DYNAMICS OF GEOMETRICALLY FINITE RATIONAL MAPS

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ABSTRACT. Geometrically finite branched covering maps of the sphere are studied. We prove that a geometrically finite branched covering map whose post-critical set is infinite is combinatorially equivalent to a rational map if and only if it is combinatorially linearizable and there is no Thurston obstruction for it. For a topological conjugacy between two geometrically finite rational maps whose post-critical sets are infinite, we prove that there is a quasiconformal conjugacy isotopic to it rel the post-critical set. Moreover, if the conjugacy is holomorphic on the Fatou set, then it is a Möbius transformation. We also show that every wandering Julia component is a simple closed curve for geometrically finite rational maps.

§1. INTRODUCTION

Let f: $S^2 \to S^2$ be a branched covering map of the sphere S^2 with degree bigger than one. Throughout this paper, we always suppose degree bigger than one when we say a branched covering map of the sphere or a rational map of the Riemann sphere $\hat{\mathbb{C}}$. We call

$$\Omega(f) = \{x : \deg_x f > 1\}$$

the *critical set* of f, and

$$P(f) = \bigcup_{n>0} f^n(\Omega(f))$$

the *post-critical set* of f. Note that $f(P(f)) \subset P(f)$ and $P(f) = P(f^n)$.

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The map f is called *critically finite* if $P(f)$ is finite, *geometrically finite* if $P(f)$ has only finitely many accumulation points. In the later case, every accumulation point is periodic.

Two branched covering maps $f, g : S^2 \to S^2$ are called *combinatorially equivalent* if there exist homeomorphisms $\phi, \psi : S^2 \to S^2$ such that ϕ is isotopic to ψ rel $P(f)$ and $\phi f = g\psi$.

A simple closed curve on $S^2 - \overline{P(f)}$ is *essential* if it does not bound a disk in $S^2 - \overline{P(f)}$, *peripheral* if it encloses a single point of $P(f)$. A *multicurve* $\Gamma = \{\gamma_i\}$ on $S^2 - \overline{P(f)}$ is a finite nonempty collection of disjoint simple closed curves, each essential and non-peripheral, and no two isotopic rel $P(f)$.

A multicurve determines a transition matrix $A(\Gamma) : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ by the formula

$$A_{\delta\gamma} = \sum_{\alpha} \frac{1}{\deg(f : \alpha \to \gamma)}$$

where the sum is taken over all components α of $f^{-1}(\gamma)$ which are isotopic to δ rel $P(f)$. Let $\lambda(\Gamma) \geq 0$ denote the spectral radius of $A(\Gamma)$. Since $A(\Gamma) \geq 0$, the Perron-Frobenius Theorem guarantees that $\lambda(\Gamma)$ is an eigenvalue of $A(\Gamma)$ with a non-negative eigenvector (see [LT]). A multicurve Γ is called a *Thurston obstruction* if $\lambda(\Gamma) \geq 1$.

Suppose f is a critically finite branched covering map of the sphere S^2. For every point x in S^2, define $v_f(x)$ (which may be ∞) as the least common multiple of the local degrees $\deg_y f^n$ for all $n > 0$ and all $y \in S^2$ such that $f^n(y) = x$. (Note that $v_f(x) = 1$ if x is not in $P(f)$.) Then

$$\mathcal{O}_f = (S^2, v_f)$$

is the orbifold of f and

$$v_f : P(f) \to \mathbb{N} \cup \{\infty\}$$

is the signature of $\mathcal{O}_f$. A well-known theorem of Thurston says that when the signature of $\mathcal{O}_f$ is not $(2, 2, 2, 2)$, then f is combinatorially equivalent to a rational map if and only if there is no Thurston obstruction. And moreover, the rational map is unique up to holomorphic conjugations (see [DH, Mc1]).

It is proposed to extend Thurston's Theorem to geometrically finite branched covering maps and to show that every component of the Julia set of a geometrically finite rational map is locally connected (see [Bi]). It is known that for a geometrically finite rational map f whose post-critical set is infinite, there is no Thurston obstruction for it [Mc1] and that all eventually periodic components of its Julia set are locally connected [TY].

Suppose that f is a geometrically finite branched covering map of the sphere S^2 and a is an accumulation point of $P(f)$ with period $p \geq 1$. We say f is *combinatorially linearizable* at a if there exist homeomorphisms $\phi, \psi : S^2 \to \hat{\mathbb{C}}$ such that $\phi(a) = 0$, ϕ is isotopic to ψ rel $P(f)$, $\phi f^p \psi^{-1}(z) = \lambda z$ for some $0 < \lambda < 1$ if $\deg_a f^p = 1$ and $\phi f^p \psi^{-1}(z) = z^d$ if $\deg_a f^p = d > 1$ on a neighborhood of the origin. If f is combinatorially linearizable at every accumulation point of $P(f)$, we say f is *combinatorially linearizable.* It is clear that the condition, combinatorially linearizable, is invariant under combinatorial equivalence. Our main result is that

Theorem A. *Let f be a geometrically finite branched covering map of the sphere S^2 whose post-critical set $P(f)$ is infinite. Then f is combinatorially equivalent to a rational map if and only if f is combinatorially linearizable and there is no Thurston obstruction.*

In the theorem the local condition, combinatorial linearizable, is non-trivial. In §3 we will construct an example showing that there is a geometrically finite branched covering map which has no Thurston obstruction but is not combinatorially linearizable.

One of the main ideas in the paper is so called *pullback argument.* It first appeared in Thurston's algorithm for the finding of a rational map in a combinatorial equivalence class of critically finite branched covering maps of the sphere (see [DH]). It is also developed in the Sullivan's study of the conjugacy theorem for hyperbolic rational maps (see [Sul]). In the later context, the idea can be explained as follows: given a hyperbolic rational map and a new local complex structure on a small domain about every accumulation point of the post-critical set of the rational map such that the new local complex structure is compatible with the map, there is a unique global complex structure on the sphere determined by pulling back the the new local complex structures to the Fatou set of the rational map. This global complex structure is compatible with the map and produces a quasiconformal conjugacy. By combining the analysis near a parabolic periodic point of a rational map with the theory of extremal quasiconformal maps (see pp.57, [Mc2]), we first extend Sullivan's Conjugacy Theorem to geometrically finite rational maps in §2.

A homeomorphism (quasiconformal or holomorphic homeomorphism) ϕ of $\hat{\mathbb{C}}$ is called a *topological (quasiconformal, holomorphic) conjugacy* between two rational maps f and g if $\phi f = g\phi$.

Theorem B. *Let ϕ be a topological conjugacy between two geometrically finite rational maps f and g whose post-critical sets are infinite. Then there is a quasiconformal conjugacy between f and g isotopic to ϕ rel $P(f)$. Moreover, if ϕ is holomorphic on the Fatou set, then it is a holomorphic conjugacy.*

Another main idea in the paper is *pullback partition.* For a geometrically finite rational map f, by considering its combinatorics, we can construct a partition for the dynamical system generated by f as follows: take a small linearization disk in each periodic Fatou domain of f and let U be their union. The geometric finiteness guarantees that there is an integer $N \geq 1$ such that every component of $f^{-n}(U)$ is *parallel* (see §4 for the definition) to a component of $f^{-m}(U)$ for all $n, m \geq N$. The complement of U in $\hat{\mathbb{C}}$ also satisfies this property. The domains of $f^{-N}(U)$ and the continua $\hat{\mathbb{C}} - f^{-N}(U)$ form a partition of $\hat{\mathbb{C}}$. We call this partition a *pullback partition* for f.

Now suppose f is a geometrically finite branched covering map of S^2 whose post-critical set is infinite. If f is combinatorially linearizable, we can construct a similar *pullback partition* by taking a small domain at every accumulation point of $P(f)$. This partition satisfies the property in the previous paragraph. First, there exists a complex structure on the small domains induced from the combinatorial linearization. Therefore, there is a complex structure on the domains of the partition by pullback. Second, for a *strictly-essential* (see §4 for the definition) continuum which is parallel to a component of its preimages under some f^n. The map f^n restricted to this component can be extended to a critically finite branched covering map of S^2. We then apply Thurston's Theorem to get a complex structure on this continuum. The condition of no Thurston obstruction guarantees that we can glue those complex structures together to get a global complex structure on the sphere S^2. In §4 and §5, we will explore this in more details.

The pullback partition also reveals the topological structure of the Julia set of a geometrically finite rational map. A Julia component K of a rational map f is a connected component of the Julia set of f which contains more than one points. A Julia component K is *eventually periodic* if there exist $k \geq 0$ and $p > 0$ such that $f^{k+p}(K) = f^k(K)$ and *periodic* if $k = 0$, and *wandering* if $f^n(K) \cap f^m(K) = \emptyset$ for all $n \neq m$. In §6, we prove the following result.

Theorem C. *For a geometrically finite rational map f, every wandering Julia component of f is a simple closed curve.*

Combining the result in [TY], we have

Corollary. *Every Julia component of a geometrically finite rational map is locally connected.*

Acknowledgement. In the paper, Theorem B is the motivation, Theorem A is the main result, and Theorem C is a consequence of the method. Theorem B is first appeared in the paper [CY] of Cui and Yin. The proof here is simpler. Theorem C is first announced and proved later in [PT] by Pilgrim and Tan for hyperbolic rational maps with disconnected Julia set. The method in the paper gives an independent proof. The first author (Cui) would like to thank Yin for suggesting a problem which leads to Theorem B and for allowing us to have the statement of Theorem B in the paper. The first and second authors (Cui and Jiang) would like to thank Tan for suggestive conversations and Pilgrim for comments and for both of them to allow us to include their statement in Theorem C in this paper. There is another relavent study by Epstein in [Ep] and by Epstein-Keen-Tresser [EKT] on dynamics of finite type complex analytic maps, in which many interesting results on the dynamics of a complex analytic map at its parabolic periodic points are obtained. The first author (Cui) also would like to thank the Einstein Chair of Sciences in the Graduate Center of CUNY for its support and hospitality during his visit. The second author (Jiang) also would like to thank The Nonlinear Centre at The University of Cambridge for its support and hospitality during his visit.

§2. Quasiconformal Conjugacy between Goemetrically Finite Rational Maps

Let f be a rational map of $\hat{\mathbb{C}}$. Denote by $F(f)$, $J(f)$ the Fatou and Julia sets of f respectively. Suppose that z_0 is a fixed point of f and $\lambda = f'(z_0)$ is the multiplier of f at z_0. We call z_0 a super-attracting, attracting, repelling, rationally neutral (also call parabolic), or irrationally neutral fixed point of f if $\lambda = 0$, $0 < |\lambda| < 1$, $|\lambda| > 1$, $|\lambda| = 1$ and $\lambda^q = 1$ for some integer $q \geq 1$, or $|\lambda| = 1$ but λ^q is never 1 for all integers $q \geq 1$. We first list some classical linearization theorems. One can refer to some standard book for their proofs (e.g., [CG], [Bl], and [Mi]).

Koenigs Theorem. *If z_0 is an attracting or repelling fixed point of f, then there is a neighbourhood D of z_0 and a conformal map $h : D \to \Delta_r = \{z \in \mathbb{C}, |z| < r\}$ such that $h(z_0) = 0$ and $hfh^{-1}(z) = \lambda z$.*

Boettcher Theorem. *If z_0 is a super-attracting fixed point of f, then there is a neighbourhood D of z_0 and a conformal map $h : D \to \Delta_r$ $(r < 1)$ such that $h(z_0) = 0$ and $hfh^{-1}(z) = z^d$, where $d = \deg_{z_0} f > 1$.*

Note that the domain D in each of the above two theorems is contained in the Fatou set $F(f)$. We call D a *linearization disk* at z_0. We now list a theorem about a parabolic fixed point (we also call it a linearization theorem).

Fatou Flower Theorem. *If z_0 is a parabolic fixed point of f such that $\lambda^q = 1$ for some integer $q \geq 1$ and $\lambda^m \neq 1$ for all $0 < m < q$. Then there is an integer $k > 0$ and kq analytic curves which pairwise tangent at z_0 and which bounded petals V_i $(1 \leq i \leq kq)$ such that f is injective on $V = \cup_{i=1}^{kq} V_i$, $\overline{f(V)} \subset V \cup \{z_0\}$ and $f^n(z)$ converges to z_0 as $n \to \infty$ uniformly for z in any compact set in V.*

We say V_i an *attractive petal* of f at z_0 and V an *attractive flower* of f at z_0. For each attractive petal V_i, $\overline{f^q(V_i)} \subset V_i \cup \{z_0\}$. So $V_i \subset F(f)$ and there is a conformal map h from V_i into $\hat{\mathbb{C}}$ such that $hfh^{-1}(z) = z + 1$. For this reason, we also call V_i a *linearization disk* of $F(f)$.

