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Determinant bundles, Quillen metrics and Mumford isomorphisms over the universal commensurability Teichmüller space

by

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1. Introduction

Let $\mathcal{T}_g$ denote the Teichmüller space comprising compact marked Riemann surfaces of genus g. Let $\mathrm{DET}_n \to \mathcal{T}_g$ be the determinant (of cohomology) line bundle on $\mathcal{T}_g$ arising from the nth tensor power of the relative cotangent bundle on the universal family $\mathcal{C}_g$ over $\mathcal{T}_g$. The bundle DET_0 is called the Hodge line bundle. The bundle DET_n is equipped with a Hermitian structure which is obtained from the construction of Quillen metrics on determinant bundles using the Poincaré metric on the relative tangent bundle of $\mathcal{C}_g$, [Q].

These natural line bundles over $\mathcal{T}_g$ carry liftings of the standard action of the mapping class group, MC_g, on $\mathcal{T}_g$. We shall think of them as MC_g-equivariant line bundles, and the isomorphisms we talk about will be MC_g-equivariant isomorphisms. By applying the Grothendieck–Riemann–Roch theorem, Mumford [Mu] had shown that DET_n is a certain fixed (genus-independent) tensor power of the Hodge bundle. More precisely,

$$\mathrm{DET}_n = \mathrm{DET}_0^{\otimes(6n^2-6n+1)} \tag{1.1}$$

this isomorphism of equivariant line bundles being ambiguous only up to multiplication by a non-zero scalar. (Any choice of such an isomorphism will be called a Mumford isomorphism in what follows.)

There is a remarkable connection, discovered by Belavin and Knizhnik [BK], between the Mumford isomorphism above for the case $n=2$ (i.e., that DET_2 is the 13th tensor power of Hodge) and the existence of the Polyakov string measure on the moduli space $\mathcal{M}_g$. (See the discussion after Theorem 5.5 for more details.) This suggests the question of finding a genus-independent formulation of the Mumford isomorphisms over some "universal" parameter space of Riemann surfaces (of varying topology).

In this paper we combine a Grothendieck–Riemann–Roch lemma (Lemma 2.9) with a new concept of $\mathbf{C}^*\otimes\mathbf{Q}$ bundles (§5), to construct a universal version of the determinant bundles and Mumford's isomorphism. Our objects exist over a universal base space $\mathcal{T}_\infty=\mathcal{T}_\infty(X)$, which is the infinite directed union of the complex manifolds that are the Teichmüller spaces of higher genus surfaces that are unbranched coverings of any (pointed) reference surface X. The bundles and the relating isomorphisms are equivariant with respect to the natural action of the universal commensurability group CM_∞—which is defined (up to isomorphism) as the group of isotopy classes of unbranched self correspondences of the surface X arising from pairs of non-isotopic pointed covering maps $X'\rightrightarrows X$ (see below and in §5).

In more detail, our universal objects are obtained by taking the direct limits using the following category $\mathcal{S}$: for each integer $g\geqslant 2$, there is one object in $\mathcal{S}$, an oriented closed pointed surface X_g of genus g, and one morphism $X_{\tilde{g}}\to X_g$ for each based isotopy class of finite unbranched pointed covering map. For each morphism of $\mathcal{S}$ (say of degree d) we have an induced holomorphic injection of Teichmüller spaces arising from pullback of complex structure:

$$\mathcal{T}(\pi)\colon \mathcal{T}_g \to \mathcal{T}_{\tilde{g}}. \tag{1.2}$$

The GRR lemma provides a natural isomorphism of the line bundle $\mathrm{DET}_{n,g}^{\otimes 12d}$ on $\mathcal{T}_g$ with the pullback line bundle $\mathcal{T}(\pi)^*\mathrm{DET}_{n,\tilde{g}}^{\otimes 12}$. We may view this isomorphism, equivalently, as a degree d homomorphism covering the injection $\mathcal{T}(\pi)$ between the principal $\mathbf{C}^*$ bundles associated to $\mathrm{DET}_{n,g}$ and $\mathrm{DET}_{n,\tilde{g}}$, respectively. Each commutative triangle in $\mathcal{S}$ yields a commutative triangular prism whose top face is the following triangle of total spaces of principal $\mathbf{C}^*$ bundles:

$$\begin{tikzcd}
\mathrm{DET}_{n,g}^{\otimes 12} \arrow[rr] \arrow[dr] & & \mathrm{DET}_{n,\tilde{g}}^{\otimes 12} \arrow[dl] \\
& \mathrm{DET}_{n,\bar{g}}^{\otimes 12} &
\end{tikzcd}$$

and whose bottom face is the commuting triangle of base spaces for these bundles:

$$\begin{tikzcd} \mathcal{T}_g \arrow[rr] \arrow[dr] & & \mathcal{T}_{\tilde{g}} \arrow[dl] \\ & \mathcal{T}_{\tilde{\tilde{g}}} & \end{tikzcd} \tag{1.3}$$

Moreover, the canonical mappings above relating these DET_n bundles over the various Teichmüller spaces preserve the Quillen Hermitian structure of these bundles in the sense that unit circles are carried to unit circles.

We explain in brief the commensurability Teichmüller space $\mathcal{T}_\infty$ and the large mapping class group CM_∞ acting thereon. For each object X in $\mathcal{S}$, consider the directed set $\{\alpha\}$ of all morphisms in $\mathcal{S}$ with range X. Then we form

$$\mathcal{T}_\infty(X) := \operatorname{dir\,lim} \mathcal{T}_{g(\alpha)}, \tag{1.4}$$

where the limit is taken over $\{\alpha\}$, and $g(\alpha)$ is the genus for the domain of morphism α. Each morphism $X_{g'} \to X_g$ induces a holomorphic *bijection* of the corresponding direct limits, and we denote any of these isomorphic "ind-spaces" (inductive limit of finite-dimensional complex spaces—see [Sh]) by $\mathcal{T}_\infty$—the universal commensurability Teichmüller space. (Compare §2 and Example 4 on p. 547 of [Su].) Notice that a pair of morphisms $X' \rightrightarrows X$ determines an automorphism of $\mathcal{T}_\infty$; we call the group of automorphisms of $\mathcal{T}_\infty$ obtained in this way the *commensurability modular group* CM_∞.

We now take the direct limit of the $\mathbf{C}^*$ principal bundles associated to $\mathrm{DET}_{n,g}^{\otimes 12}$ over $\mathcal{T}_g$ to obtain a new object—a $\mathbf{C}^* \otimes \mathbf{Q}$ bundle over $\mathcal{T}_\infty$—denoted $\mathrm{DET}(n, \mathbf{Q})$. As sets the total space with action of the group $\mathbf{C}^* \otimes \mathbf{Q}$ is defined by the direct limit construction. Continuity and complex analyticity for maps into these sets are defined by the corresponding properties for factorings through the strata of the direct system (§5).

There are the natural isomorphisms of Mumford, as stated in (1.1), at the finite-dimensional stratum levels. By our construction these isomorphisms are *rigidified* to be natural over the category $\mathcal{S}$. Therefore we have natural Mumford isomorphisms between the following $\mathbf{C}^* \otimes \mathbf{Q}$ bundles over the universal commensurability Teichmüller space $\mathcal{T}_\infty$:

$$\mathrm{DET}(n, \mathbf{Q}) \quad \text{and} \quad \mathrm{DET}(0, \mathbf{Q})^{\otimes(6n^2 - 6n + 1)}. \tag{1.5}$$

We also show that the natural Quillen metrics of the DET bundles fit together to define a natural analogue of Hermitian structure on these $\mathbf{C}^* \otimes \mathbf{Q}$ bundles; in fact, for all our canonical mappings in the direct system the unit circles are preserved. Note Theorem 5.5.

