Topological Conjugacy of Circle Diffeomorphisms

JUN HU
Department of Mathematics, Graduate Center
CUNY, NY10036, USA

DENNIS P. SULLIVAN
Department of Mathematics, Graduate Center
CUNY, NY10036, USA and IHES, FRANCE

February 1, 2008

Abstract

The classical criterion for a circle diffeomorphism to be topologically conjugate to an irrational rigid rotation was given by A. Denjoy [1]. In [5] one of us gave a new criterion. There is an example satisfying Denjoy's bounded variation condition rather than [5]'s Zygmund condition and vice versa. This paper will give the third criterion which is implied by either of the above criteria.

Contents

1. Introduction
2. Cross ratio distortion
3. Non wandering set and ergodicity
4. Proof of results
5. Three examples
6. Appendix

Published in modified form:
Ergod. Th. & Dynam. Sys. **17** (1997), 173–186

Stony Brook IMS Preprint #1995-7
May 1995

1 Introduction

Given a circle orientation preserving homeomorphism $f : S^1 \to S^1$, the rotation number

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \quad mod\ 1$$

is independent of x and the lift F of f, where $F : R^1 \to R^1$ is a lift of f and $x \in R^1$. And it is invariant under topological conjugations. The rotation number $\rho(f)$ is a rational number if and only if f has a periodic orbit. From the theory of Poincaré, for an orientation preserving homeomorphism $f : S^1 \to S^1$, if f has a periodic orbit then its dynamics turn out trivial: any two periodic orbits have the same period and any orbit tends to a periodic orbit; if f doesn't have any periodic orbit then it is semi-conjugate to an irrational rigid rotation. A natural question is whether or not the semi-conjugation could be improved to be a topological conjugation. In the following context when we say a rigid rotation we always mean an irrational rigid rotation. Denjoy proved the following.

Theorem A Given an orientation preserving homeomorphism f of the circle S^1 with an irrational rotation number, f is topologically conjugate to a rigid rotation provided f is C^1 and the logarithm of the derivative of f is of bounded variation.

There is an example (Denjoy counterexample) to show that C^1 smoothness is not enough [6]. It is shown even C^∞ smoothness is not enough yet in [15]. Actually an orientation preserving circle homeomorphism with an irrational rotation number is topologically conjugate to an irrational rigid rotation if and only if it has no wandering interval. Denjoy achieved this by controlling the variation of the derivative. Recently one of us proved the non existence of wandering interval by assuming the logarithm of the derivative satisfies the Zygmund condition.

Definition A continuous map $f : R^1 \to R^1$ satisfies the Zygmund condition if there exists $B > 0$ such that

$$\sup_{x,t} \left|\frac{f(x+t) + f(x-t) - 2f(x)}{t}\right| \le B.$$

Theorem B Given an orientation preserving homeomorphism f of the circle S^1 with an irrational rotation number, f is topologically conjugate to a rigid rotation if f is C^1 and the logarithm of the derivative satisfies the Zygmund condition.

But there is an example satisfying Denjoy's bounded variation condition and not Zygmund's condition and vice versa [section 5]. This paper gives a third criterion which is implied by either of the above two and which implies f is topologically conjugate to a rigid rotation.

Definition Let I be a closed interval of R^1. A continuous map $f : I \to R^1$ is of bounded Zygmund variation if there exists $B > 0$ such that

$$\sup_{\{x_0, x_1, \cdots, x_n\}} \sum_{i=0}^{n-1} \left|f(x_i) + f(x_{i+1}) - 2f\left(\frac{x_i + x_{i+1}}{2}\right)\right| \le B,$$

where $\{x_0, x_1, \cdots, x_n\}$ is a partition of the interval I. The supremum is called Zygmund variation of f over I. It is denoted by $ZV(f|_I)$.

Definition Let I be a closed interval of R^1. A continuous map $f : I \to R^1$ is of bounded quadratic variation if there exists $B > 0$ such that

$$\sup_{\{x_0,x_1,\cdots,x_n\}} \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i))^2 \leq B,$$

where $\{x_0, x_1, \cdots, x_n\}$ is a partition of the interval I. The supremum is called quadratic variation of f over I. It is denoted by $QV(f|_I)$.

Theorem C Given an orientation preserving homeomorphism f of the circle S^1 with an irrational rotation number, f is topologically conjugate to a rigid rotation if f is C^1 and the logarithm of the derivative has bounded Zygmund variation and bounded quadratic variation.

2 Cross ratio distortion

In this section we control cross ratio distortion for standard 4-tuples in terms of Zygmund variation and quadratic variation (compare §1 of [5]). Let $a, b, c, d \in R^1$ and $a < b < c < d$.

One cross ratio $[a, b, c, d] = \frac{(d-b)(c-a)}{(c-b)(d-a)}$ can be computed by

$$log[a, b, c, d] = \int\int_S \frac{dxdy}{(x-y)^2},$$

where S is $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$.

Another cross ratio (a, b, c, d) is $\frac{(c-b)(d-a)}{(b-a)(d-c)}$ and, obviously,

$$[a, b, c, d] = 1 + \frac{1}{(a, b, c, d)}.$$

Given a homeomorphism h, the distortion of the second cross ratio under h is

$$\frac{(ha, hb, hc, hd)}{(a, b, c, d)}.$$

In this paper, by the cross ratio distortion we mean the distortion of the second cross ratio.

