes 1) &= t^{|x|} x \otimes 1, \\ \lambda(t)(1 \otimes y) &= t^{|y|+1}(1 \otimes y), \end{aligned}$$

where $|z|$ denotes the degree of z. The lifting $\psi(t)$ of $\lambda(t)$ on $m(G/H)$ gives elements of $\mathrm{Aut}_{\mathbb{R}}(m(G/H)\otimes\mathbb{R})$ such that $\lim_{t\to 0}\psi(t)=0$. Then by Theorem 2.7 it has $\mathbb{Q}$-positive weights. QED

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