Indeed, the existence of the canonical relating morphism between determinant bundles (fixed n) in the fixed covering $\pi: X_{\tilde{g}} \to X_g$ situation was first conjectured and deduced by us utilizing the differential geometry of these Quillen metrics. Recall that the

Teichmüller spaces $\mathcal{T}_g$ and $\mathcal{T}_{\tilde{g}}$ carry natural symplectic forms (defined using the Poincaré metrics on the Riemann surfaces)—the Weil–Petersson Kähler forms—which are in fact the curvature forms of the natural Quillen metrics of these DET bundles ([Wo], [ZT], [BGS]). If the covering π is unbranched of degree d, a direct calculation shows that this natural WP form on $\mathcal{T}_{\tilde{g}}$ (appropriately renormalized by the degree) pulls back to the WP form of $\mathcal{T}_g$ by $\mathcal{T}(\pi)$ (the embedding of Teichmüller spaces induced by π). One expects therefore that if one raises the DET_n bundle on $\mathcal{T}_g$ by the tensor power d, then it extends over the larger Teichmüller space $\mathcal{T}_{\tilde{g}}$ as the DET_n bundle thereon. This intuition is, of course, what is fundamentally behind our direct limit constructions. Since it turns out to be technically somewhat difficult to actually prove that the relevant bundles are isomorphic using this differential geometric method, we have separated that aspect of our work into a different article [BN].

Can objects on $\mathcal{T}_\infty$ that are equivariant by the commensurability modular group CM_∞ be viewed as objects on the quotient $\mathcal{T}_\infty/\mathrm{CM}_\infty$? This quotient is problematical and interesting, so we work with the equivariant statement.

We end the Introduction by mentioning some problems. The universal commensurability Teichmüller space, $\mathcal{T}_\infty$, is made up from embeddings $\mathcal{T}(\pi)$ that are isometric with respect to the natural Teichmüller metrics, so it carries a natural Teichmüller metric. Our theorems give us genus independent determinant line bundles $\mathrm{DET}(n, \mathbf{Q})$, Quillen metrics and Mumford isomorphisms over $\mathcal{T}_\infty$, all compatible with each other and the commensurability group CM_∞. *Are the above structures uniformly continuous for this metric?* Then they would pass to the completion $\widetilde{\mathcal{T}}_\infty$ of $\mathcal{T}_\infty$ for the Teichmüller metric. One knows that $\widetilde{\mathcal{T}}_\infty$ is a separable complex Banach manifold which is the Teichmüller space of complex structures on the universal solenoidal surface $H_\infty = \varprojlim \widetilde{X}$, where $\widetilde{X}$ ranges, as above, over all finite covering surfaces of X. (See [Su], [NS] for the Teichmüller theory of H_∞.) We would conjecture that this continuity is true and that the $\mathbf{C}^*\otimes\mathbf{Q}$ bundles $\mathrm{DET}(n, \mathbf{Q})$, Quillen metrics and Mumford isomorphisms can be defined over $\mathcal{T}(H_\infty) = \widetilde{\mathcal{T}}_\infty$ directly by looking at compact solenoidal Riemann surfaces themselves. Now we may consider square integrable holomorphic forms on the leaves of H_∞ (which are uniformly distributed copies of the hyperbolic plane in H_∞) regarded as modules over the $\mathbf{C}^*$-algebra of H_∞ with chosen transversal. The measure of this transversal would become a real parameter extending the genus above. One expects that A. Connes' version of Grothendieck–Riemann–Roch would replace Deligne's functorial version which we are using here.

Finally one would hope that the Polyakov measure (§5) on Teichmüller space, when viewed as a metric on the canonical bundle, would also make sense at infinity in the direct limit because this measure can be constructed by applying the 13th power Mumford isomorphism ((1.1) for $n=2$) to the L^2 inner product on the Hodge line bundle. That issue remains open.

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2. A lemma on determinant bundles

Let X be a compact Riemann surface, equivalently, an irreducible smooth projective curve over $\mathbf{C}$. Let L be a holomorphic line bundle on X. The determinant of L is then defined to be the 1-dimensional complex vector space $(\bigwedge^{\mathrm{top}} H^0(X,L))\otimes(\bigwedge^{\mathrm{top}} H^1(X,L)^*)$, and will be denoted by $\det(L)$. Take a Riemannian metric g on X compatible with the conformal structure of X. Fix a Hermitian metric h on L. Using g and h, a Hermitian structure can be constructed on $\Omega^i(X,L)$, the space of i-forms on X with values in L. Moreover the vector space $H^1(X,L)$ is isomorphic, in a natural way, with the space of harmonic 1-forms with values in L. Consequently the vector spaces $H^0(X,L)$ and $H^1(X,L)$ are equipped with Hermitian structures which in turn induce a Hermitian structure on $\det(L)$—this metric on $\det(L)$ is usually called the L^2 metric. Let $\Delta:=\bar\partial^*\bar\partial$ be the Laplacian acting on the space of smooth sections of L. Let $\{\lambda_i\}_{i\geqslant 1}$ be the set of non-zero eigenvalues of Δ; let ζ denote the analytic continuation of the function $s\mapsto\sum_i 1/\lambda_i^s$. The *Quillen metric* on $\det(L)$ is defined to be the Hermitian structure on $\det(L)$ obtained by multiplying the L^2 metric with $\exp(\zeta'(0))$, [Q].

To better suit our purposes, we will modify the above (usual) definition of the Quillen metric by a certain factor. Consider the real number $a(X)$ appearing in Théorème 11.4 of [D]. This number $a(X)$ depends only on the genus of X. The statement in Remark 11.5 of [D]—to the effect that there is a constant c such that $a(X)=c\cdot\chi(X)$, where $\chi(X)$ is the Euler characteristic of X—has been established in [GS]. (The constant c is related to the derivative at (-1) of the zeta function for the trivial Hermitian line bundle on $\mathbf{C}P^1$ (4.1.7 of [GS]).) Let $H_Q(L;g,h)$ denote the Quillen metric on $\det(L)$ defined above. Henceforth, by *Quillen metric* on $\det(L)$ we will mean the Hermitian metric

$$\exp\big(\tfrac{1}{12}a(X)\big)H_Q(L;g,h). \tag{2.1}$$

Next we will describe briefly some key properties of the determinant line and the Quillen metric.

Let $\pi\colon \mathcal{X}\to S$ be a family of compact Riemann surfaces parametrized by a base S. We can work with either holomorphic (Kodaira–Spencer) families over a complex-analytic variety S, or with algebraic families over complex algebraic varieties (or, more generally, over a scheme) S. In the algebraic category one means that π is a proper smooth morphism of relative dimension one with geometrically connected fibers. In the analytic category, π is a holomorphic submersion again with compact and connected fibers. Take a Hermitian line bundle $L_S\to\mathcal{X}$ with Hermitian metric h_S. Fix a Hermitian metric g_S on the relative tangent bundle $T_{\mathcal{X}/S}$.

For any point $s\in S$, the above construction gives a Hermitian line $\det(L_s)$ (the Hermitian structure is given by the Quillen metric). The basic fact is that these lines fit together to give a line bundle on S, [KM], which is called the *determinant bundle* of L_S, and is denoted by $\det(L_S)$. Moreover the function on the total space of $\det(L_S)$ given by the norm with respect to the Quillen metric on each fiber is a C^∞ function, and hence it induces a Hermitian metric on $\det(L_S)$, [Q]. This bundle will be denoted by $\det(L_S)$. We shall make clear in Remark 2.13 below that this "determinant of cohomology" line bundle is also an algebraic or analytic bundle—according to the category within which we work.