We call a 4-tuple $a < b < c < d$ standard if $b - a = c - b = d - c$. The cross ratio distortion under h of a standard 4-tuple is bounded away from zero and from above if and only if (ha, hb, hc, hd) is also. If h is C^1 diffeomorphism, then

$$log[1 + \frac{1}{(ha, hb, hc, hd)}] = log[ha, hb, hc, hd] = \int\int_S (h \times h)^* \mu,$$

where μ is the measure $\frac{dxdy}{(x-y)^2}$.

Clearly the cross ratio distortion under f of a standard 4-tuple is bounded away from zero and from above if and only if $log[ha, hb, hc, hd]$ is also. Calculating the integrand, we get

$$\frac{h'xh'y}{(hx-hy)^2} = \frac{1}{(x-y)^2}\frac{h'xh'y}{[h']_{xy}^2},$$

where $[h']_{xy}$ is the average of h' over the interval $[x, y]$.

Since $b-a = c-b = d-c$, $\int\int_S \frac{dxdy}{(x-y)^2} = log([a, b, c, d]) = log\frac{4}{3}$. Thus a bound on $\frac{h'xh'y}{[h']_{xy}^2}$ yields a bound on the cross ratio distortions for standard 4-tuples.

We say h satisfies the *bounded Koebe condition* if one of the following equivalent conditions hold:

$$1)\ \ \frac{1}{M} \leq \frac{h'xh'y}{[h']_{xy}^2} \leq M \ for\ some\ M > 0,$$

$$2)\ \ |log\frac{h'xh'y}{[h']_{xy}^2}| \leq M' \ for\ some\ M' > 0.$$

The following proposition is trivial.

Prop. 1 *If h satisfies the bounded Koebe condition then the cross ratio distortion under h of a standard 4-tuple is bounded away from zero and from above.*

In order to estimate the log in 2), i.e.,

$$logh'x + logh'y - 2log[h']_{xy},$$

let us consider the following two terms:

$$a)\ \ logh'x + logh'y - 2[logh']_{xy}$$

and

$$b)\ \ log[h']_{xy} - [logh']_{xy}\ .$$

Remark: If both a) and b) are bounded, then 1) and 2) hold.

Expression a) can be controlled by the Zygmund variation of $logh'$ on the interval $[x, y]$ because of the following proposition.

Prop. 2 *Let ϕ be a continuous function from R^1 to R^1. Then*

$$|\phi(x) + \phi(y) - 2[\phi]_{xy}|$$

is no more than the Zygmund variation $ZV(\phi|_{[x,y]})$ of ϕcorollary over $[x, y]$.

Remark: As we define the Zygmund variation of ϕ on the interval $[a, b]$ in the introduction, we can also define the average Zygmund variation of ϕ on $[a, b]$ by replacing the value $\phi(\frac{x_i+x_{i+1}}{2})$ of ϕ at the middle point by the average $\frac{1}{|x_{i+1}-x_i|}\int_{[x_i,x_{i+1}]}\phi$ of ϕ over $[x_i, x_{1+1}]$. Prop. 2 tells us that the average Zygmund variation of ϕ over $[a, b]$ is no more than the Zygmund variation of ϕ over $[a, b]$. Conversely one can show that the Zygmund variation of ϕ over $[a, b]$ is no more than twice the average Zygmund variation of ϕ over $[a, b]$. Hence these two conditions are actually equivalent.

Proof: Without loss of generality, assume $[x, y] = [0, 1]$. Then

$$[\phi]_{01} = \int_0^1 \phi dx.$$

Suppose that we define successive approximations to the average of ϕ over $[a, b]$ by

$$A_0[a, b] = (\phi(a) + \phi(b))/2$$

and

$$A_{n+1}[a, b] = (A_n[a, m] + A_n[m, b])/2,$$

where $m = (a + b)/2$. Similarly, measure non-linearity by expressions

$$N_0[a, b] = A_0[a, b] - A_1[a, b] = (\phi(a) - 2\phi(m) + \phi(b))/4$$

and

$$N_{n+1}[a, b] = (N_n[a, m] + N_n[m, b])/2,$$

or equivalently

$$N_n[a, b] = A_n[a, b] - A_{n+1}[a, b].$$

Then

$$A_0[0, 1] - A_n[0, 1] = N_0[0, 1] + \cdots + N_{n-1}[0, 1]$$

with

$$|N_k[0, 1]| \leq ZV(\phi|_{[0,1]})/2^{k+2}$$

hence

$$2|A_0[0, 1] - \lim_{n\to\infty} A_n[0, 1]| \leq ZV(\phi|_{[0,1]}),$$

i.e.,

$$|\phi(0) + \phi(1) - 2[\phi]_{01}| \leq ZV(\phi|_{[0,1]}).$$

Next we estimate the expression b) in terms of the quadratic variation.

Lemma 1 *If $\epsilon \geq \delta > -1$, assume*

$$log(1 + \epsilon) = \epsilon - \frac{\epsilon^2}{2}\Delta(\epsilon),$$

then there exists $B(\delta) > 0$ depending on δ such that $|\Delta(\epsilon)| \leq B(\delta)$.

Proof: Since

$$\Delta(\epsilon) = \frac{\epsilon - log(1 + \epsilon)}{\epsilon^2/2},$$

The proof is an elementary calculation.