When considering f^{-1} on a neighborhood of z_0, we see that there are also kq analytic curves which are pairwise tangent at z_0 and which bounded petals V_i' $(1 \leq i \leq kq)$ such that f is injective on $V' = \cup_{i=1}^{kq} V_i'$, $f(V') \cup \{z_0\} \supset \overline{V'}$ and f^q

is holomorphically conjugated to $z \to z+1$ on each V_i'. We call V_i' an *expansive petal* of f at z_0 and V' an *expansive flower* of f at z_0. Furthermore, there are $2kq$ analytic curves which are pairwise tangent at z_0 and which bounded petals V_i'' at z_0 such that f is injective on $V'' = \cup_{i=1}^{2kq} V_i''$, $f(V'') = V''$ and f^q is holomorphically conjugated to $z \to z+1$ on each V_i''. We call V_i'' a *parabolic petal* of f at z_0 and V'' a *parabolic flower* of f at z_0 (see Fig. 1).

It is clear that the union of any attractive flower and any expansive flower forms a neighborhood of z_0 and every attractive (or expansive) petal must intersect with two parabolic petals (see Fig. 1).

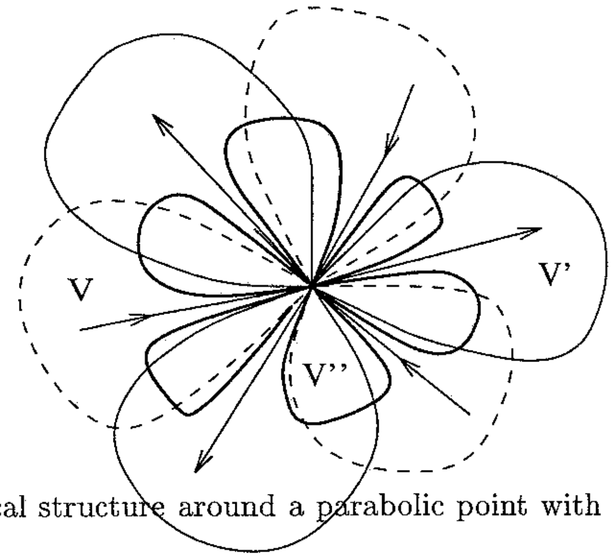

Fig. 1: Topological structure around a parabolic point with three attracting petals.

For a point z in $\hat{\mathbb{C}}$, let $\omega(z)$ mean set of all accumulation points of the forward orbit $\{f^n(z)\}_{n=1}^{\infty}$ of z. From the above discussion, if z_0 is a parabolic fixed point of f and if $\omega(z) = z_0$ for $z \in \hat{\mathbb{C}}$, then either $f^n(z) = z_0$ for some integer $n \geq 0$ or there is an integer $N \geq 0$ such that $f^n(z) \in V$ for all $n > N$ since $V \cup V'$ is a neighborhood about z_0.

In the literature, the geometrical finiteness of a rational map is usually defined as $J(f) \cap \overline{P(f)}$ is finite. The next proposition shows that our definition is equivalent to the usual definition for rational maps. To do this, we first introduce a theorem of Mañé [Ma].

A critical point c of f is called *recurrent* if $c \in \omega(c)$. A compact forward invariant set $\Lambda \subset J(f)$ is called *expanding* if there exist constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(z)| \geq C\lambda^n$ for all $z \in \Lambda$ and $n \geq 1$.

Mañé's Theorem. *Let f be a rational map and $\Lambda \subset J(f)$ a compact forward invariant set containing neither critical points nor parabolic periodic points from*

f. Then either Λ is expanding or $\Lambda \cap \omega(c) \neq \emptyset$ for some recurrent critical point c of f.

Proposition 2.1. *If g is a geometrically finite rational map, then $J(f) \cap \overline{P(f)}$ is finite.*

Proof. We only need to prove that each critical point in the Julia set $J(f)$ is eventually periodic. First, if $c \in \omega(c)$ is a recurrent critical point in $J(f)$, then c is periodic since each accumulation point of $P(f)$ is periodic. But this can not happen, so there is no recurrent critical point in $J(f)$. By Mañé's Theorem and this argument, every periodic point in $J(f)$ is either repelling or parabolic. If $c \in J(f)$ is a critical point, then there exists $p \geq 1$ such that $f^{np}(c)$ converges to a periodic point $a \in J(f)$ as n goes to ∞. By the linearization theorems above, c has to be eventually periodic. □

Before the proof of theorem B, we give two lemmas about local quasiconformal conjugacies. The next lemma appeared in [Sul] and we provide a proof here for the completeness. For a rational map f, denote

$$Q(f) = \overline{\bigcup_{n>0} f^{-n}(P(f))}.$$

Lemma 2.2. *Suppose ϕ is a topological conjugacy between rational maps f and g, and W is an attracting, super-attracting or parabolic fixed Fatou domain of f. Then there is an isotopy on W rel $W \cap Q(f)$*

$$\Phi : I \times W \to \phi(W), \qquad I = [0,1],$$

such that $\Phi(t,\cdot)f = g\Phi(t,\cdot)$ for all $t \in I$, $\Phi(0,\cdot) = \phi|_W$ and $\Phi(1,\cdot)$ is quasiconformal.

Proof. If W is an attracting or parabolic fixed Fatou domain of f. Define an equivalent relation on $W - Q(f)$ by $z_1 \sim z_2$ if there exist $n_1, n_2 \geq 0$ such that $f^{n_1}(z_1) = f^{n_2}(z_2)$. With the induced complex structure, the quotient space $R_f(W) = (W - Q(f))/\sim$ is a torus with finitely many but at least one punctures when W is attracting or the sphere with finitely many but at least three punctures when W is parabolic.

The conjugacy ϕ induces a homeomorphism $\tilde{\phi} : R_f(W) \to R_g(\phi(W))$. Since there are only finitely many punctures for $R_f(W)$, there is an isotopy $\tilde{\Phi} : I \times R_f(W) \to R_g(\phi(W))$ such that $\tilde{\Phi}(0,\cdot) = \tilde{\phi}$ and $\tilde{\Phi}(1,\cdot)$ is quasiconformal. Let Φ be the lift of $\tilde{\Phi}$ so that $\Phi(0,\cdot) = \phi$, then it satisfies the conditions in the lemma.

Now let us consider a super-attracting fixed Fatou domain W of f. Define an equivalent relation on W by $z_1 \sim z_2$ if there exist $n_1, n_2 \geq 0$ such that $f^{n_1}(z_1) = f^{n_2}(z_2)$. The closures of equivalent classes give a singular foliation on W. The singularities of this foliation occur at all the critical points and their preimages in W.

Let $D \subset W$ be a linearization disk of f. Then ∂D is a leaf. Suppose $L_1, \cdots, L_n$ are all the leaves in $D - \overline{f(D)}$ which intersect with $Q(f)$. Denote by A_i $(0 \le i \le n)$ the components of $D - \overline{f(D)} - \cup_{i=1}^n L_i$. They are annuli. For each i, The annuli A_i and $\phi(A_i)$ can be realized as the strip $B_1 = \{z : 0 < \Im z < r\}$ and $B_2 = \{z : 0 < \Im z < R\}$ factored by $< z \mapsto z+1 >$, respectively. Let $\tilde{\phi}(z) = u(z) + iv(z)$ be a lift of ϕ. Then $\tilde{\phi}(z+x) = \tilde{\phi}(z) + x$ for all $x \in \mathbb{R}$. Define

$$\begin{cases} u(t,y) = u(y)(1-t) + tRy/r \\ v(t,y) = v(y)(1-t) + t\{v(0) + [v(ir) - v(0)]y/r\}. \end{cases}$$

Then $\tilde{\Phi}(t, x+iy) = u(t,y) + iv(t,y) + x$ is an isotopy from B_1 to B_2 modulo the boundary and $\tilde{\Phi}(1,\cdot)$ is quasiconformal.

Let Φ be the projection of $\tilde{\Phi}$. define Φ on other annulus as above and define Φ on W so that $\Phi(t,\cdot)f = g\Phi(t,\cdot)$ for all t. It is easy to verify that Φ satisfies the conditions of the lemma. □

Lemma 2.3. *Suppose V is an attractive flower of a rational map f at a parabolic fixed point z_0, ϕ is a local quasiconformal conjugacy between rational maps f and g defined on the parabolic periodic Fatou domains W of f associated to z_0. Then for any $\epsilon > 0$, there is a domain $G \supset \overline{V}$ and a quasiconformal map φ defined on G such that $\varphi|_V = \phi|_V$ and $K(\varphi) < K(\phi) + \epsilon$, where $K(\cdot)$ means the maximal dilatation of a map on its definition domain.*

Proof. We only give the proof for the case that V contains only one petal. In general, the proof is similar. Let V'_f and V''_f be an expansive petal and a parabolic flower of f at z_0, respectively. Define an equivalent relation on V'_f by $z_1 \sim z_2$ if there exist $n_1, n_2 \ge 0$ such that $f^{n_1}(z_1) = f^{n_2}(z_2)$. With the induced complex structures, the quotient space $V'_f/\sim = \mathbb{C} - \{0\}$. Denote by $\pi_f : V'_f \to \mathbb{C} - \{0\}$ the projection. Then $\pi_f(V''_f \cap V'_f)$ is the union of two Jordan domains around 0 and the infinity whose closures are disjoint.

Note that $V''_f \subset W$. So $\phi(V''_f)$ is a parabolic flower of g at $\phi(z_0)$. Let V'_g be an expansive petal of g at $\phi(z_0)$. Using the same discussion, we have a projection $\pi_g : V'_g \to \mathbb{C} - \{0\}$.

The local quasiconformal conjugacy ϕ induces a quasiconformal map $\tilde{\phi}$ from $\pi_f(V''_f \cap V'_f)$ into $\mathbb{C} - \{0\}$. It is clear that $\tilde{\phi}$ can be extended to be a homeomorphism $\tilde{\psi}$ from $\mathbb{C} - \{0\}$ to itself, such that there is a local homeomorphism ψ defined on a smaller expansive petal U'_f of f at z_0 such that $\tilde{\psi}\pi_f = \pi_g\psi$ and $\psi|_{U'_f \cap V''_f} = \phi|_{U'_f \cap V''_f}$. Furthermore, by Lemma A.3 (see Appendix), for any $\epsilon > 0$, We can pick a quasiconformal homeomorphism $\tilde{\varphi}$ such that $\tilde{\varphi} = \tilde{\psi}$ on some smaller Jordan domains $A \cup B$ around 0 and the infinity and $K(\tilde{\psi}) < K(\phi) + \epsilon$, where $A \cup B$ are the projection of the intersection of some smaller expansive petal and parabolic petal of f at z_0 under π_f.

Denote by U_f'' the parabolic flower of f at z_0 generated by $\pi^{-1}(A \cup B)$. Then there is a smaller expansive petal (we still denote it by U_f') of f at z_0 and a local quasiconformal map φ defined on U_f' such that $U_f' \cap f^{-1}(V) \subset U_f' \cap U_f''$, $\tilde{\varphi}\pi_f = \pi_g \varphi$ and $\varphi|_{U_f' \cap U_f''} = \phi|_{U_f' \cap U_f''}$. Define $\varphi|_{f^{-1}(V)} = \phi|_{f^{-1}(V)}$. The ψ is well-defined and this completes the proof. □

Theorem B. *Let ϕ be a topological conjugacy between two geometrically finite rational maps f and g whose post-critical sets are infinite. Then there is a quasiconformal conjugacy between f and g isotopic to ϕ rel $P(f)$. Moreover, if ϕ is holomorphic on the Fatou set, then it is a holomorphic conjugacy.*

Proof. From Sullivan's Classification Theorem and the fact that the boundaries of Siegel disks and Herman rings are contained in the closure of the post-critical set, we see that there are only three kinds of periodic Fatou domains for f: attracting, super-attracting and parabolic. Note that $F(f) \neq \emptyset$.

Take a linearization disk in every periodic Fatou domain and denote by U their union. Then $\hat{\mathbb{C}} - \overline{U}$ is connected, $\overline{U}$ is contained in the union of $f^{-1}(U)$ with the set of parabolic periodic points and $P(f) - U$ is a finite set. Let Φ be the local isotopy constructed in Lemma 2.2 and denote $\phi_0 = \Phi(1, \cdot)$. By Lemma 2.3, there is a global isotopy $\Psi(t, z)$ rel $P(f)$, $\Psi(t, z) : I \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, such that $\Psi(0, \cdot) = \phi$, $\psi = \Psi(1, \cdot)$ is quasiconformal and $\Psi(t, \cdot)|_U = \Phi(t, \cdot)|_U$ for all $t \in I$. Furthermore we let $\psi|_{\hat{\mathbb{C}} - \overline{U} - P(f)}$ be extremal quasiconformal modulo the boundary.

Let Ψ_1 be the lift of Ψ so that $\Psi_1(0, \cdot) = \phi$. Then $\Psi_1(t, \cdot)|_U = \Phi(t, \cdot)|_U$ for all $t \in I$ and $K(\psi_1) = K(\psi)$, where $\psi_1 = \Psi_1(1, \cdot)$. We claim that $K(\psi) = K(\phi_0)$. Otherwise, by Theorem A.1 (see Appendix) and Lemma 2.3, $\psi|_{\hat{\mathbb{C}} - \overline{U} - P(f)}$ is a Teichmüller map and hence unique extremal. But $\psi_1|_{f^{-1}(U) - U} \neq \psi|_{f^{-1}(U) - U}$ since $\psi_1|_{f^{-1}(U)} = \phi_0|_{f^{-1}(U)}$. It is a contradiction.