The determinant bundle $\det(L_S)$ is functorial with respect to base change. We describe what this means. For a morphism $\gamma\colon S'\to S$ consider the bundle, $p_2^*L_S\to S'\times_S\mathcal{X}$, on the fiber product, where $p_2\colon S'\times_S\mathcal{X}\to\mathcal{X}$ is the projection onto the second factor. The Hermitian structure h_S pulls back to a Hermitian structure on $L_{S'}:=p_2^*L_S$; and, similarly, the metric g_S induces a Hermitian structure on the relative tangent bundle of $S'\times_S\mathcal{X}$. "Functorial with respect to base change" now means that in the above situation there is a *canonical isometric isomorphism*

$$\varrho_{S',S}\colon \det(L_{S'})\to\gamma^*\det(L_S)$$

such that if

$$S''\xrightarrow{\gamma'}S'\xrightarrow{\gamma}S$$

are two morphisms then the following diagram is commutative:

$$\begin{tikzcd} \det(L_{S''}) \arrow[r, "\varrho_{S'',S'}"] \arrow[d, "\varrho_{S'',S}"] & \gamma'^*(\det(L_{S'})) \arrow[d, "\gamma'^*\varrho_{S',S}"] \\ (\gamma\circ\gamma')^*\det(L_S) \arrow[r, "\mathrm{id}"] & (\gamma\circ\gamma')^*\det(L_S) \end{tikzcd} \tag{2.2}$$

The determinant of cohomology construction $\det(L_S)$ produces a bundle over the parameter space S induced by the bundle over the total space $\mathcal{X}$; now, the Grothendieck–Riemann–Roch (GRR) theorem gives a canonical isomorphism of $\det(L_S)$ with a combination of certain bundles obtained (on S) from the direct images of the bundle L_S and the relative tangent bundle $T_{\mathcal{X}/S}$. In order to relate canonically the determinant bundle obtained from a given family $\mathcal{X}\to S$ (fibers of genus g, say) with the determinant arising from a covering family $\widetilde{\mathcal{X}}$ (having fibers of some higher genus $\tilde{g}$), we shall utilize the GRR theorem in a formulation due to Deligne, [D, Theorem 9.9 (iii)].

In fact, Deligne introduces a "bilinear pairing" that associates a line bundle, denoted by $\langle L_S, M_S\rangle$, over S from any pair of line bundles L_S and M_S over the total space of the fibration $\mathcal{X}\to S$. If L_S and M_S carry Hermitian metrics then a canonically determined Hermitian structure gets induced on the Deligne pairing bundle $\langle L_S, M_S\rangle$ as well. Denoting by $\mathcal{L}=L_S$ the given line bundle over $\mathcal{X}$, the GRR theorem in Deligne's formulation reads:

$$\det(\mathcal{L})^{\otimes 12} = \langle T^*_{\mathcal{X}/S}, T^*_{\mathcal{X}/S}\rangle \otimes \langle \mathcal{L}, \mathcal{L}\otimes T_{\mathcal{X}/S}\rangle^{\otimes 6}. \tag{2.3}$$

Here $T^*_{\mathcal{X}/S}$ denotes the relative cotangent bundle over $\mathcal{X}$, and the equality asserts that there is a *canonical isomorphism, functorial with respect to base change*, between the bundles on the two sides. Furthermore, Théorème 11.4 of [D] says that the canonical identification in (2.3) is actually an *isometry* with the Quillen metric on the left side and the Deligne pairing metrics on the right. (The constant $\exp(a(X))$ in the statement of Théorème 11.4 of [D] has been absorbed in the definition (2.1).) We proceed to explain the Deligne pairing and the metric thereon in brief; details are to be found in §§ 1.4 and 1.5 of [D].

Let L and M be two line bundles on a compact Riemann surface X. For a pair of meromorphic sections l and m of L and M, respectively, with the divisor of l being disjoint from the divisor of m, let $\mathbf{C}\langle l, m\rangle$ be the 1-dimensional vector space with the symbol $\langle l, m\rangle$ as the generator. For two meromorphic functions f and g on X such that $\mathrm{div}(f)$ is disjoint from $\mathrm{div}(m)$ and $\mathrm{div}(g)$ is disjoint from $\mathrm{div}(l)$, the following identifications of complex lines are to be made:

$$\begin{aligned}\langle fl, m\rangle &= f(\mathrm{div}(m))\langle l, m\rangle,\\ \langle l, gm\rangle &= g(\mathrm{div}(l))\langle l, m\rangle.\end{aligned} \tag{2.4}$$

The Weil reciprocity law says that for any two meromorphic functions f_1 and f_2 on X with disjoint divisors, $f_1(\mathrm{div}(f_2))=f_2(\mathrm{div}(f_1))$, [GH, p. 242]. So we have

$$\langle fl, gm\rangle = f(\mathrm{div}(gm))\cdot g(\mathrm{div}(l))\langle l, m\rangle = g(\mathrm{div}(fl))\cdot f(\mathrm{div}(m))\langle l, m\rangle.$$

From the above equality it follows that the identifications in (2.4) produce a complex 1-dimensional vector space, denoted by $\langle L, M\rangle$, from the pair of line bundles L and M. If L and M are both equipped with Hermitian metrics then the Hermitian metric on $\mathbf{C}\langle l, m\rangle$ defined by

$$\log \|\langle l, m\rangle\| := \frac{1}{2\pi i}\int_X \partial\bar{\partial}(\log\|l\|\cdot\log\|m\|)+\log\|l\|(\operatorname{div}(m))+\log\|m\|(\operatorname{div}(l)) \qquad (2.5)$$

is compatible with the relations in (2.4)—hence it gives a Hermitian structure on $\langle L, M\rangle$, see [D, 1.5.1].

Consider now a family of Riemann surfaces $\mathcal{X}\to S$; let L_S and M_S be two line bundles on $\mathcal{X}$, equipped with Hermitian structures. Over an open subset $U\subset S$, let l_U, m_U be two meromorphic sections of L_S and M_S, respectively, with finite supports over U such that the support of l_U is disjoint from the support of m_U. (Support of a section is the divisor of the section.) For another open set V and two such sections l_V and m_V, the relations in (2.4) give a function

$$C_{U,V}\in\mathcal{O}^*_{U\cap V}.$$

Using the Weil reciprocity law it can be shown that $\{C_{U,V}\}$ forms a 1-cocycle on S. In other words, we get a line bundle on S, which we will denote by $\langle L_S, M_S\rangle$. The Hermitian structure on $\langle L, M\rangle$, described earlier, makes $\langle L_S, M_S\rangle$ into a Hermitian bundle.

Given a meromorphic section m of M_S, let $m^{\otimes n}$ be the meromorphic section of M_S^n obtained by taking the nth tensor power of m. Note that $\operatorname{div}(m^{\otimes n})=n\cdot\operatorname{div}(m)$. The map $\langle l, m^{\otimes n}\rangle\to\langle l, m\rangle^{\otimes n}$ can be checked to be compatible with the relations (2.4), and hence it induces an isomorphism

$$\langle L_S, M_S^n\rangle\to\langle L_S, M_S\rangle^n. \qquad (2.6)$$

From the definition (2.5) we see that (2.6) is an isometry for the metric on M_S^n induced by the metric on M_S.

We shall now see how the critical formula (2.3) follows from the general GRR theorem of [D]. Indeed, let $\mathcal{L}$ denote any rank n vector bundle on the total space of the family $\mathcal{X}$; we reproduce below the statement of Theorem 9.9 (iii) of [D]:

$$\det(\mathcal{L})^{\otimes 12}=\langle T^*_{\mathcal{X}/S}, T^*_{\mathcal{X}/S}\rangle\otimes\langle\Lambda^n(\mathcal{L}), \Lambda^n(\mathcal{L})\otimes T_{\mathcal{X}/S}\rangle^{\otimes 6}\otimes I_{\mathcal{X}/S}C^2(\mathcal{L})^{-12}. \qquad \text{(GRR-D)}$$

Now, from the definition of $I_{\mathcal{X}/S}C^2$ in [D, 9.7.2] it follows that for a line bundle $\mathcal{L}$, the bundle $I_{\mathcal{X}/S}C^2(\mathcal{L})$ is the trivial bundle on S, and the metric on it is the constant metric, [D, Theorem 10.2 (i)]. From Théorème 11.4 of [D] we conclude that that the canonical identification in the statement above is actually an isometric identification. (The factor

$\exp(a(X))$ in Théorème 11.4 of [D] is taken care of by the definition (2.1).) Thus we have obtained the isometric isomorphism stated in (2.3).