Definition 1 *A quantity C_1 (depending on parameters) is a big O of another quantity C_2 (depending on the parameters) if there exists a constant B (independent of the parameters) such that*

$$|C_1| \leq B|C_2|.$$

Prop. 3 *Suppose the derivative h' satisfies $1/C \leq h' \leq C$ for some $C > 0$. Then the expression b) is equal to the big O of the quadratic variation of $logh'$ over the interval $[x, y]$.*

Proof: Let $h'(x) = a$. The expression b) is unchanged if we multiply $h'x$ by $1/a$. Write $(1/a)h'$ on $J = [x, y]$ as $1 + \epsilon$ where ϵ is a function of $(t - x), t \in J$. Expand the two terms of b)

$$log\frac{1}{|J|}\int_J (1+\epsilon) - \frac{1}{|J|}\int_J log(1+\epsilon)$$

$$= log(1 + \frac{1}{|J|}\int_J \epsilon) - \frac{1}{|J|}\int_J [\epsilon - \frac{\epsilon^2}{2}\Delta(\epsilon)]$$

$$= [\frac{1}{|J|}\int_J \epsilon - \frac{1}{2}(\frac{1}{|J|}\int_J \epsilon)^2 \Delta(\frac{1}{|J|}\int_J \epsilon)] - [\frac{1}{|J|}\int_J \epsilon - \frac{1}{|J|}\int_J \frac{\epsilon^2}{2}\Delta(\epsilon)]$$

$$= -\frac{1}{2}(\frac{1}{|J|}\int_J \epsilon)^2 \Delta(\frac{1}{|J|}\int_J \epsilon) + \frac{1}{|J|}\int_J \frac{\epsilon^2}{2}\Delta(\epsilon).$$

By the Cauchy inequality, $(\frac{1}{|J|}\int_J \epsilon)^2 \leq \frac{1}{|J|}\int_J \epsilon^2$. Since $1/C \leq h' \leq C$ for some $C > 0$, there exists $\delta(C) > -1$ such that $\epsilon = \frac{h^{'}t}{h^{'}x} - 1 \geq \delta$ for any $t \in J$, $J = [x, y]$. Hence $\frac{1}{|J|}\int_J \epsilon \geq \delta$. By the Lemma 1, there exists $B(\delta) > 0$ such that $|\Delta(\epsilon)| \leq B(\delta)$. Hence $|\Delta(\frac{1}{|J|}\int_J \epsilon)| \leq B(\delta)$. Furthermore we can get that $\epsilon = \frac{h't}{h'x} - 1$ is a big O of $log\frac{h't}{h'x} = logh't - logh'x$. so the expression b) is a big O of the quadratic variation of $logh'$ over J. The following proposition will be used in section 4 to the iterates of a circle diffeomorphism f.

Prop. 4 *Suppose $h : I \to R^1$ is a C^1 diffeomorphism with $h' > 0$, and $logh'$ has bounded Zygmund variation and bounded quadratic variation over I. Assume $J_0 \subset I$ and J_0 $J_1 = h(J_0)$, $\cdots$, $J_n = h^n(J_0)$ are pairwise disjoint. Then the cross ratio distortion under h^n of a standard 4-tuple in the interval J_0 is the big O of the sum of the Zygmund variation and the quadratic variation of $\log h^{'}$ on $\cup_{i=0}^{n-1} J_i$.*

Proof: From the expression 2) above the Prop. 1, we want to estimate

$$log\frac{(h^n)^{'}(x)(h^n)^{'}(y)}{[(h^n)^{'}]^2_{xy}}.$$

By the chain rule of calculating the derivative of h^n,

$$log\frac{(h^n)^{'}(x)(h^n)^{'}(y)}{[(h^n)^{'}]^2_{xy}} = \sum_{i=0}^{n-1} log\frac{h^{'}(h^i(x))h^{'}(h^i(y))}{[h^{'}]^2_{h^i(x)h^i(y)}}.$$

Each summand can be decomposed into the expression a) and expression b), by the Prop. 2 and Prop. 3, each summand is the big O of the sum of the Zygmund variation and the quadratic variation of $logh^{'}$ over the interval $[h^i(x), h^i(y)]$, where $i = 0, 1, 2, \cdots, n-1$. So the cross ratio distortion under h^n of a standard 4-tuple in J_0 is the big O of the sum of the Zygmund variation and the quadratic variation of $logh^{'}$ on $\cup_{i=0}^{n-1} J_i$.

3 Nonwandering set and ergodicity

In this section we review some basic techniques due to Denjoy [1]. Suppose $f : S^1 \to S^1$ is an orientation preserving homeomorphism with an irrational rotation number.

For $x \in S^1$, let

$$\omega(x) = \cap_{n\in N} Cl(\{f^k(x)|k \geq n\}),$$
$$\alpha(x) = \cap_{n\in N} Cl(\{f^{-k}(x)|k \geq n\}),$$

where $Cl(A)$ means the closure of the set A. They are called ω limit set of the orbit of x and α limit set of the orbit of x respectively.

$x \in S^1$ is called a wandering point of f if there exists a neighborhood U of x such that

$$f^k(U) \cap U = \emptyset, \forall k \in Z \setminus \{0\}.$$

A point is called a nonwandering point if it is not a wandering point. $\Omega(f)$ denotes the set of all nonwandering points, which is called nonwandering set. Clearly it is a closed subset.