Inductively, let Ψ_n be the lift of Ψ_{n-1} so that $\Psi(0, \cdot) = \phi$. Then $\Psi_n(t, \cdot)|_{f^{-n+1}(U)} = \Psi_{n-1}(t, \cdot)|_{f^{-n+1}(U)}$, $\psi_n = \Psi_n(1, \cdot)$ is quasiconformal and $K(\psi_n) = K(\phi_0)$. So $\{\psi_n\}$ is a normal family. It is easy to check that ψ_n converges to a quasiconformal map φ of $\hat{\mathbb{C}}$, $\varphi f = g \varphi$ and φ is isotopic to ϕ rel $P(f)$.

If ϕ is holomorphic on $F(f)$, let $\Phi(t, z) = \phi(z)$ on U, then $K(\phi_0) = 1$. So $K(\varphi) = 1$ and $\varphi = \phi$ is a Möbius transformation. □

§3. Combinatorial Linearization for Geometrically Finite Branched Covering Maps

Suppose that $f : S^2 \to S^2$ is a geometrically finite branched covering map and that a is an accumulation point of $P(f)$. Then a is a periodic point of f of period

$p \geq 1$. We say $\Gamma = \{\gamma_n\}_{n=0}^{\infty}$ is a *curve family nested at* a if $\gamma_n \subset S^2 - \overline{P(f)}$ are pairwise disjoint simple closed curves and γ_{n+1} separates γ_n from a. It is f*-invariant* if there is a component of $f^{-p}(\gamma_{n+1})$ isotopic to γ_n rel $P(f)$ for all $n \geq 0$, and *shrinking* if for any neighborhood U of a, there is $N \geq 0$ and a simple closed curve $\beta \subset U - \overline{P(f)}$ such that β is isotopic to γ_N rel $P(f)$.

Proposition 3.1. *Suppose that a is an accumulation point of $P(f)$ with period $p \geq 1$ and $\deg_a f^p = 1$. Then f is combinatorially linearizable at a if and only if there is a shrinking f-invariant curve family nested at a.*

Proof. Suppose f is combinatorially linearizable at a, i.e., there exist homeomorphisms $\phi, \psi : S^2 \to \hat{\mathbb{C}}$ such that $\phi(a) = 0$, ϕ is isotopic to ψ rel $P(f)$ and $g(z) = \phi f^p \psi^{-1}(z) = \lambda z$ for some $0 < \lambda < 1$ on a neighborhood U of the origin.

For any $0 < r < 1$, let $\Delta_r = \{z \in \mathbb{C}; |z| < r\}$ and $C_r = \partial \Delta_r$. There is a $0 < r < 1$ such that $\Delta_r \subset U$, such that $g^n(C_r) \cap \overline{\phi(P(f))} = \emptyset$ for all $n \geq 0$. Denote $\gamma_n = \phi^{-1} g^n(C_r)$, then $\{\gamma_n\}$ is a shrinking f-invariant curve family nested at a.

Conversely, we may choose an f-invariant curve family $\{\gamma_n\}$ nested at a such that γ_n converges to a as $n \to \infty$. Let A_n be the annulus enclosed by γ_n and γ_{n+1} and let $P_n = P(f) \cap A_n$. So there exists an integer $N \geq 1$ such that the component $D_a(\gamma_N)$ of $S^2 - \gamma_N$ containing a contains no critical value of f^p and such that $P_n = f^{p(n-N)}(P_N)$ for all $n \geq N$. We assume $N = 1$ for the conveniency.

Since $D_a(\gamma_1)$ contains no critical value of f^p, $f_a^{-p}(D_a(\gamma_1))$ is also simply-connected where f_a^{-p} denotes the inverse branch along a. Let $\beta_{n-1} = f_a^{-p}(\gamma_n)$ for every $n > 0$, then β_n is isotopic to γ_n rel $P(f)$ and β_n converges to a as $n \to \infty$. So there exists a homeomorphism θ of S^2 such that θ is isotopic to the identity rel $P(f)$ and $\theta(\gamma_n) = \beta_n$, i.e., $f^p \theta(\gamma_n) = \gamma_{n+1}$ for all $n \geq 0$.

Let $0 < r < 1$ be a real number and $g(z) = \lambda z$ for some $0 < \lambda < 1$. Let $B = \{z \in \mathbb{C}; g(r) \leq |z| \leq r\}$. There is a a homeomorphism ϕ_1 from $\overline{A_1}$ to $\overline{B}$ such that $\phi_1 f^p \theta = g \phi_1$ on γ_1. Inductively, let ϕ_n be the homeomorphism from $\overline{A_n}$ to $g^{n-1}(B)$ such that $\phi_n (f^p \theta)^{n-1} = g^{n-1} \phi_1$. Finally, let ϕ be a homeomorphism from S^2 to $\hat{\mathbb{C}}$ such that $\phi = \phi_n$ on A_n for all $n \geq 1$. Then $\phi f^p \theta \phi^{-1}(z) = g(z)$ on Δ_r. □

Suppose f is a geometrically finite rational map. If a is an attractive periodic point, Koenigs Theorem implies that it is combinatorial linearizable. If a is super-attractive, Boettcher Theorem implies that it is combinatorial linearizable. Suppose a is a parabolic periodic point of f with period $p \geq 1$, V is an attractive flower of f^p at a. Let $\gamma_1 \subset \hat{\mathbb{C}} - \overline{P(f)}$ be a simple closed curve such that one component $D(\gamma_1)$ of $\hat{\mathbb{C}} - \gamma_1$ contains $\overline{f^p(V)}$ and such that $D(\gamma_1) \cap \overline{P(f)} = \overline{f^p(V)} \cap \overline{P(f)}$. Inductively, let $\gamma_n \subset D(\gamma_{n-1}) - \overline{P(f)}$ be a simple closed curve such that $D(\gamma_n) \supset \overline{f^{np}(V)}$ and $D(\gamma_n) \cap \overline{P(f)} = \overline{f^{np}(V)} \cap \overline{P(f)}$ for all $n \geq 2$. Then $\{\gamma_n\}$ is a shrinking f-

invariant curve family nested at a. So f is combinatorially linearizable at a. This shows that

Proposition 3.2. *If a geometrically finite branched covering map of S^2 is combinatorially equivalent to a rational map, then it is combinatorially linearizable.*

The following lemmas give some conditions about combinatorially linearizable for a geometrically finite branched covering maps.

Lemma 3.3. *Suppose that f is a geometrically finite branched covering map of S^2, that a is an accumulation fixed point of $P(f)$, and that $\deg_a f = 1$. If f is combinatorially linearizable at a, then there is a neighborhood V of a such that any f-invariant curve family on V is shrinking.*

Proof. Since f is combinatorially linearizable at a and $\deg_a f = 1$, there exist homeomorphisms $\phi, \psi : S^2 \to \hat{\mathbb{C}}$ such that $\phi(a) = 0$, ϕ is isotopic to ψ rel $P(f)$ and $\phi f \psi^{-1}(z) = g(z)$ on a neighborhood U of a, where $g(z) = \lambda z$ for some $0 < \lambda < 1$.

Let $0 < r < 1$ be a constant such that $\Delta_r = \{z \in \mathbb{C}; |z| < r\} \subset U$ and such that $\psi(P(f)) \cap \partial A_n = \emptyset$ and $\psi(P(f)) \cap A_n = g^n(\psi(P(f)) \cap A_0)$ for all $n \geq 0$, where

$$A_n = \{z \in \mathbb{C} : g^{n+1}(r) < |z| < g^n(r)\}.$$

Then for any simple closed curves α_1, α_2 in $\Delta_r - \overline{\psi(P(f))}$, if α_1 is isotopic to α_2 rel $\psi(P(f))$, then $g(\alpha_1)$ is isotopic to $g(\alpha_2)$ rel $\phi(P(f))$.

Denote $V = \psi^{-1}(\Delta_{g(r)})$. For any f-invariant curve family $\{\gamma_n\}$ in V, $\beta_n = \phi(\gamma_n) \subset \Delta_{g(r)}$ and $g^{-1}(\beta_{n+1}) \cap \Delta_r$ is isotopic to β_n rel $\psi(P(f))$. Thus β_{n+1} is isotopic to $g(\beta_n)$ rel $\psi(P(f))$. Hence it is also isotopic to $g^n(\beta_1)$ rel $\psi(P(f))$ for all $n \geq 1$. But we know that $g^n(\beta_1)$ converges to the origin as $n \to \infty$ and that γ_{n+1} is isotopic to $\psi^{-1} g^n(\beta_1)$ rel $P(f)$. So $\{\gamma_n\}$ is shrinking. □

Lemma 3.4. *Suppose that f is a geometrically finite branched covering map of S^2 and that a is an accumulation point of $P(f)$. Suppose $f(a) = a$ and $deg_a f = 1$. If there is only one forward orbit of critical points converging to a, then there is at most one shrinking f-invariant curve family nested at a up to isotopies rel $P(f)$, i.e., if $\{\gamma_n\}$ and $\{\beta_n\}$ are shrinking f-invariant curve families nested at a, then there exist $k \in \mathbb{Z}$ and $N \geq 1$ such that γ_{n+k} is isotopic to β_n rel $P(f)$ for all $n \geq N$.*

Proof. Consider the Riemann surface $R = \mathbb{C} - \{0, 1, 2^{\pm 1}, \cdots\}$ and let $g(z) = z/2$ be a conformal map from R to itself. If α, δ are simple closed geodesics on R under Poincaré metric such that $g^n(\alpha) \cap \alpha = \emptyset$ and $g^n(\delta) \cap \delta = \emptyset$ for all $n \in \mathbb{Z}$. We claim that there is $k \in \mathbb{Z}$ such that $g^k(\alpha) = \delta$. This observation implies the lemma. (Compare this lemma with the example (2) in [p.35, Mc2].)

Now we prove the claim. Let A be the annulus enclosed by α and $g(\alpha)$. Then $A \cap \cup_{n\in\mathbb{Z}} g^n(\delta)$ has finitely many disjoint pairwise arcs. If $A\cap\cup_{n\in\mathbb{Z}}g^n(\delta)$ contains a whole curve $g^k(\delta)$, then $g^k(\delta)$ is isotopic to either α or $g(\alpha)$. So $g^{k+n}(\delta)$ is isotopic to either $g^n(\alpha)$ or $g^{n+1}(\alpha)$ for all $n \in \mathbb{Z}$. Otherwise, suppose $A\cap\cup_{n\in\mathbb{Z}}g^n(\delta)$ dose not contain any whole curve $g^k(\delta)$. One of the arcs in $A\cap\cup_{n\in\mathbb{Z}}g^n(\delta)$ must connect two points in α and another must connect two points in $g(\alpha)$. Since there is only one puncture in A, there exists an arc homotopic to a segment of either α or $g(\alpha)$ modulo the end points. But they are all geodesics. It is a contradiction. □

The following counterexample is one of the main motivations of this paper.

Proposition 3.5. *There is a geometrically finite branched covering map which is not combinatorially linearizable and there is no Thurston obstruction for it.*

Proof. Let $Q(z) = \lambda z + z^2$, $0 < |\lambda| < 1$. The origin is the bounded attracting fixed point of Q, $z_0 = -\lambda/2$ is the bounded critical point of Q and $P = \{z_n = Q^n(z_0)\}_{n\geq 1}$ converges to the origin as $n \to \infty$. There is a domain $D \subset \mathbb{C}$ and a conformal map $\varphi : D \to \Delta_r$ such that $\overline{P} \subset D$, $\varphi(0) = 0$ and $\varphi Q \varphi^{-1}(z) = \lambda z$ on Δ_r. Let $\alpha_0 \subset D - \overline{P}$ be a simple closed curve such that one component $D(\alpha_0)$ of $\mathbb{C} - \alpha_0$ contains $\overline{P}$ and $Q(\alpha_0) \subset D(\alpha_0)$. Since Q is injective on D, $\alpha_n = Q^n(\alpha_0)$ for all $n \geq 0$ form an Q-invariant curve family nested at the origin and α_n converges to the origin as $n \to \infty$.

We can construct a branched covering map f of $\hat{\mathbb{C}}$ of degree 2 such that $f = Q$ on $(\hat{\mathbb{C}} - D(\alpha_0)) \cup P \cup \cup_{n=0}^{\infty} \alpha_{2n}$ and such that $f(\alpha_{2n+1})$ satisfying the following condition (see Fig. 2):

(*) There exist paths $\delta_n \subset D(f(\alpha_{2n+1}))$ and $\delta_{n+1} \subset D(\alpha_{2n+2})$ which have common endpoints z_{2n+3} and z_{2n+4}, such that $\delta_{n+1}\delta_n$ runs around z_{2n+2} twice, where $\delta_{n+1}\delta_n$ means the closed path going through δ_n and δ_{n+1}.

Obviously, $P(f) = P \cup \{\infty\}$ and $f(\alpha_{2n+1})$ is not isotopic to α_{2n+2} rel P. Denote $\gamma_n^n = \alpha_{2n}$ and $\gamma_{n+1}^n = \alpha_{2n+1}$ $(n \geq 0)$, then $\Gamma^n = \{\gamma_n^n, f(\gamma_n^n) = \gamma_{n+1}^n, \cdots\}$ is an f-invariant curve family nested at the origin for all $n \geq 0$.