With this background behind us we can formulate our main lemma. Let $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ be two families of Riemann surfaces over S (say with fibers of genus g and $\tilde{g}$, respectively), and $p: \widetilde{\mathcal{X}} \to \mathcal{X}$ be an étale (i.e. unramified) covering of degree d, commuting with the projections onto S. In other words, the map p fits into the following commutative diagram:

$$\begin{tikzcd} \widetilde{\mathcal{X}} \arrow[rr, "p"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd} \tag{2.7}$$

The situation implies that each fiber of the family $\widetilde{\mathcal{X}}$ is a degree $d=(\tilde{g}-1)/(g-1)$ holomorphic covering over the corresponding fiber of the family $\mathcal{X}$. Fix also a Hermitian metric g on $T_{\mathcal{X}/S}$. Since p is étale, $p^*T_{\mathcal{X}/S}=T_{\widetilde{\mathcal{X}}/S}$, and hence g induces a Hermitian metric p^*g on $T_{\widetilde{\mathcal{X}}/S}$. Let $\mathcal{X}' \to S$ be a third family of Riemann surfaces which is again an étale cover of $\widetilde{\mathcal{X}}$ and fits into the following commutative diagram:

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "q"] \arrow[dr] & \widetilde{\mathcal{X}} \arrow[r, "p"] \arrow[d] & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd} \tag{2.8}$$

We want to prove the following:

LEMMA 2.9. (i) *Let $\mathcal{L}$ be a Hermitian line bundle on $\mathcal{X}$ and let $p^*\mathcal{L} \to \widetilde{\mathcal{X}}$ be the pullback of $\mathcal{L}$ equipped with the pullback metric. Then there is a canonical isometric isomorphism*

$$\det(p^*(\mathcal{L}))^{\otimes 12} \cong \det(\mathcal{L})^{\otimes 12 \cdot \deg(p)}$$

of bundles on S. This isomorphism is functorial with respect to base change.

(ii) *Denoting the isometric isomorphism obtained in* (i) *by $\Gamma(p)$, and similarly defining $\Gamma(q)$ and $\Gamma(p \circ q)$, the following diagram commutes*:

$$\begin{tikzcd} \det((p\circ q)^*(\mathcal{L}))^{\otimes 12} \arrow[r, "\Gamma(q)"] \arrow[d, "\Gamma(p\circ q)"] & \det(p^*(\mathcal{L}))^{\otimes 12\cdot\deg(q)} \arrow[d, "\Gamma(p)^{\otimes \deg(q)}"] \\ \det(\mathcal{L})^{\otimes 12\cdot\deg(p\circ q)} \arrow[r, "\mathrm{id}"] & \det(\mathcal{L})^{\otimes 12\cdot\deg(p\circ q)} \end{tikzcd}$$

where $\Gamma(p)^{\otimes \deg(q)}$ is the isomorphism on appropriate bundles, obtained by taking the $\deg(q)$*-th tensor product of the isomorphism $\Gamma(p)$.*

(The terminology "functorial with respect to base change" was explained earlier. We will use "canonical" to mean functorial with respect to base change.)

Proof of Lemma 2.9. The idea of the proof is to relate—utilizing GRR in form (2.3)—the determinant bundles, which are difficult to understand, with the more tractable "Deligne pairings".

Let $\mathcal{M}$ be any line bundle on $\widetilde{\mathcal{X}}$ equipped with a Hermitian structure. First we want to show that there is a canonical isometric isomorphism

$$\langle p^*\mathcal{L}, \mathcal{M}\rangle \to \langle \mathcal{L}, N(\mathcal{M})\rangle, \tag{2.10}$$

where $N(\mathcal{M}) \to \mathcal{X}$ is the norm of $\mathcal{M}$. We recall the definition of $N(\mathcal{M})$. The direct image $R^0p_*(\mathcal{M})$ is locally free on $\mathcal{X}$, and the bundle $R^0p_*(\mathcal{M})$ admits a natural reduction of structure group to the *monomial group* $G \subset \mathrm{GL}(\deg(p), \mathbf{C})$. (The group G is the semi-direct product of permutation group, $P_{\deg(p)}$, with the invertible diagonal matrices, defined using the permutation action of $P_{\deg(p)}$.) Mapping $g \in G$ to the permanent of g (on G it is simply the product of all non-zero entries) we get a homomorphism to $\mathbf{C}^*$, which is denoted by μ. Using this homomorphism μ we have a holomorphic line bundle on $\mathcal{X}$, associated to $R^0p_*(\mathcal{M})$, which is defined to be $N(\mathcal{M})$. Clearly the fiber of $N(\mathcal{M})$ over a point $x \in \mathcal{X}$ is the tensor product

$$N(\mathcal{M})_x = \bigotimes_{y \in p^{-1}(x)} \mathcal{M}_y. \tag{2.11}$$

The Hermitian metric on $\mathcal{M}$ gives a reduction of the structure group of $R^0p_*(\mathcal{M})$ to the maximal compact subgroup $G_U \subset G$. Since $\mu(G_U) = U(1)$, we have a Hermitian metric on $N(\mathcal{M})$. Note that the Hermitian metric on $N(\mathcal{M})$ is such that the above equality (2.11) is actually an isometry.

For a meromorphic section m of M, the above identification of fibers gives a meromorphic section of $N(M)$ which is denoted by $N(m)$. Given sections l and m of $\mathcal{L}$ and $\mathcal{M}$, respectively, with finite support over $U \subset S$ (the support of p^*l and m being assumed disjoint) we map $\langle p^*l, m\rangle$ to $\langle l, N(m)\rangle$. It can be checked that this map is compatible with the relations in (2.4). Hence we get a homomorphism from the bundle $\langle p^*\mathcal{L}, \mathcal{M}\rangle$ to $\langle \mathcal{L}, N(\mathcal{M})\rangle$; this is our candidate for (2.10). To check that it is an isometry, we evaluate the (logarithms of) norms of the sections $<p^*l, m>$ and $<l, N(m)>$ given by definition (2.5). It is easy enough to see from (2.5) that the norms of these two sections coincide.

Therefore for a Hermitian line bundle $\mathcal{L}'$ on $\mathcal{X}$, the isomorphism (2.10) implies that

$$\langle p^*\mathcal{L}, p^*\mathcal{L}'\rangle = \langle \mathcal{L}, N(p^*\mathcal{L}')\rangle.$$

But $N(p^*\mathcal{L}') = \mathcal{L}'^d$, where $d := \deg(p)$, and moreover the Hermitian metric on $N(p^*\mathcal{L}')$ coincides with that of $\mathcal{L}'^d$. Hence from the isometric isomorphism obtained in (2.6) we get the following identification of Hermitian line bundles (the isomorphism so created being again functorial with respect to change of base space):

$$\langle p^*\mathcal{L}, p^*\mathcal{L}'\rangle = \langle \mathcal{L}, \mathcal{L}'\rangle^d. \tag{2.12}$$