A subset A is invariant under f if

$$f(A) \subset A,\ f^{-1}(A) \subset A.$$

A non-empty subset A is minimal for f if it is closed, invariant under f and there is no non-empty proper closed subset of A which is invariant under f.

Prop. 5 *Suppose f has no periodic point, then*
(1) $\Omega(f) = \omega(x) = \alpha(x)$, $\forall x \in S^1$;
(2) $\Omega(f)$ is a minimal set of f;
(3) either $\Omega(f)$ is a nowhere dense perfect subset of S^1 or $\Omega(f) = S^1$.

Proof: (1) $\omega(x)$ is a non-empty closed invariant subset of S^1. Let (γ, δ) be a component of $S^1 \setminus \omega(x)$, then $f^j((\gamma, \delta))$ is also a component of $S^1 \setminus \omega(x)$ for any $j \in Z$. Since f has no periodic point, $\{f^j([\gamma, \delta])|j \in Z\}$ must be pairwise disjoint and hence (γ, δ) is a wandering interval of f. So $S^1 \setminus \omega(x) \subset S^1 \setminus \Omega(f)$. Hence $\Omega(f) \subset \omega(x)$. Clearly $\omega(x) \subset \Omega(f)$. So $\Omega(f) = \omega(x)$. Similarly $\Omega(f) = \alpha(x)$.

(2) Clearly from (1).

(3) Let $\partial\Omega$ denote the boundary of Ω, $\partial\Omega$ is closed. Since

$$\partial\Omega \subset \Omega,\ f(\partial\Omega) = \partial f(\Omega) = \partial\Omega,$$

either $\partial\Omega = \emptyset$ hence $\Omega = S^1$ or $\partial\Omega = \Omega$ hence Ω is nowhere dense. For the second case, Ω is perfect since $\Omega = \omega(y)$, $\forall y \in \Omega$.

Definition 2 *Suppose f has no periodic point. We say f is ergodic if $\Omega(f) = S^1$, otherwise we say f is not ergodic.*

The following result is well known and its proof can be found in several references ([1], [2], [3], [4] and etc.).

Prop. 6 *Suppose an orientation preserving homeomorphism $f : S^1 \to S^1$ has no periodic point and is ergodic, $\alpha = \rho(f)$. Then f is topologically conjugate to an irrational rigid rotation $\tau_\alpha : S^1 \to S^1$ given by*

$$\tau_\alpha(\xi) = e^{2\pi i\alpha}\xi.$$

4 Proofs of results

A circle homeomorphism with an irrational rotation number is topologically conjugate to a rigid rotation if and only if it is ergodic, in other words if and only if it has no wandering interval. Denjoy's $C^{1+b.v}$-condition and [5]'s C^{1+Z}-condition both guarantee the nonexistence of a wandering interval. In this section we prove that the C^1-plus bounded Zygmund variation and bounded quadratic variation guarantee the nonexistence of a wandering interval. Before we get into the proofs of these results, we need the following technique lemmas.

Prop. 7 (Contraction Principle) *([10], [6]) Suppose $f : S^1 \to S^1$ is a circle homeomorphism has no periodic orbits and I is a subinterval of S^1. If $\inf_{n\geq 0}\{|f^n(I)|\} = 0$, then I is a wandering interval of f.*

Proof: Let $I_n = f^n(intI)$ and $\Sigma = \cup_{n\geq 0} I_n$.

Case 1: If $\Sigma = S^1$, then S^1 is covered by finite $I_{n_i}, i = 1, 2, ..., k$. Since $\inf_{n\geq 0} |I_n| = 0$, the Lebesgue's lemma implies that there is I_l, $l \in N$, contained in one of $I_{n_i}, i = 1, 2, ..., k$, name it I_j. Without loss of generality we assume $l > j$. Note $I_l = f^{l-j}(I_j)$. So f^{l-j} will have a periodic point in the closure of I_j. This is contradiction.

Case 2: Suppose $\Sigma \neq S^1$. If there is no component U of Σ such that some iterate of U intersects with U, then U and hence I are wandering intervals. If there is a component U of Σ such that $f^n(U) \cap U \neq \emptyset$ for some $n \geq 0$, then $f^n(U) \subseteq U$ and hence f^n has a periodic point in the closure of U. This is again a contradiction.

Lemma 2 (Real Koebe Principle) *If $h : I \to R^1$ does not increase the cross ratio distortions for standard 4-tuples too much then the quasisymmetric distortions for standard interior triples are controlled. More precisely, if x, $y \in I$ satisfy $|x - y|$ is as small as the distance to the boundary ∂I of I and $z = (x + y)/2$, then*

$$\frac{1}{C} \leq |h(x) - h(z)|/|h(z) - h(y)| \leq C,$$

where C only depends on the bound of the cross ratio distortions for standard 4-tuples.

Proof: See §2 of [5]. The idea to prove this lemma is to use the four interval arguement. Let J, L, M, R be four contiguous equal lenth intervals. Suppose the lenth of $h(L)$ is much smaller than $h(M)$. Since the cross ratio distortion $\frac{|h(M)||h(T)|}{h(L)||h(R)|}/3$ on L, M, R is greater than the ratio distortion $\frac{|h(M)|}{|h(L)|}$, no bound of ratio distortions implies no bound of cross ratio distortions.

It is easy to use the Real Koebe Principle to get the following Macroscopic Koebe Distortion Principle.