Since $f(\gamma_{n+1}^n)$ is not isotopic to γ_{n+1}^{n+1} rel P, each curve in Γ^n is not isotopic to any curve in Γ^{n+1}. If Γ^N is shrinking for some N, so is Γ^{N+1} since γ_{N+1}^{N+1} separates γ_N^N and the origin. By Lemma 3.4, Γ^n is not shrinking for all $n \geq 0$. Because γ_n^n converges to the origin, f is not combinatorially linearizable at the origin by Lemma 3.3.

Note that $D(\gamma_1^0) \supset D(\gamma_1^1) \supset \{z_3, z_4, \cdots\}$, thus $f^{-1}(D(\gamma_1^1)) \supset \{z_2, z_3, \cdots\}$, where f^{-1} is the inverse branch along the origin. From (*), $f^{-2}(D(\gamma_1^1)) \supset P$ since z_1 is a critical value of degree 2. By induction, since $D(\gamma_{n+1}^n) \supset D(\gamma_{n+1}^{n+1}) \supset \{z_{2n+3}, z_{2n+4}, \cdots\}$, $f^{-2n}(D(\gamma_n^n)) = f^{-2n-1}(D(\gamma_{n+1}^n)) \supset P$ and $f^{-2n-1}(D(\gamma_{n+1}^{n+1})) \supset P - \{z_1\}$. By (*), $f^{-2n-2}(D(\gamma_{n+1}^{n+1})) = f^{-2n-3}(D(\gamma_{n+2}^{n+1})) \supset P$.

For any multicurve Γ, since γ_n^n converges to zero as $n \to \infty$, there is $N > 0$ such that $\beta \cap \overline{D(\gamma_N^N)} = \emptyset$ for all $\beta \in \Gamma$, where $D(\gamma_N^N)$ is the bounded component of $\hat{\mathbb{C}} - \gamma_N^N$. But $f^{-2N}(D(\gamma_N^N))$ contains P. Thus each component of $f^{-N}(\beta)$ is either non-essential or peripheral. Denote by $A(\Gamma)$ the transition matrix determined by Γ. That means $A(\Gamma)^{2N} = 0$. Hence its spectral radius $\lambda(\Gamma) = 0$. So Γ is not a Thurton obstruction. $\square$

Remark. Let δ be a simple closed curve in the annulus between α_0 and α_2 which encloses z_1 and z_2, T_1 be the simple Dehn twist along δ, T_{2n+1} be the simple Dehn twist along $Q^{2n}(\delta)$ so that $T_{2n+1}Q^{2n} = Q^{2n}T_1$. Then $f_n = QT_1^2T_3^2 \cdots T_{2n-1}^2$ converges to a geometrically finite branched covering map. It is just the above constructed map f.

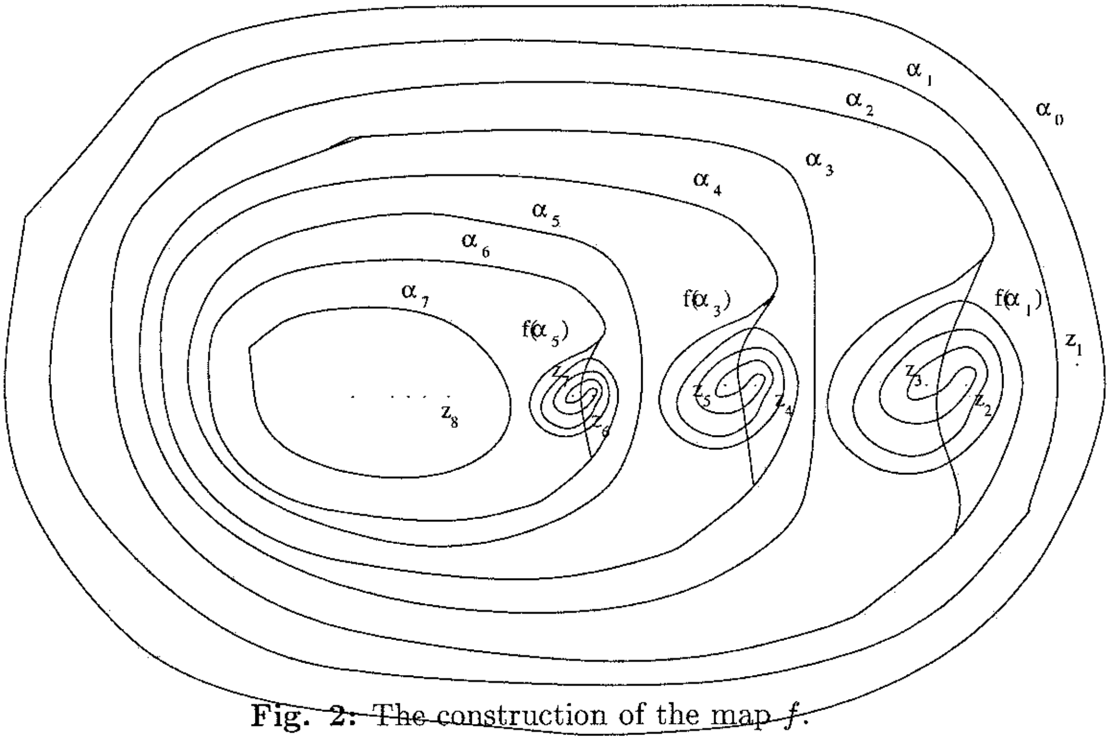

Fig. 2: The construction of the map f.

§4. Pullback Partitions of A Geometrically Finite Branched Covering Map

Let f be a branched covering map of S^2. A connected set E is called *inessential (essential)* if there is (not) a simply-connected domain $D \supset \overline{E}$ such that $D \cap P(f) = \emptyset$. An essential set $E \subset S^2$ is called *semi-essential (strictly-essential)* if there is (not) an annulus $A \supset \overline{E}$ such that $A \cap P(f) = \emptyset$, *peripheral* around $a \in P(f)$ if there is a simply-connected domain $D \supset \overline{E}$ such that $D \cap P(f) = \{a\}$.

Two sets $E_1, E_2 \subset S^2$ are called *parallel* (denote by $E_1 \sim E_2$) if there exist continuous maps $\phi_i : S^2 \to S^2$ $(i = 1, 2)$ such that $\phi_i^{-1}(x) = x$ for every $x \in P(f)$, $\phi_i|_{S^2 - P(f)}$ is homotopic to the identity, $\phi_1(E_1) \subset E_2$ and $\phi_2(E_2) \subset E_1$.

The following is easy to verify. If two connected sets $E_1, E_2 \subset S^2$ are parallel and E_1 is inessential (semi-essential, strictly-essential, peripheral), then so is E_2. Since $f(P(f)) \subset P(f)$, $E_1 \sim E_2$ implies $f^{-1}(E_1) \sim f^{-1}(E_2)$. (This says that pullback will not destroy the parallel property of two sets.) Every component of the preimage of an inessential set is also inessential. Each component of the preimage of a semi-essential set is either inessential or semi-essential, each component of the preimage of a peripheral set is either inessential or peripheral. (This says that pullback will make the property of a set better.)

A non-trivial open set $U \subset S^2$ is called *tame* if there is a branched covering map f of S^2 and a Jordan domain $D \subset S^2$ such that $U = f^{-1}(D)$. From the definition, if U is a tame open set, then the interior $\mathrm{In}(S^2 - U)$ of $S^2 - U$ is tame and the boundary of the interior $\mathrm{In}(S^2 - U)$ is ∂U; if g is a branched covering map, then $g^{-1}(U)$ is also tame. The next lemma is easy to prove:

Lemma 4.1. *Let U be a tame open set. Then the boundary ∂U of U can be uniquely decomposed as the union of simple closed curves $\gamma_1, \cdots, \gamma_n$ such that:*

(i) For any γ_i $(1 \le i \le n)$, there is a unique component D of U so that $\gamma_i \subset \partial D$.

(ii) $\gamma_i \cap \gamma_j$ is either empty or finite for all $1 \le i < j \le n$.

We call $\gamma_1, \cdots, \gamma_n$ the boundary curves of U.

Suppose $\mathcal{U}$ is a tame open set and $\mathcal{H} = S^2 - \mathcal{U}$. We call $(\mathcal{U}, \mathcal{H})$ a *pullback partition* for f if each essential component of $\mathcal{U}$, $f^{-1}(\mathcal{U})$, $\mathcal{H}$ and $f^{-1}(\mathcal{H})$) is parallel to a component of $f^{-1}(\mathcal{U})$, $\mathcal{U}$, $f^{-1}(\mathcal{H})$ and $\mathcal{H}$, respectively. Denote $\mathcal{U}^n = f^{-n}(\mathcal{U})$ and $\mathcal{H}^n = f^{-n}(\mathcal{H})$ for any $n > 0$. It is clear that $(\mathcal{U}^n, \mathcal{H}^n)$ is also a pullback partition for f.

Proposition 4.2. *Suppose f is a geometrically finite rational map whose post-critical set is infinite. Then there is a pullback partition $(\mathcal{U}, \mathcal{H})$ for f such that:*

(1) $\overline{\mathcal{U}} \subset f^{-1}(\mathcal{U}) \cup \mathcal{N}$, where $\mathcal{N}$ is a finite set, $f(\mathcal{N}) \subset \mathcal{N}$ and there is $n \ge 0$ such that $f^n(\mathcal{N})$ is the set of all the parabolic periodic points, and

(2) $\cup_{n=1}^{\infty} f^{-n}(\mathcal{U}) = F(f)$.

Proof. Since $F(f) \neq \emptyset$, take a linearization disk in each periodic Fatou domain and denote by U their union so that the boundary of U meets $\overline{P(f)}$ only at parabolic periodic points. We claim that there is $N > 0$ such that $\big(\mathcal{U} = f^{-N}(U), \mathcal{H} = S^2 - \mathcal{U}\big)$ forms a pullback partition for f. Obviously, U is tame and both (1) and (2) holds.

First note that $f^{-n-1}(U) \supset f^{-n}(U)$ and $\overline{P(f)} \cap (\hat{\mathbb{C}} - f^{-n}(U))$ is a finite set for all $n \ge 0$. Second there is a finite set $\mathcal{N} \subset \overline{P(f)}$ such that any boundary curve does not meet $\overline{P(f)} - \mathcal{N}$ and there is a constant $M > 0$ such that there are at most M boundary curves of $f^{-n}(U)$ meet one point in $\mathcal{N}$ for all $n > 0$. Third, for any two boundary curves α, β of $f^{-n}(U)$ and $f^{-m}(U)$ respectively, there is a component of $\hat{\mathbb{C}} - \alpha$ contained in a component of $\hat{\mathbb{C}} - \beta$.

Combine these facts, there is a constant $N_0 > 0$ such that any boundary curve of $f^{-n}(U)$ is isotopic to a boundary curve of $f^{-m}(U)$ rel $P(f)$ for all $n, m \geq N_0$. Since each semi-essential domain is parallel to a simple closed curve in $\hat{\mathbb{C}} - \overline{P(f)}$, every semi-essential component of $f^{-n}(U)$ is parallel to a component of $f^{-m}(U)$ for all $n, m \geq N_0$.

For any strictly-essential component V of $f^{-n}(U)$, there is a component W of $f^{-n-1}(U)$ containing V and hence is strictly-essential. When $n \geq N_0$, W contains no other strictly-essential component of $f^{-n}(U)$. So the number of strictly-essential components of $f^{-n}(U)$ increases for $n \geq N_0$. But this number is bounded, thus there is $N_1 \geq N_0$ such that for all $n \geq N_1$, each strictly-essential component of $f^{-n-1}(U)$ contains a strictly-essential component of $f^{-n}(U)$ and hence they are parallel.

It is easy to verify that for $n, m \geq N_1$, each component of $\hat{\mathbb{C}} - f^{-n}(U)$ is parallel to a component of $\hat{\mathbb{C}} - f^{-m}(U)$. Denote $\mathcal{U} = f^{-N_1}(U)$ and $\mathcal{H} = \hat{\mathbb{C}} - \mathcal{U}$. So $(\mathcal{U}, \mathcal{H})$ forms a pullback partition for f. □

Remark. One may verify that if $(\mathcal{U}', \mathcal{H}')$ is another pullback partition of f satisfying (1) and (2) in the proposition, then each essential component of $\mathcal{U}'$, $\mathcal{H}'$ is parallel to a component of $\mathcal{U}$ and $\mathcal{H}$, respectively.