To prove part (i) of the lemma we apply the GRR isomorphism (2.3) to both $\mathcal{L}$ and $p^*\mathcal{L}$, and compare the Deligne pairing bundles appearing on the right hand sides using the result (2.12). To simplify notation set $\omega = T^*_{\mathcal{X}/S}$. By applying (2.3) to $p^*\mathcal{L}$, and noting that since the map p is étale, the relative tangent bundle $T_{\widetilde{\mathcal{X}}/S} = p^*T_{\mathcal{X}/S}$, we deduce that $\det(p^*\mathcal{L})^{\otimes 12}$ is canonically isometrically isomorphic to $\langle p^*\mathcal{L}, p^*(\mathcal{L}\otimes\omega^{-1})\rangle^{\otimes 6} \otimes \langle p^*\omega, p^*\omega\rangle$. Taking $\mathcal{L}'$ to be $\mathcal{L}\otimes\omega$ in (2.12) we have $\langle \mathcal{L}, \mathcal{L}\otimes\omega\rangle^d = \langle p^*\mathcal{L}, (p^*\mathcal{L}\otimes\omega)\rangle$. Substituting ω in place on $\mathcal{L}$ and $\mathcal{L}'$ in (2.12) we have $\langle\omega,\omega\rangle^d = \langle p^*\omega, p^*\omega\rangle$. Therefore the bundle $\langle p^*\mathcal{L}, p^*(\mathcal{L}\otimes\omega^{-1})\rangle^{\otimes 6}\otimes\langle p^*\omega, p^*\omega\rangle$ is isometrically isomorphic to $\langle \mathcal{L}, \mathcal{L}\otimes\omega^{-1}\rangle^{6d}\otimes\langle\omega,\omega\rangle^d$. But now applying (2.3) to $\mathcal{L}$ itself we see that this last bundle is isometrically isomorphic to $\det(\mathcal{L})^{\otimes 12d}$. That completes the proof. Notice that since all isomorphisms used in the above proof were canonical (functorial with base change), the final isomorphism asserted in part (i) is also canonical in the same sense.

In order to prove part (ii) of the lemma, we first note that the isometric isomorphisms in (2.10) and (2.12) actually fit into the following commutative diagram:

$$\begin{array}{ccc}
\langle (p\circ q)^*\mathcal{L}, \mathcal{M}\rangle & \longrightarrow & \langle p^*\mathcal{L}, N(\mathcal{M})_q\rangle \\
\downarrow & & \downarrow \\
\langle \mathcal{L}, N(\mathcal{M})\rangle & = & \langle \mathcal{L}, N(\mathcal{M})\rangle
\end{array}$$

where $\mathcal{L}$ is a Hermitian line bundle on $\mathcal{X}$, and $\mathcal{M}$ is a Hermitian line bundle on $\mathcal{X}'$, $N(\mathcal{M})\to\mathcal{X}$ is the norm of $\mathcal{M}$ for the covering $p\circ q$, and $N(\mathcal{M})_q\to\widetilde{\mathcal{X}}$ is the norm of $\mathcal{M}$ for the covering q. Indeed, the commutativity of the above diagram is straightforward to deduce from the fact that the following two bundles on $\mathcal{X}$: namely, $N(\mathcal{M})$ and the norm of $N(\mathcal{M})_q$, are isometrically isomorphic. The isomorphism can be defined, for example, using (2.11). Now using (2.3), and repeatedly using the above commutative diagram, we obtain part (ii). □

We will have occasion to use this general lemma in concrete situations.

Remark 2.13. In [KM] and in [D] the basic context is the algebraic families category, and the determinant of cohomology bundle as well as the Deligne pairing bundles are constructed in this category. However, since the constructions of the determinant bundles and of the Deligne pairing are *canonical* and *local*, they work equally well for holomorphic families of Riemann surfaces also. The point is that if $\mathcal{X}\to S$ is a holomorphic family of

Riemann surfaces parametrized by a complex manifold S, and $\mathcal{L}\to\mathcal{X}$ is a holomorphic line bundle, then $\det(\mathcal{L})\to S$ is a holomorphic line bundle which is functorial with respect to holomorphic base changes. And if $\mathcal{L}$ and $\mathcal{M}$ are two holomorphic line bundles on $\mathcal{X}$ then $\langle\mathcal{L},\mathcal{M}\rangle$ is a holomorphic line bundle on S—again functorial with respect to holomorphic base changes. In fact, an analytic construction of the determinant bundle and the Quillen metric is to be found in [BGS].

Since the constructions of the Quillen metric and the metric on the Deligne pairing, (using (2.5)), also hold true for holomorphic families, consequently, Lemma 2.9 holds in the holomorphic category as well as in the algebraic category.

Remark 2.14. The statement that $\det(p^*(\mathcal{L}))^{\otimes 12}\cong\det(\mathcal{L})^{\otimes 12\cdot\deg(p)}$ as line bundles actually holds for curves over any field. The statement about isometry makes sense only when we have Riemann surfaces.

3. Determinant bundles over Teichmüller spaces

Our aim in this section is to apply Lemma 2.9 to the universal family of marked Riemann surfaces of genus g over the Teichmüller space $\mathcal{T}_g$. The situation of Lemma 2.9 is precipitated by choosing any finite covering space over a topological surface of genus g.

Let $\pi\colon\widetilde{X}\to X$ be an unramified covering map between two compact connected oriented surfaces $\widetilde{X}$ and X of genera $\tilde{g}$ and g, respectively. Assume that $g\geqslant 2$. The degree of the covering π, which will play an important role, is the ratio of the respective Euler characteristics; namely, $\deg(\pi)=(\tilde{g}-1)/(g-1)$.

We recall the basic deformation spaces of complex (conformal) structures on smooth closed oriented surfaces—the Teichmüller spaces. Let $\mathrm{Conf}(X)$ (or $\mathrm{Conf}(\widetilde{X})$) denote the space of all smooth conformal structures on X (or $\widetilde{X}$). Define $\mathrm{Diff}^+(X)$ (or $\mathrm{Diff}^+(\widetilde{X})$) to be the group of all orientation preserving diffeomorphisms of X (or $\widetilde{X}$), and denote by $\mathrm{Diff}_0^+(X)$ (or $\mathrm{Diff}_0^+(\widetilde{X})$) the subgroup of those that are homotopic to the identity.

The group $\mathrm{Diff}^+(X)$ acts naturally on $\mathrm{Conf}(X)$ by pullback of conformal structure. We define

$$\mathcal{T}(X)=\mathcal{T}_g:=\mathrm{Conf}(X)/\,\mathrm{Diff}_0^+(X) \tag{3.1}$$

to be the Teichmüller space of genus g (marked) Riemann surfaces. Similarly obtain $\mathcal{T}_{\tilde{g}}:=\mathrm{Conf}(\widetilde{X})/\,\mathrm{Diff}_0^+(\widetilde{X})$—the Teichmüller space for genus $\tilde{g}$. The Teichmüller space $\mathcal{T}_g$ carries naturally the structure of a $(3g-3)$-dimensional complex manifold which is embeddable as a contractible domain of holomorphy in the affine space $\mathbf{C}^{3g-3}$. The mapping class group of the genus g surface, namely the discrete group $\mathrm{MC}_g:=\mathrm{Diff}^+(X)/\,\mathrm{Diff}_0^+(X)$, acts properly discontinuously on $\mathcal{T}_g$ by holomorphic automorphisms, the quotient being the moduli space $\mathcal{M}_g$. For these basic facts see, for example, [Na].

The Teichmüller spaces are fine moduli spaces. In fact, the total space $X \times \mathcal{T}_g$ admits a natural complex structure such that the projection to the second factor

$$\psi_g \colon \mathcal{C}_g := X \times \mathcal{T}_g \to \mathcal{T}_g \tag{3.2}$$

gives the universal Riemann surface over $\mathcal{T}_g$. This means that for any $\eta \in \mathcal{T}_g$, the submanifold $X \times \eta$ is a complex submanifold of $\mathcal{C}_g$, and the complex structure on X induced by this embedding is represented by η. As is well-known ([Na, Chapter 5]), the family $\mathcal{C}_g \to \mathcal{T}_g$ is the *universal* object in the category of holomorphic families of genus g marked Riemann surfaces.