Definition 3 *Let M and T be two intervals with $M \subset T$, and L and R be components of $T \setminus M$. If $\epsilon > 0$ we say T is an ϵ-scaled neighborhood of M if*

$$\frac{|L|}{|M|} \geq \epsilon \textit{ and } \frac{|R|}{|M|} \geq \epsilon.$$

Prop. 8 (Macroscopic Koebe Distortion Principle) *Given any $B > 0$, $\epsilon > 0$, there exists $\delta > 0$ only depending on B and ϵ such that, for any homeomorphism f of the circle, any subintervals $M \subset T$ and any $n \geq 0$, if the cross ratio distortion under f^n of any standard 4-tuple in T is bounded by B and $f^n(T)$ contains an ϵ-scaled neighborhood of $f^n(M)$ then T contains a δ-scaled neighborhood of M.*

Proof: Let $T \setminus M = L \cup R$. Without loss of generality, we only need to prove $\frac{|M|}{|L|}$ can not be very large. Suppose $\frac{|M|}{|L|}$ is large, we cut M into pieces L_i from left to right with lengths $2^{i-1}|L|, i = 1, 2, 3, \cdots$. We also denote $L_0 = L$. From the Real Koebe Principle, there exists a constant C only depending B such that

$$\frac{|f^n(L_i)|}{|\cup_{j=0}^{i-1} f^n(L_j)|} \geq \frac{1}{C},$$

where $i = 1, 2, 3, \cdots$. Hence

$$\frac{|\cup_{j=0}^{i} f^n(L_i)|}{|\cup_{j=0}^{i-1} f^n(L_j)|} \geq 1 + \frac{1}{C},$$

where $i = 1, 2, 3, \cdots$. So

$$\frac{|\cup_{j=0}^{i} f^n(L_i)|}{|f^n(L_0)|} \geq (1 + \frac{1}{C})^i,$$

where $i = 1, 2, 3, \cdots$. This means

$$\frac{|\cup_{j=1}^{i} f^n(L_i)|}{|f^n(L_0)|} \geq (1 + \frac{1}{C})^i - 1,$$

where $i = 1, 2, 3, \cdots$.

Clearly i can not be very large, otherwise $f^n(T)$ can not be an ϵ-scaled neighborhood of $f^n(M)$. Hence we can find a bound of i only depending on B and ϵ, which means there exists $\delta > 0$ only depending on B and ϵ such that T contains a δ-scaled neighborhood of M.

Definition 4 *The intersection multiplicity of a collection of sets $X_{\alpha \in \Lambda}$ is the maximal cardinality of a subcollection with non-empty intersection.*

Use the Contraction Principle (Prop. 7), it is easy to get the following proposition.

Prop. 9 *Suppose $f : S^1 \to S^1$ is an orientation preserving homeomorphism without periodic orbits. Let I be a wandering interval and not contained in any larger wandering interval. If I is a proper subset of an interval J, then the intersection multiplicity of the pullbacks $\{f^{-i}(J) : i = 0, 1, 2, ...\}$ is infinity.*

Proof: Suppose the intersection multiplicity of the pullbacks $\{f^{-i}(J) : i = 0, 1, 2, ...\}$ is finite. Then $|f^{-n}(J)| \to 0$ as $n \to \infty$. Now apply the contraction Principle to J and the map f^{-1}. It says that J is a wandering interval. But this is false because I was a maximal wandering interval.

Definition 5 *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism. The variation of the logarithm of cross ratio distortion under f is defined as*

$$\sup_{\{x_0,x_1,\cdots,x_n\}} \sum_{i=0}^{n-1} \sup_{b_i,c_i \in (x_i,x_{i+1})} \log \frac{(f(x_i), f(b_i), f(c_i), f(x_{i+1}))}{(x_i, b_i, c_i, x_{i+1})}$$

where b_i and c_i belong to the open interval (x_i, x_{i+1}) from x_i to x_{i+1} counter clockwisely and $\{x_0, x_1, \cdots, x_n\}$ is a partition of S^1.

Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with an irrational rotational number. Let I be a wandering interval for f. The following combinatorial machinery on wandering intervals, $I_n = f^n(I) : n = 0, 1, 2, ...$, was developed in [10] and can be found in [6].

Definition 6 *If $n \in N$, we say I_k is a left (or right) predecessor of I_n if there is no $I_l, 0 \leq l < n$, in the gap (I_k, I_n) (or (I_n, I_k)), where (I_k, I_n) denotes the counter-clockwise gap from I_k to I_n. We denote them by $I_{L(n)}$ and $I_{R(n)}$.*

I_n has a successor I_{n+a} if

1. I_{n-a} is a left (or right) predecessor (with $0 < a \leq n$);

2. $f^a|_{[I_{n-a}, I_{n+a}]}$ (or $f^a|_{[I_{n+a}, I_{n-a}]}$) contains no predecessor of I_n;

3. if I_n is to the left (or right) of I_{n+a}, then there is no $I_k, 0 \leq k < n + a$ in the gap (I_n, I_{n+a}) (or (I_{n+a}, I_n)).

Furthermore we define the natural neighborhood T_n of I_n to be the biggest closed interval containing I_n which contains no $I_i, i \in N$, except its nearest predecessor or successor.