Proposition 4.3. *Suppose f is a geometrically finite branched covering map and $P(f)$ is an infinite set. If f is combinatorially linearizable, then there are a branched covering map g from $\hat{\mathbb{C}}$ to itself which is combinatorially equivalent to f and a pullback partition $(\mathcal{U}, \mathcal{H})$ for g such that*

1) $\overline{\mathcal{U}} \subset g^{-1}(\mathcal{U})$,
2) $P(g) \cap \mathcal{H}$ *is a finite set,*
3) g *is holomorphic on* $g^{-1}(\mathcal{U})$,
4) *If a component V of $\mathcal{U}$ satisfies $g^{-k}(V) \supset V$ for some $k \geq 1$, then $V \cap P(g)$ is infinite.*

Proof. Since f is combinatorially linearizable, there is a branched covering map g from $\hat{\mathbb{C}}$ to itself which is combinatorially equivalent to f such that for each accumulation point a_i of $P(g)$, there is a Jordan domain $D_i \ni a_i$ such that whose closures are disjoint pairwise, $\overline{U} \subset g^{-1}(U)$ (where $U = \cup D_i$), the boundary of U does not meet $\overline{P(g)}$ and g is holomorphic on U.

Following the discussion above, there is an integer $N \geq 0$ such that $\big(\mathcal{U} = g^{-N}(U), \mathcal{H} = \hat{\mathbb{C}} - \mathcal{U}\big)$ forms a pullback partition for g. Obviously, 1) and 2) hold.

Pulling back the complex structure to $g^{-1}(\mathcal{U}) = g^{-N-1}(U)$. Then g is holomorphic on $g^{-1}(\mathcal{U})$.

For any component V of $\mathcal{U}$, $g^N(V)$ is a component D_j of U. If $g^{-k}(V) \supset V$, let $i > 0$ be an integer so that $ik \geq N$, then $g^{ik}(V) = g^{ik-N}(D_j) \subset V$. Thus V contains an accumulation point of $P(g)$. □

Proposition 4.4. *Suppose g is a branched covering map of $\hat{\mathbb{C}}$ and $(\mathcal{U}, \mathcal{H})$ is a pullback partition of g satisfying the conditions of Proposition 4.3. Let $\mathcal{U}^n = g^{-n}(\mathcal{U})$ and $\mathcal{H}^n = g^{-n}(\mathcal{H})$ for all the integer $n > 0$. If there are homeomorphisms ϕ and ψ of $\hat{\mathbb{C}}$ such that ϕ is isotopic to ψ rel $P(g)$, $\psi(\mathcal{H}^2)$ is contained in the interior $In(\phi(\mathcal{H}^1))$ of $\phi(\mathcal{H}^1)$ and $\phi g \psi^{-1}$ is holomorphic on $In(\psi(\mathcal{H}^2))$, then g is combinatorially equivalent to a rational map.*

Proof. Since $\psi(\mathcal{H}^2) \subset \mathrm{In}(\phi(\mathcal{H}^1))$, we may assume $\phi|_{\mathcal{U}^1} = \psi|_{\mathcal{U}^1}$ and $\partial\psi(\mathcal{H}^2)$ is the union of quasicircles. By Koebe Theorem (refer to [BB]), there is a one-to-one map $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that:

(a) φ is conformal on $\mathrm{In}(\phi(\mathcal{H}^1))$ and continuous on the boundary,

(b) $\varphi\phi(\partial\mathcal{H}^1)$ is the union of round circles (a round circle is the image of the unit circle under a Möbius transformation), and

(c) $\varphi\phi$ is conformal on $\mathcal{U}^1$ and for any component V of $\mathcal{U}^1$, $\varphi\phi(\partial V) = \partial\varphi\phi(V)$.

Let ϕ_1, ψ_1 be homeomorphisms of $\hat{\mathbb{C}}$ such that ϕ_1 is isotopic to ψ_1 rel $P(g)$, $\phi_1 = \varphi\phi$ on $\mathcal{U} \cup \mathcal{H}^1$ and $\psi_1 = \varphi\psi$ on $\mathcal{U}^1 \cup \mathcal{H}^2$. Then $g_1 = \phi_1 g \psi_1^{-1}$ is holomorphic on $\Im(\varphi\psi(\mathcal{U}^1 \cup \mathcal{H}^2))$. Particularly, we may modify ψ_1 on the complement of $\mathcal{U}^1 \cup \mathcal{H}^2$ such that g_1 is quasiregular (a quasiconformal map composed with a holomorphic function).

Note that every grand orbit of g_1 goes through the complement of $\psi_1(\mathcal{U}^1 \cup \mathcal{H}^2)$ only one times. Pulling back the complex structure, we get a quasiconformal map which conjugates g_1 to a rational map. □

Remark. Note the rational maps obtained in this proposition are sub-hyperbolic. A natural problem is finding all the topological conjugate classes with given combinatorics.

§5. Holomorphic embedding

Let g be a branched covering map of $\hat{\mathbb{C}}$ and $(\mathcal{U}^1, \mathcal{H}^1)$ be a corresponding pullback partition of g satisfying the conditions of Proposition 4.3. Let $\mathcal{U}^n = g^{-n}(\mathcal{U}^1)$ and $\mathcal{H}^n = g^{-n}(\mathcal{H}^1)$ for $n > 1$. Suppose there is no Thurston obstruction for g, we will prove that g with $(\mathcal{U}^1, \mathcal{H}^1)$ and $(\mathcal{U}^2, \mathcal{H}^2)$ satisfies the assumption of Proposition 4.4. Theorem A follows.

For any connected set $E \subset S^2$, define the enclosure $\mathrm{Encl}(E)$ of E the union of E and the inessential components of its complement. Denote by $\mathcal{H}^1_*$ the union of enclosures of essential components of $\mathcal{H}^1$, $\mathcal{H}^n_*$ the union of essential components of $g^{-n+1}(\mathcal{H}_*^1)$. Then $\mathcal{H}_*^{n+1} \subset \Im(\mathcal{H}_*^n)$, $g(\mathcal{H}_*^{n+1}) \subset \mathcal{H}_*^n$ and each component of $\mathcal{H}_*^n$ is parallel to a component of $\mathcal{H}_*^m$ for any $n, m \geq 1$, where $\Im(\cdot)$ means the interior. Note that if a component H of $\mathcal{H}_*^1$ is semi-essential, then its enclosure is an annulus, if H is strictly-essential, each boundary curve of it is essential and no two parallel, if H is strictly-essential and peripheral, then its enclosure is simply-connected and intersects with $P(g)$ on a point.

Proposition 5.1. *Suppose $\mathcal{H}_*^1$ is the disjoint union of domains $H_1^1, \cdots, H_t^1$. If there exist embeddings $\phi_i : H_i^1 \to \hat{\mathbb{C}}$ and homeomorphisms $\theta_i : H_i^1 \to H_i^1$ for all $1 \leq i \leq t$ such that θ_i is isotopic to the identity rel $H_i^1 \cap P(g)$ and $\phi_j g(\phi_i\theta_i)^{-1}$ is holomorphic whenever it is defined, then g with $(\mathcal{U}^1, \mathcal{H}^1)$ and $(\mathcal{U}^2, \mathcal{H}^2)$ satisfies the assumption of Proposition 4.4. In particular, if $\mathcal{H}_*^1$ is empty, then g always satisfies the assumption of Proposition 4.4.*

Proof. First note that there is a homeomorphism ϕ_0 of $\hat{\mathbb{C}}$ so that $\phi_1\phi_0^{-1}$ is holomorphic on $\mathrm{In}(\phi_0(H_i^1))$ for all i. Thus $\phi_0 g(\phi_0\theta_i)^{-1}$ is holomorphic on $\mathrm{In}(\phi_0\theta_i(H_i^1 \cap \mathcal{H}_*^2))$ for all i.

Since θ_i is isotopic to the identity, there is a homeomorphism ψ_0 of $\hat{\mathbb{C}}$ so that $\psi_0 = \phi_0\theta_i$ on H_i^1 and ψ_0 is isotopic to ϕ_0 rel $P(g)$. So $\phi_0 g\psi_0^{-1}$ is holomorphic on $\mathrm{In}(\mathcal{H}_*^2)$.

From the definition of $\mathcal{H}_*^1$, there is a homeomorphism δ of $\mathcal{H}_*^1$ which is isotopic to the identity rel $\mathcal{H}_*^1 \cap P(g)$ such that $\mathrm{In}(\delta(\mathcal{H}^1 \cap \mathcal{H}_*^1)) \supset \theta_i(\mathcal{H}^2 \cap \mathcal{H}_*^1)$ for all i.

By Pullback argument, there is a homeomorphism ζ of $\mathcal{H}_*^2$ which is isotopic to the identity rel $\mathcal{H}_*^2 \cap P(g)$, such that $\delta g\zeta^{-1} = g$ on $\mathcal{H}_*^2$. Extend δ and ζ to global homeomorphisms so that they are isotopic the identity rel $P(g)$. Then

$$\phi_0\delta g\zeta^{-1}\psi_0^{-1} = \phi_0 g\psi_0^{-1}$$

is holomorphic on $\mathrm{In}(\mathcal{H}_*^2)$ and

$$\mathrm{In}(\phi_0\delta(\mathcal{H}^1 \cap \mathcal{H}_*^1)) \supset \psi_0\zeta(\mathcal{H}^2 \cap \mathcal{H}_*^1).$$

Since every component of $\mathcal{H}^1 - \mathcal{H}_*^1$ is inessential, we can modify ϕ_0 and ψ_0 in $\hat{\mathbb{C}} - \mathcal{H}_*^1$ so that $\phi_0\delta g\zeta^{-1}\psi_0^{-1}$ is holomorphic on $\mathrm{In}(\mathcal{H}^2)$ and $\mathrm{In}(\phi_0\delta(\mathcal{H}^1)) \supset \psi_0\zeta(\mathcal{H}^2)$.
□

The following lemmas will be used in the sequel.

Lemma 5.2. *Let γ be a peripheral boundary curve of $\mathcal{H}_*^1$ around a fixed point $a \in P(g)$ and let δ be the component of $g^{-1}(\gamma)$ which parallel to γ. Then δ separates γ and a. Moreover, δ and γ are contained in either different component of $\mathcal{H}_*^1$ or a peripheral and strictly-essential component of $\mathcal{H}_*^1$.*

Proof. Note that $a \in \mathcal{H}_*^1$. By (4) of Proposition 4.3, if V is a component of $\mathcal{U}^1$ peripheral around a, then there is only one component of $g^{-1}(V)$ peripheral around a, which separates V and a. So if β is a boundary curve of $\mathcal{H}_*^1$ peripheral around a, then there is only one component of $g^{-1}(\beta)$ peripheral around a and which separates β and a. Now the lemma follows. □

Lemma 5.3. *Let h be a critically finite branched covering map of S^2. Suppose the signature of $\mathcal{O}_h$ is not $(2,2,2,2)$, $P \supset P(h)$ is a finite set and $h(P) \subset P$. If for any multicurve Γ on $S^2 - P$, $\lambda(\Gamma) < 1$, then there exist homeomorphisms $\phi, \psi : S^2 \to \hat{\mathbb{C}}$ such that ϕ is isotopic to ψ rel P and $\phi g \psi^{-1}$ is a rational map.*

Here we use the isotopy rel P instead of $P(h)$ when we say multicurves and their transition matrices. One may verify this lemma following the proof of Thurston's theorem (refer to [DH]). Now we begin to prove Theorem A.

Proof of Theorem A. Step 1. Let $H_1^1, \cdots, H_s^1$ $(s \le t)$ be all the non-peripheral and strictly-essential components of $\mathcal{H}_*^1$. Then for every $1 \le i \le s$ and $n \ge 2$, there is only one component of $\mathcal{H}_*^n$, denote by H_i^n, parallel to H_i^1 and $H_i^n \subset \mathrm{In}(H_i^1)$. So for each i, there is $k \ge 0$ and $p \ge 1$ such that $g^k(H_i^{k+p+1}) = H_j^{p+1}$ and $g^p(H_j^{p+1}) = H_j^1$.

Suppose $g(H_i^2) = H_{i+1}^1$ $(1 \le i < r)$ and $g(H_r^2) = H_1^1$, then $g^r(H_1^{r+1}) = H_1^1$. For each boundary curve γ^1 of H_1^1, denote by γ^n the boundary curve of H_1^n parallel to γ^1, there are $k \ge 0$ and $p \ge 1$ such that $g^{kr}(\gamma^{kr+pr+1}) = \alpha^{pr+1}$ and $g^{pr}(\alpha^{pr+1}) = \alpha^1$, where α^1 is a boundary curve of H_1^1. By Lemma 5.2, α^1 is non-peripheral and hence $\deg(g^{pr} : \alpha^{pr+1} \to \alpha^1) > 1$ since there is no Thurston obstruction for g. So $\deg(g^r|_{H_1^{r+1}}) > 1$.

For each H_i^1, $1 \le i \le s$, choose a point in each component of $\hat{\mathbb{C}} - H_i^1$ and denote by P_i the union of $P(g) \cap H_i^1$ with these points. If $g(H_i^2) = H_j^1$, there is a branched covering map (or a homeomorphism) h_i from $\hat{\mathbb{C}}$ to itself such that $h_i|_{H_i^2} = g|_{H_i^2}$, $\deg(h_i) = \deg(g|_{H_i^2})$, $h_i(P_i) \subset P_j$ and such that there is at most one critical point in each component of $\hat{\mathbb{C}} - H_i^2$ whose image is contained in P_j.