Given a complex structure on X, using π we may pull back this to a complex structure on $\widetilde{X}$. This gives an injective map $\mathrm{Conf}(X) \to \mathrm{Conf}(\widetilde{X})$. Given an element $f \in \mathrm{Diff}_0^+(X)$, from the homotopy lifting property, there is a unique diffeomorphism $\tilde{f} \in \mathrm{Diff}_0^+(\widetilde{X})$ such that $\tilde{f}$ is a lift of f. Mapping f to $\tilde{f}$ defines an injective homomorphism of $\mathrm{Diff}_0^+(X)$ into $\mathrm{Diff}_0^+(\widetilde{X})$. We therefore obtain an injection

$$\mathcal{T}(\pi) \colon \mathcal{T}_g \to \mathcal{T}_{\tilde{g}}. \tag{3.3}$$

It is known that this map $\mathcal{T}(\pi)$ is a *proper holomorphic embedding* between these finite-dimensional complex manifolds; $\mathcal{T}(\pi)$ respects the quasiconformal-distortion (=Teichmüller) metrics. From the definitions it is evident that this embedding between the Teichmüller spaces depends only on the (unbased) isotopy class of the covering π.

Remark 3.4. In fact, we see that $\mathcal{T}$ is thus a contravariant functor from the category of closed oriented topological surfaces, morphisms being covering maps, to the category of finite-dimensional complex manifolds and holomorphic embeddings. We shall have more to say about this in §5.

Over each genus Teichmüller space we have a sequence of natural determinant bundles arising from the powers of the relative (co-)tangent bundles along the fibers of the universal curve. Indeed, let $\omega_g \to \mathcal{C}_g$ be the relative cotangent bundle for the projection ψ_g in (3.2). The determinant line bundle over $\mathcal{T}_g$ arising from its nth tensor power is fundamental, and we shall denote it by

$$\mathrm{DET}_{n,g} := \det(\omega_g^n) \to \mathcal{T}_g, \quad n \in \mathbf{Z}. \tag{3.5}$$

Applying Serre duality shows that there is a canonical isomorphism $\mathrm{DET}_{n,g} = \mathrm{DET}_{1-n,g}$, for all n. $\mathrm{DET}_{0,g} = \mathrm{DET}_{1,g}$ is called the *Hodge* line bundle over $\mathcal{T}_g$.

These holomorphic line bundles carry natural *Quillen Hermitian structure* arising from the Poincaré metrics on the fibers of the universal curve. Recall that any Riemann surface Y of genus $g \geqslant 2$ admits a unique conformal Riemannian metric of constant curvature -1, called the Poincaré metric of Y. This metric depends smoothly on the conformal structure (because of the uniformization theorem with moduli parameters), and hence, for a family of Riemann surfaces of genus at least two, the Poincaré metric induces a Hermitian metric on the relative tangent/cotangent bundle. We thus obtain Quillen metrics on each $\mathrm{DET}_{n,g}$. The metric functorially assigned by the Quillen metric on any tensor power of $\mathrm{DET}_{n,g}$ will also be referred to as the Quillen metric on that tensor power.

Observe that by the naturality of the above constructions it follows that the action of MC_g on $\mathcal{T}_g$ has a natural lifting as unitary automorphisms of these DET bundles.

We invoke back into play the unramified finite covering $\pi: \widetilde{X} \to X$. Let

$$\mathcal{T}(\pi)^* \mathcal{C}_{\tilde{g}} \to \mathcal{T}_g \tag{3.6}$$

be the pull-back to $\mathcal{T}_g$ of the universal family $\mathcal{C}_{\tilde{g}} \to \mathcal{T}_{\tilde{g}}$ using the map $\mathcal{T}(\pi)$. Given the topological covering space π we therefore obtain the following étale covering map between families of Riemann surfaces parametrized by $\mathcal{T}_g$:

$$\pi \times \mathrm{id}: \mathcal{T}(\pi)^* \mathcal{C}_{\tilde{g}} \to \mathcal{C}_g := X \times \mathcal{T}_g.$$

This is clearly a holomorphic map. In fact, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{T}(\pi)^* \mathcal{C}_{\tilde{g}} \arrow[rr, "\pi \times \mathrm{id}"] \arrow[dr] & & \mathcal{C}_g \arrow[dl] \\ & \mathcal{T}_g & \end{tikzcd} \tag{3.7}$$

exactly as in the general situation (2.7) above Lemma 2.9.

Now let

$$\mathrm{id} \times \mathcal{T}(\pi): \mathcal{T}(\pi)^* \mathcal{C}_{\tilde{g}} \to \mathcal{C}_{\tilde{g}}$$

denote the tautological lift of the map $\mathcal{T}(\pi)$. From the definition of the Poincaré metric it is clear that for an unramified covering of Riemann surfaces, $\widetilde{Y} \to Y$, the Poincaré metric on $\widetilde{Y}$ is the pull-back of the Poincaré metric on Y. If $\omega_{\tilde{g}}$ is the relative cotangent bundle on $\mathcal{C}_{\tilde{g}}$ then this compatibility between Poincaré metrics implies that the two Hermitian line bundles on $\mathcal{T}(\pi)^* \mathcal{C}_{\tilde{g}}$, namely, $(\pi \times \mathrm{id})^* \omega_g$ and $(\mathrm{id} \times \mathcal{T}(\pi))^* \omega_{\tilde{g}}$, are canonically isometric.

But since the determinant bundle of a pullback family is the pullback of the determinant bundle, the holomorphic Hermitian bundle $\mathcal{T}(\pi)^*(\det(\omega_{\tilde{g}}^n)) \to \mathcal{T}_g$ is canonically isometrically isomorphic to the determinant bundle of $(\mathrm{id}\times\mathcal{T}(\pi))^*\omega_{\tilde{g}}^n\to\mathcal{T}(\pi)^*\mathcal{C}_{\tilde{g}}$. Using this and simply applying Lemma 2.9 to the commutative diagram (3.7) we obtain the following theorem. (All the Quillen metrics are with respect to the Poincaré metric on fibers.)

THEOREM 3.8a. *The two holomorphic Hermitian line bundles* $\det(\omega_g^n)^{12\cdot\deg(\pi)}$ *and* $\mathcal{T}(\pi)^*(\det(\omega_{\tilde{g}}^n))^{12}$ *on* $\mathcal{T}_g$ *are canonically isometrically isomorphic for every integer* n. *In other words, there is a canonical isometrical line bundle morphism* $\Gamma(\pi)$ *lifting* $\mathcal{T}(\pi)$ *and making the following diagram commute*:

$$\begin{tikzcd}
\mathrm{DET}_{n,g}^{\otimes 12\cdot\deg(\pi)} \arrow[r, "\Gamma(\pi)"] \arrow[d] & \mathrm{DET}_{n,\tilde{g}}^{\otimes 12} \arrow[d] \\
\mathcal{T}_g \arrow[r, "\mathcal{T}(\pi)"] & \mathcal{T}_{\tilde{g}}
\end{tikzcd}$$

Remark 3.9. The bundle morphism $\Gamma(\pi)$ has been obtained from Riemann–Roch isomorphisms—as evinced by the proof of Lemma 2.9. We shall therefore, in the sequel, refer to these canonical mappings as GRR morphisms. Tensor powers of the GRR morphisms will also be referred to as GRR morphisms. The functoriality of these morphisms is explained below in Theorem 3.8b.