Remark: Of course I_n can have at most one predecessor on each side. Moreover I_n has at most one successor, denote it by $I_{S(n)}$. Therefore $T_n = [I_{L(n)}, I_{R(n)}]$ if I_n has two predecessors and no successor and $T_n = [I_{L(n)}, I_{S(n)}]$ (or $T_n = [I_{S(n)}, I_{R(n)}]$) if I_n has a successor.

One can prove the following lemmas ([6], p. 309).

Lemma 3 *For every $n \in N$, I_n can have at most one successor.*

Lemma 4 *Assume the interval I_n has two predecessors $I_{L(n)}, I_{R(n)}$ and a successor $I_{S(n)}$, If this successor is to the right of I_n then the predecessors of $I_{S(n)}$ are I_n and $I_{R(n)}$ and if $I_{S(n)}$ has a successor then this successor must be again to the right of $I_{S(n)}$.*

Remark: This lemma implies that if I_n has a successor $I_{S(n)}$ and $I_{S(n)}$ also has a successor $I_{S(S(n))}$ then $S(n) - n = S(S(n)) - S(n)$ and $I_{S(n)}$ is between I_n and $I_{S(S(n))}$. Continuing this if there exists a maximal integer k such that $I_{S^{i+1}(n)}$ is a successor of $I_{S^i(n)}$ for $0 \leq i \leq k-1$, then the intervals $I_{S^i(n)}$, $0 \leq i \leq k-1$, are ordered and f^a acts as a translation on these intervals, where $a = S(n) - n$.

Theorem 1 *([6], p. 310) Let $n \in N$ and assume that I_n has two predecessors $I_{L(n)}$ and $I_{R(n)}$. Let $M_n \supset I_n$ be an interval contained either in $[I_{L(n)}, I_n]$ or in $[I_n, I_{R(n)}]$. Assume that $\{M_{t_0}, M_{t_0+1}, ..., M_n\}$ are pullbacks of M_n. If the intersection multiplicity of this collection is at least $2m$ and $m \geq 2$ then there exists $t \in \{t_0, ..., n\}$ such that*

(1) $I_{S(t)}, I_{S^2(t)}, ..., I_{S^{2m-2}(t)}$ are defined;

(2) $n = S^m(t)$ and $I_{S^j(t)}$ is contained in M_n for $j = m, ..., 2m-2$.

Corollary 1 *Assume an interval $T \supset I_n$ and T is contained in the natural neighborhood T_n of I_n. Then the intersection multiplicity of the pullbacks of T is at most* 15.

Proof: Consider the pullbacks of $T \cap [I_{L(n)}, I_n]$ and $T \cap [I_n, I_{R(n)}]$ seperately. Suppose the intersection multiplicity of the pullbacks of T is at least 16. Then either the pullbacks of $T \cap [I_{L(n)}, I_n]$ or the pullbacks of $T \cap [I_n, I_{R(n)}]$ has intersection multiplicity ≥ 8. Take $m = 4$, the previous theorem imples that $I_{S^2(n)}$ is contained in $T \cap [I_n, I_{R(n)}]$. This is impossible because of $T \subset T_n$.

Now we can prove the following theorem.

Theorem 2 *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with an irrational rotation number. If the logarithm of the cross ratio distortion under f has bounded variation B, then f has no wandering interval, hence it is topologically conjugate to a rigid rotation.*

Proof: Suppose I is a maximal wandering interval for f and $I_n = f^n(I), n \geq 0, n \in Z$. There exists arbitrarily large $n \in N$ and $l, r < n$ such that $I_n \subset (I_l, I_r)$, $I_k \cap (I_l, I_r) = \emptyset, 0 \leq k < n$, and $|I_n| \leq \min\{|I_l|, |I_r|\}$. This property is proved as follows. Pick up I_l and I_r such that the gap (I_l, I_r) contains no I_k for $0 \leq k \leq \max\{l, r\}$. By the density of any orbit under an irrational rotation, there exists I_n first gets into the gap (I_l, I_r). If $|I_n| \leq \min\{|I_l|, |I_r|\}$ then it is done, otherwise replace I_r by I_n and go on. Since the sum of the lenths of I_k is bounded, eventually we will get $|I_n| \leq |I_r|$, furthermore we get $|I_n| \leq |I_l|$. We have seen I_l and I_r are two predecessors of I_n.

Let T_n be the natural neighborhood of I_n. If I_n has no successor, then $T_n = [I_l, I_r]$. By the corollary 1, the intersection multiplicity of the pullbacks of T_n is bounded by 15. Use the Macroscopic Koebe Distortion Principle, we get a bigger wandering interval J strickly containing I. This contradicts with the maximality of I. Hence I_n has a successor $I_{s(n)}$. Use the same way as the above, we get $|I_{s(n)}| < |I_n|$. Inductively we get infinitely many successors $I_{s^i(n)}, i = 1, 2, ...$, and by theorem 1, all successors are contained in $[I_l, I_r]$ and are ordered. Moreover $s^i(n) - s^{i-1}(n)$ is a constant $a = s(n) - n$. It follows $I_{s^i(n)}$ converges to a fixed point of f^a as $i \to \infty$. This contradicts with f has no periodic points.

Proof of Theorem C : Let I be a maximal wandering interval for f and $I_n = f^n(I), n \geq 0, n \in Z$. Let T_n be the natural neighborhood of I_n. The intersection multiplicity of the pullbacks of T_n is bounded by 15. By the Prop. 4 the cross ratio distortion of f^n on T_0 is uniformly bounded by a constant B. The rest of the proof follows the proof of the above theorem.