By this extension, $h = h_r \cdots h_2 h_1$ is a critically finite branched covering map, $\deg(h) > 1$, $P(h) \subset P_1$ and $h(P_1) \subset P_1$. Note that the signature of $\mathcal{O}_h$ is not $(2,2,2,2)$. By the choice of P_1, for any multicurve Γ on $\hat{\mathbb{C}} - P_1$, each $\gamma \in \Gamma$ is isotopic to a curve $\beta \in \hat{\mathbb{C}} - P_1$ rel P_1. So the transition matrices of Γ under h and g^r are equal and hence its spectral radius is less than 1. By Lemma 5.3, there exist homeomorphisms $\Phi, \Psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that Φ is isotopic to Ψ rel P_1 and $\Phi h \Psi^{-1} = R$ is a rational map.

There is a homeomorphism Φ_r from $\hat{\mathbb{C}}$ to itself such that $\Phi_1 h_r \Phi_r^{-1} = R_r$ is holomorphic. Inductively, there are homeomorphisms Φ_i $(1 < i < r)$ from $\hat{\mathbb{C}}$ to itself such that $\Phi_{i+1} h_i \Phi_i^{-1} = R_i$ is holomorphic. In particular, $\Phi_2 h_1 \Psi_1^{-1} = R_1$ is holomorphic and $R_r \cdots R_2 R_1 = R$.

If $g(H_j^2) = H_i^1$ $(t \ge j > s$, $1 \le i \le s)$, let Φ_j be a homeomorphism of $\hat{\mathbb{C}}$ to itself such that $\Phi_i h_j \Phi_j^{-1} = R_j$ is holomorphic. In other words, for every i, $1 \le i \le t$, we have homeomorphisms $\Phi_i, \Psi_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that Φ_i is isotopic to Ψ_i rel P_i and $\Phi_i h_j \Psi_j^{-1} = R_j$ is holomorphic when $g(H_j^2) = H_i^1$.

To avoid confusion, we denote $\Phi_i(\hat{\mathbb{C}}) = \hat{\mathbb{C}}_i$ the different copies of $\hat{\mathbb{C}}$ for all $1 \le i \le t$. Note that $\Phi_1(P_1 - P(g)) \subset F(R)$ since each point of them is eventually superattracting periodic. Because $R : \hat{\mathbb{C}}_1 \to \hat{\mathbb{C}}_1$ is critically finite, the Fatou domains meet $\Phi_1(P_1 - P(g))$ are simply-connected. We call them *marked disks.* For each marked disk W, there is a conformal map $\varphi : W \to \Delta_1$ such that $\varphi(W \cap \Phi_1(P_1)) = 0$. The map φ is unique up to rotations. We call $\varphi^{-1}(\{z \in \Delta_1, |z| = r\})$ *marked circles* of radius r.

If $R_j(\hat{\mathbb{C}}) = \hat{\mathbb{C}}_1$, since $R_j(\Omega(R_j)) \subset \Phi_1(P_1)$, the components of the preimages of marked disks under R_j^{-1} which meet $\Phi_j(P_j)$ are also simply-connected. We also call them marked disks. Marked circles of radius r can be defined similarly.

Step 2. Take all the non-peripheral boundary curves of H_i^1 for all $1 \le i \le s$ and take a boundary curve of H_j^1 when it is semi-essential and non-peripheral. Let Γ be the collection of these curves. Define the transition matrix $A(\Gamma)$ by the formula:

$$A_{\delta\gamma} = \sum_{\alpha} \frac{1}{\deg(g : \alpha \to \gamma)}$$

where the sum is taken over all components α of $g^{-1}(\gamma)$ which are isotopic to δ rel $P(g)$ and contained in the same component of $\mathcal{H}_*^1$ with δ.

Let $\Gamma' \subset \Gamma$ be a multicurve such that for any $\gamma \in \Gamma$, there is $\gamma' \in \Gamma'$ such that γ' is isotopic to γ rel $P(g)$. Then for each $\gamma \in \Gamma$,

$$\sum_{\delta \sim \delta'} A_{\delta\gamma} = A'_{\delta'\gamma'}$$

where A' is the transition matrix of Γ'. If $v = \{v_\gamma\}_{\gamma \in \Gamma}$ is an non-negative eigenvector of A corresponding to the eigenvalue $\lambda(A)$, let $u_{\gamma'} = \sum_{\gamma \sim \gamma'} v_\gamma$, then

$$\begin{aligned}(A'u)_{\delta'} &= \sum_{\gamma'} A'_{\delta'\gamma'} u_{\gamma'} = \sum_{\gamma'} \sum_{\gamma \sim \gamma'} v_\gamma \sum_{\delta \sim \delta'} A_{\delta\gamma} \\ &= \sum_{\delta \sim \delta'} \sum_{\gamma} v_\gamma A_{\delta\gamma} = \sum_{\delta \sim \delta'} \lambda(A) v_\delta = \lambda(A) u_{\delta'}.\end{aligned}$$

So $\lambda(A)$ is an eigenvalue of A' and hence $\lambda(A) \le \lambda(A') < 1$.

Lemma 5.4. *Let A be a non-negative matrix, $\lambda(A)$ the spectral radius of A. Then for any $\epsilon > 0$, there is a positive vector v such that $Av \le (\lambda(A) + \epsilon)v$.*

The proof of this lemma is easy. Thus there is an vector $v = \{v_\gamma\}_{\gamma\in\Gamma}$ such that $v_\gamma \neq 0$ for any $\gamma \in \Gamma$ and

$$(Av)_\gamma \leq \frac{\lambda(A)+1}{2} v_\gamma.$$

Take all the boundary curves of H_i^1 when it is strictly-essential and non-peripheral and take a boundary curve of H_i^1 when it is semi-essential. Let Γ'' be the collection of such curves. By Lemma 5.2 and the discussion before this paragraph, there is a function $v : \Gamma'' \to \mathbb{R}_+$ such that

$$(**) \qquad \sum_\alpha \frac{v(g(\alpha))}{\deg(g|_\alpha)} \leq \frac{\lambda(A)+1}{2} v(\gamma)$$

for all $\gamma \in \Gamma''$, where the sum is taken over the boundary curves of $\mathcal{H}_*^2$ which are isotopic to γ rel $P(g)$ and contained in the same component of $\mathcal{H}_*^1$ with γ.

Step 3. Now we want to embed H_i^1 into $\hat{\mathbb{C}}_i$ for all $1 \leq i \leq t$. If $1 \leq i \leq s$, define $\phi_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}_i$ as a homeomorphism isotopic to Φ_i rel P_i such that $\phi_i(\gamma)$ is a marked curve of radius $\exp(-v(\gamma)M)$ for each boundary curve γ of H_i^1, where $M > 0$ is a constant. If H_i^1 is semi-essential, define $\phi_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}_i$ such that $\phi_i(H_i^1)$ is an annulus of modulus $v(\gamma)M$ (we use the definition of modulus as $m(\{z \in \mathbb{C}, r < |z| < 1\}) = -\log r$), where γ is a boundary curve of H_i^1. If H_1^i is strictly-essential and peripheral, define $\phi_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}_i$ such that $\phi_i(H_i^1) = \Delta_1$ and $\phi_i(H_i^1 \cap P(g)) = 0$.

For $1 \leq i \leq s$, H_i^1 and $H_j^1 = g(H_i^2)$ are strictly-essential and non-peripheral. Since $\Phi_j h_i \Psi_i^{-1} = R_i$ is holomorphic and ϕ_j is isotopic to Φ_j rel P_j, there is a homeomorphism $\psi_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}_i$ isotopic to ϕ_i rel P_i such that $\phi_j h_i \psi_i^{-1} = R_i$ is holomorphic. For each boundary curve γ^1 of H_i^1, let γ^2 be the boundary curve of H_i^2 isotopic to γ^1 rel P_i, then $\psi(\gamma^2)$ is a marked circle in $\hat{\mathbb{C}}_i$ with radius $\exp[-v(g(\gamma^2))M/\deg(g|_{\gamma^2})]$.

The other components H_k^2 of $\mathcal{H}_*^2$ contained in H_i^1 are semi-essential. Let α_k, β_k be the essential boundary curves of H_k^2. If $g(H_k^2) = H_l^1$ is strictly-essential (then it must be non-peripheral), there exists a branched covering map h_k of $\hat{\mathbb{C}}$ such that $h_k|_{H_k^2} = g|_{H_k^2}$, $\deg(h_k) = \deg(g|_{H_k^2})$, such that there is at most one critical point in each component of $\hat{\mathbb{C}} - H_k^2$, and such that $h_k(\Omega(h_k)) \subset P_l$. Let ψ_k be a homeomorphism of $\hat{\mathbb{C}}$ such that $\phi_l h_k \psi_k^{-1}$ is holomorphic, then the modulus m of $\psi(\mathrm{Encl}(H_k^2))$ satisfies:

$$m \leq \frac{v(g(\alpha_k))M}{\deg(g|_{\alpha_k})} + \frac{v(g(\beta_k))M}{\deg(g|_{\beta_k})} + C$$

where $C > 0$ is a constant independent of M.

If $g(H_k^2) = H_l^1$ is semi-essential, let $\psi_k : H_k^2 \to \hat{\mathbb{C}}$ be an embedding such that $\phi_l g \psi_k^{-1}$ is holomorphic, then the modulus of $\psi_k(H_k^2)$ is $v(g(\alpha_k))M/\deg(g|_{H_k^2})$.

By (**), when $M > 0$ is big enough, there exists a homeomorphism θ_i of H_i^1 isotopic to the identity rel $P(g) \cap H_i^1$ such that $\phi_i\theta_i|_{H_i^2} = \psi_i|_{H_i^2}$ and such that $\phi_i\theta_i\psi_k^{-1}$ is holomorphic on $\mathrm{In}(\psi_k(H_k^2))$. Thus $\phi_l g(\phi_i\theta_i)^{-1}$ is holomorphic on $\mathrm{In}(\phi_i\theta_i(H_k^2))$.

If H_i^1 is semi-essential, the same argument also guarantees the existence of θ_i. If H_i^1 is strictly-essential and peripheral, since the modulus of $\phi(H_i^1 - P(g))$ is infinity, there is no obstruction for the holomorphic embedding.

The maps g and ϕ_i and θ_i for $1 \leq i \leq t$ satisfy Proposition 5.1. We complete the proof. □

§6. Geometrically finite Julia sets

Suppose f is a geometrically finite rational map whose post-critical set is infinite, $(\mathcal{U}, \mathcal{H})$ is a pullback partition for f satisfying the conditions of Proposition 4.1. Then $J(f) = \cap_{n=1}^{\infty}\mathcal{H}^n$ where $\mathcal{H}^n = f^{-n}(\mathcal{H})$.

Let K be a Julia component and $H^n(K)$ be the component of $\mathcal{H}^n$ containing K. Then $K = \cap_{n=1}^{\infty} H^n(K)$. In Particular, $K \subset \mathrm{In}(H^n(K))$ for all $n \geq 1$ when K does not meet preimages of parabolic periodic points. If K is inessential (or semi-essential), then there is $N \geq 0$ such that $H^n(K)$ is inessential (or semi-essential) for all $n \geq N$. If K is strictly-essential, then $H^n(K)$ is strictly-essential for all $n > 0$. Since any component of $\mathcal{H}^n$ is parallel to a component of $\mathcal{H}^m$ for every $n, m > 0$, there are exactly $k \geq 0$ strictly-essential components for all $\mathcal{H}^n$. So there are exactly k strictly-essential Julia components.

Lemma 6.1. *Let $D \subset \hat{\mathbb{C}}$ be a simply-connected domain and $D \cap P(f) = \emptyset$. Then for any domain U compactly contained in D, diam$(f_i^{-n}(U)) \to 0$ as $n \to \infty$ independently of i (where diam denotes spherical diameter and f_i^{-n} denotes the inverse branches on D).*

Proof. For any domain U such that $\overline{U} \subset D$, we have a domain D_0 such that $\overline{U} \subset D_0$ and such that $\overline{D_0} \cap \overline{P(f)} = \emptyset$. Since $P(f)$ is infinite and $f_i^{-n}(D_0) \cap P(f) = \emptyset$ for all $n \geq 0$ and i, $\{f_i^{-n}\}$ is a normal family on D_0. If there is a subsequence which converges to a non-constant function g, then $g(D_0)$ is a non-empty domain because a holomorphic map is an open map. We first claim that $g(D_0) \cap J(f) = \emptyset$. Otherwise, if $g(D_0) \cap J(f) \neq \emptyset$, there is a domain W compactly contained in $g(D_0)$ such that $W \cap J(f) \neq \emptyset$, so there is $N > 1$ such that $f^n(W)$ covers $\hat{\mathbb{C}}$ except at most two points when $n \geq N$. But $f^n(W) \subset D_0$ for infinitely many $n \in \mathbb{N}$. It is a contradiction. So $g(D_0) \subset F(f)$. However, in this case, for any domain W compactly contained in $g(D_0)$, $f^n(W)$ converges to a periodic orbit in $\overline{P(f)}$. It contracts to that $\overline{D_0} \cap \overline{P(f)} = \emptyset$. So any subsequence of $\{f_i^{-n}\}$ converges to a constant function. This implies that $\operatorname{diam}(f_i^{-n}(U)) \to 0$ as $n \to \infty$ independently of i. □

Proposition 6.2. *Every periodic Julia component of f is essential. All but finitely many of them are quasicircles.*

Proof. If a periodic Julia component is inessential, Lemma 6.1 implies that it has to consist of one point. This contracts the definition of a Julia component. So the first statement follows. For the second statement, we need only to prove that each semi-essential periodic Julia component is a quasicircle since there are only finitely many strictly-essential Julia components.