Let $\overline{X}\xrightarrow{\varrho}\widetilde{X}\xrightarrow{\pi}X$ be two unramified coverings between closed surfaces of respective genera $\bar{g}$, $\tilde{g}$ and g. By applying the Teichmüller functor we have the corresponding commuting triangle of embeddings between the Teichmüller spaces:

$$\begin{tikzcd}
\mathcal{T}_g \arrow[rr, "\mathcal{T}(\pi)"] \arrow[dr] & & \mathcal{T}_{\tilde{g}} \arrow[dl] \\
& \mathcal{T}_{\bar{g}} &
\end{tikzcd} \tag{3.10}$$

Here the two slanting embeddings are, of course, $\mathcal{T}(\pi\circ\varrho)$ and $\mathcal{T}(\varrho)$. Applying Lemma 2.9 (ii) we have

THEOREM 3.8b. *The following triangle of GRR line bundle morphisms commutes*:

$$\begin{tikzcd}
\mathrm{DET}_{n,g}^{\otimes 12\cdot\deg(\pi\circ\varrho)} \arrow[rr] \arrow[dr] & & \mathrm{DET}_{n,\tilde{g}}^{\otimes 12\cdot\deg(\varrho)} \arrow[dl] \\
& \mathrm{DET}_{n,\bar{g}}^{\otimes 12} &
\end{tikzcd} \tag{3.11}$$

All three maps in the diagram are obtained by applications of Theorem 3.8a, *and raising to the appropriate tensor powers. The triangle above sits over the triangle of Teichmüller spaces* (3.10), *and the entire triangular prism is a commutative diagram.*

Remark 3.12. The nagging factor of 12 in Theorems 3.8a and 3.8b can be dealt with as follows. The Teichmüller space being a contractible Stein domain, any two line bundles on $\mathcal{T}_g$ are isomorphic. Choose an isomorphism between $\delta\colon \det(\omega_g^n)^{\deg(\pi)} \to \mathcal{T}(\pi)^*(\det(\omega_{\tilde{g}}^n))$. Hence

$$\delta^{\otimes 12}\colon \det(\omega_g^n)^{12\cdot\deg(\pi)} \to \mathcal{T}(\pi)^*(\det(\omega_{\tilde{g}}^n))^{12}$$

is an isomorphism. Let

$$\tau\colon \det(\omega_g^n)^{12\cdot\deg(\pi)} \to \mathcal{T}(\pi)^*(\det(\omega_{\tilde{g}}^n))^{12}$$

be the isomorphism given by Theorem 3.8a. So $f{:=}\tau\circ(\delta^{\otimes 12})^{-1}$ is a nowhere zero function on $\mathcal{T}_g$. Since $\mathcal{T}_g$ is simply connected, there is a function h on $\mathcal{T}_g$ such that $h^{12}{=}f$. Any two such choices of h will differ by a 12th root of unity. Consider the homomorphism $\bar{\tau}{:=}h\cdot\delta$. Clearly $\bar{\tau}^{\otimes 12}{=}\tau$. It is easy to see that for two different choices of the isomorphism δ, the two $\bar{\tau}$'s differ by multiplication with a 12th root of unity. Moreover, if we consider a similar diagram to that in Theorem 3.8b with the factor 12 removed and all the homomorphisms being replaced by the corresponding analogues of $\bar{\tau}$, then the diagram commutes up to multiplication with a 12th root of unity.

Remark 3.13. Recall from above that the action of MC_g in $\mathcal{T}_g$ lifts to the total space of $\det(\omega_g^n)$ as bundle automorphisms preserving the Quillen metric. There is no action, a priori, of MC_g on the total space of the the pullback bundle $\mathcal{T}(\pi)^*(\det(\omega_{\tilde{g}}^n))$. However, from Theorem 3.8a the bundle $\mathcal{T}(\pi)^*(\det(\omega_{\tilde{g}}^n))^{12}$ gets an action of MC_g which preserves the pulled back Quillen metric. Theorem 3.8b ensures the identity between the MC_g actions obtained by different pullbacks.

In [BN] we will consider two special classes of coverings, namely characteristic covers and cyclic covers. In such situations the map between Teichmüller spaces, induced by the covering, actually descends to a map between moduli spaces (possibly with level structure). As mentioned in the Introduction, in that context we were able to give a proof of the existence of the GRR morphism of Theorem 3.8a using Weil–Petersson geometry and topology.

4. Power law (principal) bundle morphisms over Teichmüller spaces

We desire to obtain certain canonical geometric objects over the inductive limits of the finite-dimensional Teichmüller spaces by coherently fitting together the determinant line bundles $\mathrm{DET}_{n,g}$ thereon; the limit is taken as g increases by running through a universal tower of covering maps. To this end it is necessary to find canonical mappings relating $\mathrm{DET}_{n,g}$ to $\mathrm{DET}_{n,\tilde{g}}$ where genus $\tilde{g}$ covers genus g.

Now, given any complex line bundle $\lambda \to T$ over any base T, there is a certain canonical mapping of λ to any positive integral (dth) tensor power of itself, given by

$$\omega_d: \lambda \to \lambda^{\otimes d}, \tag{4.1}$$

where ω_d on any fiber of λ is the map $z \mapsto z^d$. Observe that ω_d maps λ minus its zero section to $\lambda^{\otimes d}$ minus its zero section by a map which is of degree d on the $\mathbf{C}^*$ fibers. We record the following properties of these maps:

(4.1a) The map ω_d is defined independent of any choices of basis, and it is evidently compatible with base change. (Namely, if we pull back both λ and λ^d over some base $T_1 \to T$, then the connecting map ω_d (over T) also pulls back to the corresponding ω_d over T_1.)

(4.1b) The map ω_d is a homomorphism of the corresponding $\mathbf{C}^*$ principal bundles. When T is a complex manifold, and λ is a line bundle in that category, then the map ω_d is a holomorphic morphism between the total spaces of the source and target bundles.

(4.1c) If λ is equipped with a Hermitian fiber metric, and its tensor powers are assigned the corresponding Hermitian structures, the map ω_d carries the unit circles to unit circles. (The choice of a unit circle amongst the natural family of zero-centered circles in any complex line is clearly equivalent to specifying a Hermitian norm. In this section we will think of Hermitian structure on a line bundle as the choice of a smoothly varying family of unit circles in the fibers.)

Thus, given a topological covering $\pi: \widetilde{X} \to X$, as in the situation of Theorem 3.8a, we may define a canonical map

$$\Omega(\pi) := \Gamma(\pi) \circ \omega_{\deg(\pi)}: \mathrm{DET}_{n,g}^{\otimes 12} \to \mathrm{DET}_{n,\tilde{g}}^{\otimes 12}, \tag{4.2}$$

where $\Gamma(\pi)$ is the canonical GRR line bundle morphism found in Theorem 3.8a. Translating Theorems 3.8a and 3.8b in terms of these holomorphic maps $\Omega(\pi)$ of positive integral fiber degree, we get:

THEOREM 4.3a. *For each integer n, there is a canonical isometrical holomorphic bundle morphism $\Omega(\pi)$ lifting $\mathcal{T}(\pi)$ and making the following diagram commute*:

$$\begin{tikzcd}
\mathrm{DET}_{n,g}^{\otimes 12} \arrow[r, "\Omega(\pi)"] \arrow[d] & \mathrm{DET}_{n,\tilde{g}}^{\otimes 12} \arrow[d] \\
\mathcal{T}_g \arrow[r, "\mathcal{T}(\pi)"] & \mathcal{T}_{\tilde{g}}
\end{tikzcd}$$

By "isometrical" we mean that the unit circles of the Quillen Hermitian structures are preserved by the $\Omega(\pi)$.

THEOREM 4.3b. *Let* π *and* ϱ *denote two composable covering spaces between surfaces, as in the situation of Theorem* 3.8b. *The following triangle of non-linear isometrical holomorphic bundle morphisms commutes*:

$$\begin{tikzcd} \mathrm{DET}_{n,g}^{\otimes 12} \arrow[rr] \arrow[dr] & & \mathrm{DET}_{n,\bar{g}}^{\otimes 12} \arrow[dl] \\ & \mathrm{DET}_{n,\tilde{g}}^{\otimes 12} & \end{tikzcd}$$

The horizontal map is $\Omega(\pi)$, *and the two slanting maps are* (*reading from left to right*) $\Omega(\pi\circ\varrho)$ *and* $\Omega(\varrho)$. *The triangle above sits over the triangle of Teichmüller spaces* (3.10), *and the entire triangular prism is a commutative diagram.*

The canonical and functorial nature of these connecting maps, $\Omega(\pi)$, will now allow us to produce direct systems of line/principal bundles over direct systems of Teichmüller spaces.