The remainder of this section explains why the conditions of the theorem C is weaker than Denjoy's condition and [5]'s condition. It is almost trivial that Denjoy's condition implies the conditions of the theorem C.

Prop. 10 *Let $h : I \to R^1$ be a C^1 smooth function and $logh'$ is of bounded variation, then $logh'$ is of bounded Zygmund variation and bounded quadratic variation.*

Proof: By the triangle inequality, the Zygmund variation of $logh'$ is no more than the variation of $logh'$ on the interval I. Let M be the maximal value of $|logh'|$ on the interval I, then the quadratic variation of $logh'$ is no more than $2M$ multiplied by the variation of $logh'$ on the interval I.

Clearly the Zygmund condition implies bounded Zygmund variation. Furthermore, the Zygmund condition implies α-Hölder continuous for $0 < \alpha < 1$. The 1/2-Hölder continuity implies the bounded quadratic variation.

Lemma 5 *If $\phi : I \to R^1$ satisfies the Zygmund condition: there exists $B > 0$ such that*

$$\sup_{x,t} |\frac{\phi(x+t) + \phi(x-t) - 2\phi(x)}{t}| \leq B,$$

then ϕ is α-hölder continuous for any $0 < \alpha < 1$.

Proof: Denote $D(x,t) = \frac{\phi(x+t)-\phi(x)}{t}$. Then

$$D(x,t/2) + D(x+t/2,t/2) = 2D(x,t), \ |D(x,t/2) - D(x+t/2,t/2)| \leq B,$$

$$D(x,t/4) + D(x+t/4,t/4) = 2D(x,t/2), \ |D(x,t/4) - D(x+t/4,t/4)| \leq B,$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$D(x,\frac{t}{2^n}) + D(x+\frac{t}{2^n},\frac{t}{2^n}) = 2D(x,\frac{t}{2^{n-1}}), \ |D(x,\frac{t}{2^n}) - D(x+\frac{t}{2^n},\frac{t}{2^n})| \leq B.$$

These give us

$$|D(x,t/2^n)| \leq |D(x,t)| + nB,$$

i. e.,

$$|\frac{\phi(x+t/2^n) - \phi(x)}{t/2^n}| \leq |D(x,t)| + nB.$$

Then

$$|\frac{\phi(x+t/2^n) - \phi(x)}{(t/2^n)^\alpha}| \leq (|D(x,t)|/n + B)n(|t|/2^n)^{1-\alpha},$$

which tells us that ϕ is α-Hölder continuous for any $0 < \alpha < 1$.

Prop. 11 *If $\phi : I \to R^1$ satisfies the Zygmund condition, then ϕ is of bounded Zygmund variation and bounded quadratic variation over the interval I.*

5 Three examples

In the introduction it is mentioned that there exists an example satisfying Denjoy's bounded variation condition but [5]'s Zygmund condition and vice versa. In this section, we will give these two examples and also we will give an example to show that there is an example being of bounded quadratic variation but not being of bounded Zygmund variation.

Example 1 *Let $\phi : [-1,\ 1] \to [-1,\ 1]$ be the following function*

$$\phi(x) = x, \quad x \in [-1,\ 0],$$

$$\phi(x) = \sqrt{x}, \quad x \in (0,\ 1].$$

Clearly ϕ is monotone hence it is of bounded variation, but the Zygmund condition fails since the right derivative of ϕ at the point 0 is infinite but the left derivative is 1.

Example 2 *Let $\phi_0(x) = 2x$ for $x \in [0, \frac{1}{2}]$ and $\phi_0(x) = 2 - 2x$ for $x \in [\frac{1}{2}, 1]$. And let*

$$\phi_n = \frac{\phi(2^n x - i)}{2^n} \ for\ x \in [\frac{i}{2^n}, \frac{i+1}{2^n}],$$

where $i = 0, 1, ..., 2^n - 1$.

Let $\phi(x) = \sum_{n=0}^{\infty} \phi_n(x)$. ϕ is differentiable at a set of measure 0 only. It can't be of bounded variation, otherwise it is differentiable almost everywhere which is a contradiction. [7] and [8] study the general theory about the differentiability of a function satisfying Zygmund condition.

Example 3 *Let $\phi : [0,\ 1] \to [0,\ 1]$ is defined by the figure 1. It is easy to get the quadratic variation of ϕ on $[0,\ 1]$ is equal to $\sum_{n=1}^{\infty} \frac{2}{n^2}$ which is finite. But the difference between the left derivative and the right derivative of ϕ at $1/2^n$ is equal to*

$$\frac{2^{n+2}}{n+1} - \frac{2^{n+1}}{n} = \frac{2^{n+1}}{n} \frac{n-1}{n+1}$$

which tends to ∞ as $n \to \infty$. Therefore it has bounded quadratic variation but has no bounded Zygmund variation.

Actually the left question is to study whether or not the bounded Zygmund variation property implies the bounded quadratic variation property.