Suppose K is a semi-essential periodic Julia component with period $p \geq 1$, then there are $N \geq 0$ and $k \geq 1$ such that $H^n(K)$ is semi-essential for all $n \geq N$ and $H^{N+kp}(K) \subset \operatorname{In}(H^N(K))$ since K does not meet the preimages of parabolic periodic points. Note that $f^{kp}(H^{N+ikp}(K)) = H^{N+(i-1)kp}(K)$. The map $f^{kp}|_{H^{N+kp}(K)}$ can be extended to be a critically finite branched covering map h of $\hat{\mathbb{C}}$ such that $P(h)$ consists of two super-attracting fixed points, such that $\deg(h) = \deg(f^{kp}|_{H^{N+kp}(K)})$, and such that h^n is uniformly quasiregular. Then h is quasiconformally conjugate to the map $z \to z^d$ for $d = \deg(h)$. In particular, K is quasiconformally mapped to the unit circle. So K is a quasicircle. □

Lemma 6.3. *For any Julia component K, there is $N \geq 0$ such that $f^n(K)$ is essential for all $n \geq N$.*

We say an inessential continuum $E \subset \hat{\mathbb{C}}$ is *ϵ-inessential* if there is simply-connected domain $D \supset E$ such that $D \cap P(f) = \emptyset$ and the modulus of $D - \operatorname{Encl}(E)$ is bigger than ϵ.

Lemma 6.4. *There is a constant $\epsilon > 0$ depending only upon f such that if K_0 is a semi-essential Julia component, if for some $l \geq 1$ $f^l(K_0)$ is also semi-essential but $f^{l+1}(K_0)$ is strictly-essential, and if $\deg(f^l|_{K_0}) > 1$, then each inessential Julia component $K \subset \operatorname{Encl}(K_0)$ is ϵ-inessential.*

Proof. First by the pullback partition for f and the proof of Theorem A, there is a constant integer $l_0 \geq 1$ such that for any semi-essential Julia component K, $\deg(f^{l_0}|_K) > 1$. Second $f^{l+1}(K_0)$ is eventually periodic because there are only finitely many strictly-essential Julia components. Now for K_0 in the lemma, let $m \geq 0$ be the biggest integer such that $\deg(f^m_{K_0}) = 1$ and let $K_1 = f^m(K_0)$. There are only finitely many such K_1 because $\deg(f^{l-m}_{K_1}) > 1$ and $f^{l-m+1}(K_1)$ is strictly-essential, because there are finitely many strictly-essential Julia components, and because there are finitely many critical points of f.

Let $H^n(K_0)$, $H^n(K_1)$, and $H^n(f^l(K_0))$ be the components in $\mathcal{H}^n$ containing K_0, K_1, $f^l(K_0)$ for every $n > 0$, respectively. There is an integer $N > 0$ (only depending on K_1) such that $H^n(K_0)$, $H^n(K_1)$, and $H^n(f^l(K_0))$ are all semi-essential for all $n \geq N$.

Since $f^l(K_0)$ is eventually periodic, it is locally connected (refer to [TY]) and hence arc-wise connected (refer to [WD], p.75). So there is a simple arc $\alpha \subset H^n(f^l(K_0))$ only cutting $f^l(K_0)$ at one point and connecting the two essential components of $\hat{\mathbb{C}} - H^n(f^l(K_0))$ for some $n = N + l$.

Because $\deg(f^{l-m}|_{H^n(K_1)}) = d > 1$. It follows that $H^{n-(l-m)}(K_1) - f^{m-l}(\alpha)$ has d connected components. So there is a constant $\epsilon > 0$ such that the closure of each component is ϵ-inessential.

If $K \subset \text{Encl}(K_0)$, then $f^m(K) \subset \text{Encl}(K_1)$ and thus contained in one component of $H^{n-(l-m)}(K_1) - f^{m-l}(\alpha)$. Hence $f^m(K)$ is ϵ-inessential, so is K. Since there are only finitely many K_1, ϵ depends only upon f. □

Proof of Lemma 6.3. If $f^n(K)$ is inessential for all $n \geq 0$, we will prove that there is a constant $\epsilon > 0$ such that $f^n(K)$ is ϵ-inessential for all $n \geq 0$.

Denote by $H(i,n)$ ($i \geq 1$, $n \geq 0$) the component of $\mathcal{H}^i$ contains $f^n(K)$, then $f(H(i,n)) = H(i-1, n+1)$. Since $f^n(K)$ is inessential, let $i(n) \geq 0$ be the smallest integer such that $H(i(n)+1, n)$ is inessential. We assume $i(n) \to \infty$ as $n \to \infty$. Otherwise, there is $i_0 \geq 1$ and infinitely many $n \in \mathbb{N}$ such that $i(n) < i_0$, i.e., $H(i_0, n)$ is inessential for these n. But $\mathcal{H}^{i_0}$ has only finitely many components, so $f^n(K)$ is ϵ-inessential for some $\epsilon > 0$ and infinitely many n. Hence it holds for all $n \geq 0$.

Define $n_1 \geq 1$ the smallest integer satisfying $i(n) \geq l_0 + 2$ (where l_0 is defined in the proof of Lemma 6.4) for all $n \geq n_1$ and $i(n_1) \geq i(n_1 - 1)$, $n_{k+1} > n_k$ the smallest integer satisfying $i(n_{k+1}) \geq i(n_{k+1} - 1)$ ($k \geq 1$). They are well-defined and $n_k \to \infty$ as $k \to \infty$. Moreover, $i(n_{k+1} - 1) = i(n_k) - (n_{k+1} - n_k - 1) \geq 1$. We claim that (refer to Fig. 3):

(I) $H(i(n_k) + 1 - (n_{k+1} - n_k),\, n_{k+1}) = f^{n_{k+1}-n_k}(H(i(n_k)+1,\, n_k))$ is semi-essential, and

(II) $H(i(n_k) - (n_{k+1} - n_k),\, n_{k+1}) = f^{n_{k+1}-n_k}(H(i(n_k),\, n_k))$ is strictly-essential.

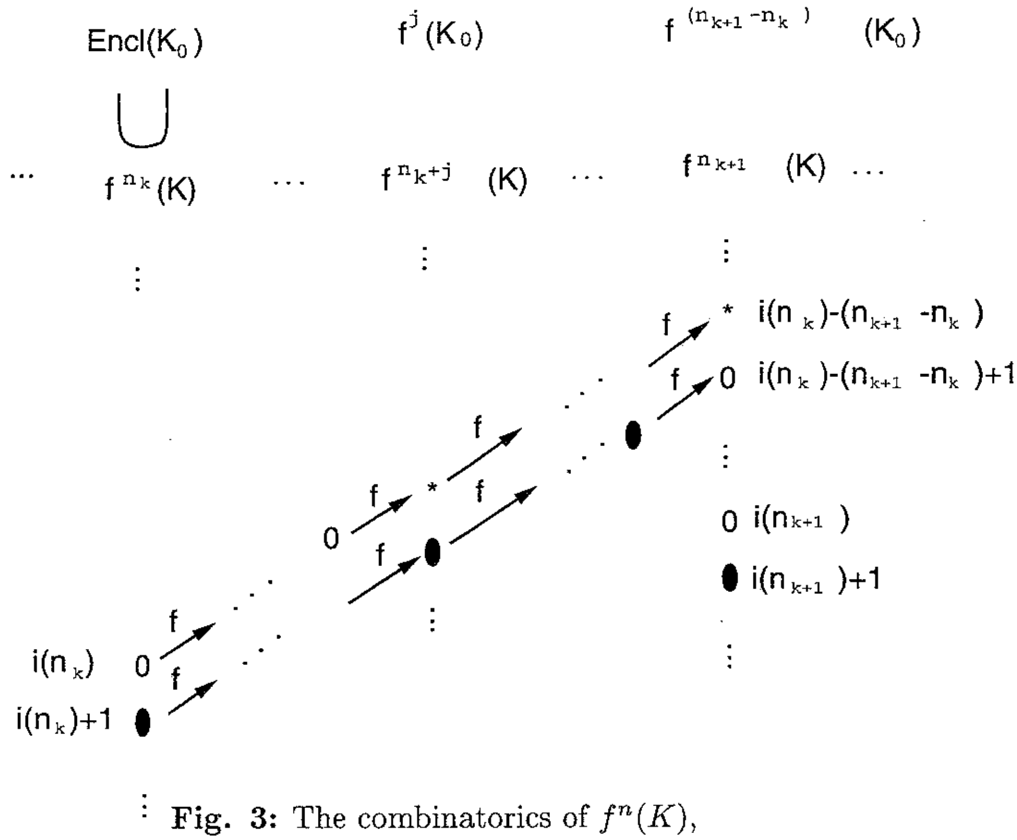

Fig. 3: The combinatorics of $f^n(K)$,
$*$: strictly-essential, 0: semi-essential, $\bullet$: inessential.

In fact, the definition of n_{k+1} implies that both of them are essential. If they are parallel, so are $H(i(n_k), n_k)$ and $H(i(n_k)+1, n_k)$. This contradicts to the definition of $i(n_k)$.

Since $H(i(n_k)-(n_{k+1}-n_k))$ is strictly-essential, it contains only one strictly-essential Julia component, so there is only one semi-essential Julia component K_0 contained in $H(i(n_k), n_k)$ such that $f^{n_{k+1}-n_k}(K_0)$ is strictly-essential. By (I), $f^{n_k}(K) \subset \mathrm{Encl}(K_0)$.

Let $1 \leq j \leq n_{k+1}-n_k$ be the smallest integer such that $f^j(K_0)$ is strictly-essential, then $j \geq l_0+1$. Otherwise, since $H(i(n_k)+1, n_k-1)$ is inessential (because $i(n_{k-1}) \leq i(n_k)$), $H(i(n_k)-j, n_k+j) = f^{j+1}(H(i(n_k)+1, n_k-1)) = f^j(H(i(n_k), n_k)) \supset f^j(K_0)$ is strictly-essential and parallel to $H(1, n_{k+j})$. So $H(j+2, n_k-1)$ is parallel to $H(i(n_k)+1, n_k-1)$ and hence inessential. Thus $j+2 \geq i(n_k-1)+1 \geq l_0+3$. It is a contradiction. So $\deg(f^{j-1}|_{K_0}) > 1$. By Lemma 6.4, $f^{n_k}(K)$ is ϵ-inessential for every k, so is $f^n(K)$ for all $n \geq 0$.

Let W_n be a simply-connected domain such that $W_n \cap P(f) = \emptyset$, $W_n \supset f^n(K)$ and the modulus $m(W_n - \mathrm{Encl}(f^n(K))) > \epsilon$, D_n is the component of $f^{-n}(W_n)$

containing K, then $m(D_n - \mathrm{Encl}(K)) > \epsilon$ and $D_n \cap f^{-n}(P(f)) = \emptyset$ for all $n \geq 0$. But $K \subset \overline{\cup_{n=1}^{\infty} f^{-n}(P(f))}$, this implies that K is a single point. It contracts to the definition of a Julia component. □

Theorem C. *For geometrically finite rational maps, every wandering Julia component is a simple closed curve.*

Before to prove Theorem C, we first recall the *prime end theorem* (refer to [CG]). A crosscut C of a simply-connected domain G whose boundary contains more than one point is an open arc in G such that $\bar{C} = C \cup \{a, b\}$ with $a, b \in \partial G$. $\{C_n\}$ is called a null-chain of G if:

(i) C_n is a crosscut of G $(n = 0, 1, \cdots)$,
(ii) $\bar{C}_n \cap \bar{C}_{n+1} = \emptyset$,
(iii) C_n separates C_0 and C_{n+1} and
(iv) $\mathrm{diam}(C_n) \to 0$ as $n \to 0$.

A prime end is an equivalent class of a null-chain by the equivalent relation: $\{C_n\}$ is equivalent to $\{C'_n\}$ if for any n, there exits m such that C'_m separates C_n from C_0 and C_m separates C'_n from C'_0.

The set C_n separates G into two domains V_n and $G - \bar{V}_n$ and $V_n \supset V_{n+1} \supset \cdots$. Then $I(p) = \cap_{n=1}^{\infty} \overline{V}_n$ is a point or a continuum, it is independent of the choice of the null-chain and is called the impression of the prime end p.