6 Appendix

The nonexistence of wandering domains for any rational map of the complex sphere was proved by Dennis Sullivan in 1985 [9]. The analogue of this theorem for one-dimensional dynamical systems was done for certain smooth multimodal maps by Martens, de Melo and van Strien in 1992 [10]. In the latest publication [6], the smooth condition used by de Melo

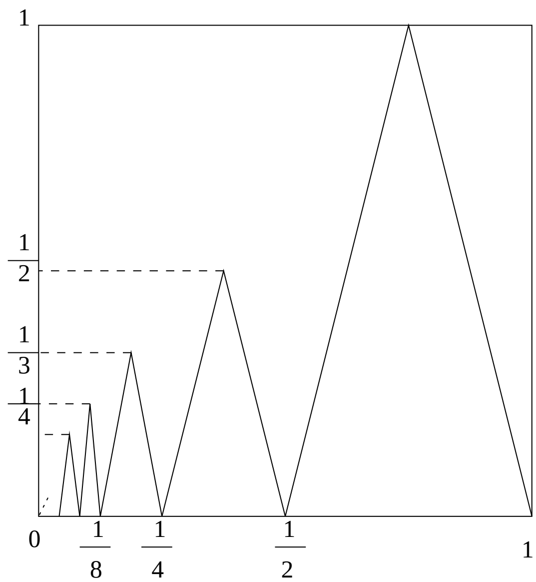

Figure 1.

and van Strien is that a multimodal map piecewise satisfies $C^{1+b.v}$ (or C^{1+Z}) and the map can be written as a power map ($x \mapsto |x|^\alpha, \alpha > 1$) composed by a $C^{1+b.v}$ (or C^{1+Z}) diffeomorphism around every turning point. Combine the analysis work of getting a bound of cross ratio distortions in this paper and the combinatorial machinery on wandering intervals in [10] or [6, p. 308-312], we can get a weak version of Martens, de Melo and van Strien's theorem of no wandering intervals for multimaodal maps. Before we state the theorem, let us give the definition of a wandering interval for a multimodal map of an interval.

Definition 7 *Let $f : I \to I$ be a continuous map of an interval I. An open interval $J \subset I$ is called a wandering interval of f if*

1) $f^n(J) \cap f^m(J) = \emptyset$ for any $n \neq m, n, m \in N$;

2) $f^n(J)$ does not converge to a periodic orbit.

Theorem 3 [11] *Let $f : I \to I$ be a C^1 smooth map satisfying*

1) f is $C^{1+b.Z.v+b.q.v}$ away from critical points;

2) Let K_f be the set of critical points of f. For each $x_0 \in K_f$, there exist $\alpha > 1$, a neighborhood $U(x_0)$ of x_0 and a $C^{1+b.Z.v+b.q.v}$ diffeomorphism $\phi : U(X_0) \to (-1, 1)$ such that $\phi(x_0) = 0$ and

$$f(x) = f(x_0) \pm |\phi(x)|^\alpha, \forall x \in U(x_0).$$

Then f has no wandering intervals.

Norton, Sullivan and Velling [12, 13 and 14] have begun the work of generalizing the setting of Denjoy's theorem to two dimensional dynamical systems by considering diffeomorphisms of the torus. The quasiconformal theory has found a place there.

Acknowledgements. Both authors wish to thank Frederick P. Gardiner for his help in writing, and thank Yunping Jiang and Meiyu Su for some discussions. We are grateful to John Milnor for his suggestions to improve the writing and especially for his simple setting of the proof of Prop. 2.

References

[1] A. Denjoy. Sur les courbes definies par les equations differentielles a la surface du tore. *J. de Math. Pure et Appl.* **11**, 1992, 333-375.

[2] W. Szlenk. *An Introduction to the theory of Smooth Dynamical Systems*, PWN-Polish Scientific Publishers. Warszawa, 1984.

[3] Z. Nitecki. *Differentiable Dynamics* . MIT Press, Cambridge, MA, 1971.

[4] Z. Zhang. *An Introduction to Differentiable Dynamical Systems* . Chinese Scientific Press, Beijing, 1987.

[5] D. Sullivan. Bounds, quadratical differentials and renormalization conjectures. *Mathematics into Twenty-first Century,* **Vol.2**. Providence, RI, Amer. Math. Soc., 1992.

[6] W. de Melo and S. J. van Strien. *One dimensional dynamics*. Springer-Verlag, 1993.

[7] E. Stein and A. Zygmund. On the differentiability of functions. *Studia Mathematica*, T. XXIII, 1964, 247-283.

[8] A. Zygmund. *Trigonometric Series* . 2nd ed., 2 vols. Cambridge, 1959.

[9] D. Sullivan. Quasiconformal homeomorphisms and dynamics I: a solution of Fatou-Julia problem on wandering domains. *Ann. Math.*, 122, 1985, 401-418.

[10] M. Martens, W. de Melo and S. van Strien. Julia-Fatou-Sullivan theory for real one-dimensional dynamics. *Acta Math.*, 1992, 273-318.

[11] J. Hu. One property for the renormalization on multimodal maps. Preprint, 1993.

[12] A. Norton. Minimal sets, wandering domains, and rigidity in 2-torus. *Amer. Math. Soc., Contemporary Math.*, no.117, 1991, 129-138.

[13] A. Norton and J. A. Velling. Conformal irregularity for Denjoy diffeomorphisms of the 2-torus. *Rocky Mountain J. Math.* (to appear).

[14] A. Norton and D. Sullivan. Wandering domains and invariant conformal structures for mappings of the 2-torus. Preprint, 1993.

[15] G. R. Hall. A C^∞ counterexample. *Erg. Th. and Dyn. Sys.* **1**, 1983, 261-272.