Prime End Theorem. *If all the prime end has single point impression, then ∂G is locally connected.*

The next lemma will be used to prove Theorem C. Suppose that A_1, $A_2 \subset \hat{\mathbb{C}}$ are annuli and that A_2 is contained essentially in A_1 (i.e., A_2 separates the two boundary components of A_1). Define

$$\omega(A_1, A_2) = \sup_{z \in \partial A_2} \inf_{\delta} \{\mathrm{diam}(\delta) : \delta \subset \overline{A_1} - A_2 \text{ is a path connecting } z \text{ and } \partial A_1\}.$$

Lemma 6.5. *Suppose that $A_n \subset \hat{\mathbb{C}}$ $(n \geq 1)$ are annuli and that A_{n+1} is contained essentially in A_n and $A = \cap_{n=1}^{\infty} A_n$. If*

(1) $\sum_{n=1}^{\infty} \omega(A_n, A_{n+1}) < \infty$ and
(2) $\hat{\mathbb{C}} - A$ is the disjoint union of two domains G and G' and A is the common boundary of G and G',

then A is a simple closed curve.

Proof. We only need to prove that $\partial G = A$ is locally connected. If there is a prime end $p = \{C_n\}$ such that $I(p)$ is a continuum with $\mathrm{diam}(I(p)) = \epsilon > 0$. Then $\mathrm{diam}(V_n) \geq \epsilon > 0$ for all $n \geq 0$.

By (1), there is an integer $N > 0$ such that for any $z \in G$, there is a path $\delta \subset G$ connecting z and $D_N = G - A_N$ such that $\mathrm{diam}(\delta) < \epsilon/4$. Because $\mathrm{diam}(C_n) \to 0$, there is an integer $M > 0$ such that as $n \geq M$, $\mathrm{diam}(C_n) < \epsilon/6$ and $\overline{V_n} \cap D_N = \emptyset$.

Note that $\mathrm{diam}(V_n) \geq \epsilon$, so there exists $z \in V_M$ such that the distance$(z, C_M) > \epsilon/2 - \epsilon/6 = \epsilon/3$. But C_M separates z and D_N, so $\delta \cap C_N \neq \emptyset$. Thus $\mathrm{diam}(\delta) \geq$ distance$(z, C_M) > \epsilon/3$. This is a contradiction. □

Proof of Theorem C. Instead of the spherical metric, we use the hyperbolic metric on $\hat{\mathbb{C}} - \overline{P(f)}$. For each path $\delta \subset \hat{\mathbb{C}} - \overline{P(f)}$, its diameter is defined by

$$d(\delta) = \sup_{z,w \in \delta} l[\delta(z,w)]$$

where $l[\delta(z,w)]$ is the length of the geodesics connecting z and w and homotopic to $\delta(z,w)$ in $\hat{\mathbb{C}} - \overline{P(f)}$. We use this diameter when we apply Lemma 6.5.

In the sequel, we always denote by H^n a component of $\mathcal{H}^n$, and $\hat{H}^n$ the interior of the enclosure of H^n.

Since $\|f'(z)\| > 1$ for all $z \in \hat{\mathbb{C}} - \overline{f^{-1}(P(f))}$, there exist constants $M_0 > 0$ and $\lambda > 1$ satisfying the following conditions.

1. $\|(f^{-1})'(z)\| < 1/\lambda$ for all z in the union of the enclosures of semi-essential components of $\mathcal{H}^1$.
2. For any semi-essential components $H^2 \subset H^1$, if $\hat{H}^2$ is an annulus, then $\omega(\hat{H}^1, \hat{H}^2) < M_0$.
3. For any strictly-essential component H^1, if $H^2 \subset H^1$ and γ is an essential boundary curve of $\hat{\mathbb{C}} - H^2$ which separates H^2 from the boundary curve β of $\hat{\mathbb{C}} - H^1$ (β is parallel to γ) such that $\gamma \cap \overline{P(f)} = \emptyset$, then for any $z \in \gamma$, there is a path $\delta \subset \mathrm{Encl}(H^1) - \hat{H}^2$ connecting z and the boundary of $\mathrm{Encl}(H^1)$ such that $d(\delta) < M_0$. Moreover, $\|(f^{-1})'(z)\| < 1/\lambda$ for all z in the component of $\mathrm{Encl}(H^1) - \hat{H}^2$ which are parallel to γ.
4. For any strictly-essential component H^1, let $K_0 \subset H^1$ be the strictly-essential Julia component. Then for any boundary curve γ of $\hat{\mathbb{C}} - \mathrm{Encl}(H^1)$, there is a point $z \in \gamma$ and a path δ in the closure of $\mathrm{Encl}(H^1) - K_0$ connecting z and K_0 such that $d(\delta) < M_0$. If $H^2 \subset H^1$ is semi-essential, let γ be the boundary curve of $\hat{\mathbb{C}} - \mathrm{Encl}(H^2)$ which separates K_0 from H^2, then for any $z \in \gamma$, there is a path δ in the closure of a component of $\mathrm{Encl}(H^1) - (K_0 \cup H^2)$, which connecting z and K_0 such that $d(\delta) < M_0$.
5. For any semi-essential Julia component K_0, if $f(K_0)$ is strictly-essential, then for any $z_1, z_2 \in K_0$, there is an arc δ in K_0 connecting z_1 and z_2 such that $d(\delta) < M_0$.
6. From 3.-5., one may verify that there is a constant $M_1 > M_0$, which depends only upon M_0 and λ, such that if $H^n \subset H^{n-1}$ are semi-essential,

if $\hat{H}^n$ is an annulus, and if $f(H^n)$ is still semi-essential but $f(H^{n-1})$ is strictly-essential, then $\omega(\hat{H}^m, \hat{H}^n) < M_1$ for any semi-essential $H^m \supset H^{n-1}$.

Suppose K is a wandering Julia component of f. By Lemma 6.3, we may assume K is semi-essential. Notice that $K = \cap_{n=1}^{\infty} H^n(K)$. Define $n_1 \geq 1$ as the smallest integer such that $H^{n_1}(K)$ is semi-essential, define $m_1 \leq n_1$ as the biggest integer such that $f^{m_1-1}(H^{n_1}(K))$ is semi-essential, define $n_2 \geq m_1 + 1$ as the smallest integer such that $f^{m_1}(H^{n_2}(K))$ is semi-essential, and define $m_2 \leq n_2$ as the biggest integer such that $f^{m_2-1}(H^{n_2}(K))$ is semi-essential. Inductively, define

(i) $n_k \geq m_{k-1} + 1$ is the smallest integer such that $f^{m_{k-1}}(H^{n_k}(K))$ is semi-essential, and
(ii) $m_k \leq n_k$ is the biggest integer such that $f^{m_k-1}(H^{n_k}(K))$ is semi-essential for $k \geq 2$.

Then $m_{k+1} > m_k$ and $n_{k+1} > n_k$. Denote $A_k = \hat{H}^{n_k}(K)$. If for some $k \geq 1$, $m_k = n_k$ and $n_{k+1} = n_k + 1$, then $\omega(A_k, A_{k+1}) \leq M_0 \lambda^{-m_k+1}$ by 1. and 2.. Otherwise, $f^{m_k-1}(H^{n_k}(K))$ is semi-essential, $f^{m_k}(H^{n_{k+1}}(K))$ is still semi-essential but $f^{m_k}(H^{n_{k+1}-1}(K))$ is strictly-essential. By 1. and 6., we also have $\omega(A_k, A_{k+1}) \leq M_1 \lambda^{-m_k+1}$. Since $m_{k+1} \leq m_k + 1$,

$$\sum_{k=1}^{\infty} w(A_k, A_{k+1}) < \infty.$$

We now complete the proof by Lemma 6.5. To apply Lemma 6.5, we also need to prove that $\hat{\mathbb{C}} - K$ has two component and that K is their common boundary. From the above discussion, since $m_k \to \infty$, there is a semi-essential component H^1 such that there exist infinitely many integer n such that $f^n(K) \subset H^1$. For each point $z \in K$, there is a subsequence of $f^n(z)$ converging to a point z_∞ in the interior of H^1. Take a simply connected domain $D \ni z_\infty$ such that $D \cap P(f) = \emptyset$ and both of the components of $\hat{\mathbb{C}} - \hat{H}^1$ intersect with D. By Lemma 6.1, there exist $\{x_n\}$ and $\{y_n\}$ which contained in the two components of $\hat{\mathbb{C}} - K$ respectively such that they converge to z. This also show that $K = \cap_{k=1}^{\infty} A_k$. By Lemma 6.5, K is a simple closed curve. □

Appendix. Extremal quasiconformal maps

Let $f : R \to R'$ be a quasiconformal homeomorphism between open Riemann surfaces. We denote by

$$\mu_f(z) \frac{d\bar{z}}{dz} = \frac{f'_{\bar{z}} d\bar{z}}{f'_z dz}$$

the Beltrami differential of f,

$$K_f(z) = \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|}$$

the dilatation of f and $K(f) = \|K_f\|_\infty$ the maximal dilatation of f.

Denote by $\mathcal{F}$ the set of all the quasiconformal homeomorphisms homotopic to f modulo the boundary. The map f is called extremal if $K(f) \le K(g)$ for all $g \in \mathcal{F}$.

A quasiconformal map f is called a Teichmüller map associated with an integrable holomorphic quadratic differential φ if $\mu_f(z) = k\overline{\varphi(z)}/|\varphi(z)|$ for some $0 < k < 1$. A Teichmüller map is unique extremal.

Theorem A.1 (Existence of Teichmüller maps). *Let $\mathcal{F}$ be a family of quasiconformal homeomorphisms of R to R' which are homotopic modulo the boundary. Let f_0 with maximal dilatation $K(f_0) > 1$ be extremal in $\mathcal{F}$ (there always exists an extremal map in $\mathcal{F}$). If there exists a quasiconformal map $f \in \mathcal{F}$ whose dilatation in some neighborhoods of ∂R $K_f < K(f_0)$, then f is a Teichmüller map associated with an integrable holomorphic quadratic differential and hence is unique extremal.*

See [Str]. The next theorem appeared in [RS].

Theorem A.2 (Main inequality). *Let f and g be quasiconformal homeomorphisms of R to R' which are homotopic modulo the boundary and that φ be an integrable holomorphic quadratic differential on R, then*

$$\int_R |\varphi| dx\, dy \le \int_R |\varphi| \frac{|1-\mu_f \frac{\varphi}{|\varphi|}|^2}{1-|\mu_f|^2} K_{g^{-1}} \circ f(z) dx\, dy.$$

The next lemma is a simple application of the above theorems and is used in Section 2.

Lemma A.3. *Let f be a quasiconformal homeomorphism of $\mathbb{C} - \{0\}$ to itself. Then for any $\epsilon > 0$, there exist $0 < r_0 < 1$ and a quasiconformal homeomorphism g of $\mathbb{C} - \{0\}$ to itself such that*

(1) $g = f$ on $\mathbb{C} - R_{r_0}$, where $R_{r_0} = \{z \in \mathbb{C},\ r_0 < |z| < 1/r_0\}$,
(2) $K(g_r) < K(f|_{\mathbb{C}-R_{r_0}}) + \epsilon$ and
(3) $f|_{R_{r_0}}$ is homotopic to $g|_{R_{r_0}}$ modulo the boundary.

Proof. For $r > 0$ small enough, let g_r be an extremal quasiconformal map from R_r to $f(R_r)$ in the homotopy class modulo the boundary. Given any $\epsilon > 0$, if there is $r > 0$ such that $\|\mu_{g_r}\|_\infty$ satisfies (2), then it is derived. Otherwise, g_r is a Teichmüller map associated with an integrable holomorphic quadratic differential

φ_r (we suppose $\|\varphi_r\| = 1$) for all $r > 0$ small enough by Theorem A.1. Then φ_r converges to zero as $r \to 0$ uniformly on any compact set in $\mathbb{C} - \{0\}$. Otherwise, there is a sequence g_{r_n} converges to a Teichmüller map of $\mathbb{C} - \{0\}$ with maximal dilatation bigger than 1. This is a contradiction.

For a fixed $r_1 > 0$, since $\|\mu_{g_r}\|_\infty \le \|\mu_f\|_\infty$, there exists a compact set $C \subset \mathbb{C} - \{0\}$ such that for all r, $g_r^{-1}(R_{r_1}) \subset C$. Since φ_r converges to zero, there is $r_0 > 0$ such that as $r < r_0$, $C \subset R_r$ and

$$\int_{g_r^{-1}(R_{r_1})} |\varphi_r| < \frac{\epsilon}{K(g_{r_1})}.$$

Apply Theorem A.2 for g_{r_1}, g_r and φ_r,

$$1 = \int_{R_r} |\varphi_r| \le \int_{R_r} |\varphi| \frac{|1 - \mu_{g_r} \frac{\varphi_r}{|\varphi_r|}|^2}{1 - |\mu_{g_r}|^2} K_{g_{r_1}}^{-1} \circ g_r.$$

Thus,

$$\begin{aligned} K(g_r) &\le \int_{R_r} |\varphi_r| K_{g_{r_1}}^{-1} \circ g_r \\ &\le \int_{g_r^{-1}(R_{r_1})} |\varphi_r| K_{g_{r_1}}^{-1} \circ g_r + \int_{R_r - g_r^{-1}(R_{r_1})} |\varphi_r| K_{g_{r_1}}^{-1} \circ g_r \\ &< \epsilon + K(f|_{\mathbb{C} - R_r}), \end{aligned}$$

as $r \le r_0$. It is again a contradiction. Therefore, there is a $r > 0$ such that $\|\mu_{g_r}\|_\infty$ satisfies (2). □

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