5. Commensurability Teichmüller space and its automorphism group

We construct a category $\mathcal{S}$ of certain topological objects and morphisms: the objects, $\mathrm{Ob}(\mathcal{S})$, are a set of compact oriented topological surfaces each equipped with a base point $(*)$, there being exactly one surface of each genus $g\geqslant 0$; let the object of genus g be denoted by X_g. The morphisms are based isotopy classes of pointed covering mappings

$$\pi\colon (X_{\tilde{g}}, *) \to (X_g, *),$$

there being one arrow for each such isotopy class. Note that the monomorphism of fundamental groups induced by (any representative of the based isotopy class) π, is unambiguously defined.

Fix a genus g and let $X=X_g$. Observe that all the morphisms with the fixed target X_g:

$$M_g = \{\alpha \in \mathrm{Mor}(\mathcal{S}) : \mathrm{Range}(\alpha) = X_g\}$$

constitute a *directed set* under the partial ordering given by factorisation of covering maps. Thus if α and β are two morphisms from the above set, then $\beta\succ\alpha$ if and only if the image of the monomorphism $\pi_1(\beta)$ is contained within the image of $\pi_1(\alpha)$; that happens if and only if there is a commuting triangle of morphisms of $\mathcal{S}$ as follows:

$$\begin{tikzcd} X_{g(\beta)} \arrow[rr, "\theta"] \arrow[dr] & & X_{g(\alpha)} \arrow[dl] \\ & X_g & \end{tikzcd}$$

Here $X_{g(\alpha)}$ denotes the domain surface for α (similarly $X_{g(\beta)}$), and the two slanting arrows are (reading from left to right), β and α. It is important to note that the factoring morphism θ is *uniquely* determined because we are working with base points. The directed property of M_g follows by a simple fiber-product argument. (Remark: Notice that the object of genus 1 in $\mathcal{S}$ only has morphisms to itself—so that this object together with all its morphisms (to and from) form a subcategory.)

Recall from §3 that each morphism of $\mathcal{S}$ induces a proper, holomorphic, Teichmüller-metric preserving embedding between the corresponding finite-dimensional Teichmüller spaces. We can thus create the natural *direct system of Teichmüller spaces* over the above directed set M_g, by associating to each $\alpha \in M_g$ the Teichmüller space $\mathcal{T}(X_{g(\alpha)})$, and for each $\beta \succ \alpha$ the corresponding holomorphic embedding $\mathcal{T}(\theta)$ (with θ as in the diagram above). Consequently, we may form the *direct limit Teichmüller space over* $X = X_g$:

$$\mathcal{T}_\infty(X_g) = \mathcal{T}_\infty(X) := \operatorname{ind} \lim \mathcal{T}(X_{g(\alpha)}), \tag{5.1}$$

the inductive limit being taken over all α in the directed set M_g. This is our *commensurability Teichmüller space.*

Remark. Over the same directed set M_g we may also define a natural *inverse system of surfaces*, by associating to $\alpha \in M_g$ a certain copy, S_α of the pointed surface $X_{g(\alpha)}$. (Fix a universal covering over $X = X_g$. S_α can be taken to be the universal covering quotiented by the action of the subgroup $\operatorname{Im}(\pi_1(\alpha)) \subset \pi_1(X, *)$.) If $g \geqslant 2$, then the inverse limit of this system is the *universal solenoidal surface* H_∞ whose Teichmüller theory was studied in [Su], [NS]. The completion of $\mathcal{T}_\infty(X)$ in the Teichmüller metric is $\mathcal{T}(H_\infty)$.

A remarkable but obvious fact about this construction is that every morphism $\pi: Y \to X$ of $\mathcal{S}$ induces a natural Teichmüller metric preserving *homeomorphism*,

$$\mathcal{T}_\infty(\pi): \mathcal{T}_\infty(Y) \to \mathcal{T}_\infty(X). \tag{5.2}$$

$\mathcal{T}_\infty(\pi)$ is invertible simply because the morphisms of $\mathcal{S}$ with target Y are cofinal with those having target X. If we consider objects and maps to be continuous/holomorphic on the inductive limit spaces when they are continuous/holomorphic when restricted to the finite-dimensional strata, then it is clear that $\mathcal{T}_\infty(\pi)$ is a biholomorphic identification. (Note that $\mathcal{T}_\infty$ acts covariantly, since it is defined by a morphism of direct systems, although the Teichmüller functor $\mathcal{T}$ of (3.3) was contravariant.)

It follows that each $\mathcal{T}_\infty(X)$ (and its completion $\mathcal{T}(H_\infty)$) is equipped with a large *automorphism group*—one from each (undirected) cycle of morphisms of $\mathcal{S}$ starting from X and returning to X. By repeatedly using pull-back diagrams (i.e., by choosing the appropriate connected component of the fiber product of covering maps), it is easy to see that the automorphism arising from any (many-arrows) cycle can be obtained simply from a two-arrow cycle $\widetilde{X} \rightrightarrows X$. Namely, whenever we have (the isotopy class of) a "self-correspondence" of X given by two non-isotopic coverings, say α and β,

$$\widetilde{X} \rightrightarrows X, \tag{5.3}$$

we can create an automorphism of $\mathcal{T}_\infty(X)$ defined as the composition: $\mathcal{T}_\infty(\beta)\circ(\mathcal{T}_\infty(\alpha))^{-1}$. Therefore each of these automorphisms—arising from any arbitrarily complicated cycle of coverings (starting and ending at X)—is obtained as one of these simple "two-arrow" compositions. These automorphisms form a group that we shall call the *commensurability modular group*, $\mathrm{CM}_\infty(X)$, acting on the universal commensurability Teichmüller space $\mathcal{T}_\infty(X)$.

We make some further remarks regarding this large new mapping class group. Consider the abstract graph (1-complex), $\Gamma(\mathcal{S})$, obtained from the category $\mathcal{S}$ by looking at the objects as vertices and the (undirected) arrows as edges. It is clear from the definition above that the fundamental group of this graph, viz. $\pi_1(\Gamma(\mathcal{S}), X)$, is acting on $\mathcal{T}_\infty(X)$ as these automorphisms. In fact, we may fill in all the "commuting triangles"—i.e., fill in the 2-cells in this abstract graph whenever two morphisms (edges) compose to give a third edge; the thereby-reduced fundamental group of this 2-complex produces on $\mathcal{T}_\infty(X)$ the action of $\mathrm{CM}_\infty(X)$.

It is interesting to observe that this new modular group $\mathrm{CM}_\infty(X)$ of automorphisms on $\mathcal{T}_\infty(X)$ corresponds exactly to "virtual automorphisms" of the fundamental group $\pi_1(X)$, generalizing the classical situation where the usual automorphism group $\mathrm{Aut}(\pi_1(X))$ appears as the action via modular automorphisms on $\mathcal{T}(X)$.

Indeed, given any group G, one may define its associated group of "virtual" automorphisms; as opposed to usual automorphisms, for virtual automorphisms we demand that they be defined only on some finite index subgroup of G. To be precise, consider isomorphisms $\varrho\colon H \to K$ where H and K are subgroups of finite index in G. Two such isomorphisms (say, ϱ_1 and ϱ_2) are considered equivalent if there is a finite index subgroup (sitting in the intersection of the two domain groups) on which they coincide. The equivalence class $[\varrho]$—which is like the *germ* of the isomorphism ϱ—is called a *virtual auto-*

morphism of G; clearly the virtual automorphisms of G constitute a group, Vaut(G), under the obvious law of composition (namely, compose after passing to deeper finite index subgroups, if necessary).

We shall apply this concept to the fundamental group of a surface of genus g ($g>1$). It is clear from definition that the group $\mathrm{Vaut}(\pi_1(X_g))$ *is genus independent*, as is to be expected in our constructions.

In fact, Vaut presents us a neat way of formalizing the "two-arrow cycles" which we introduced to represent elements of CM_∞. Letting $G=\pi_1(X)$ (recall that X is already equipped with a base point), the two-arrow diagram (5.3) above corresponds to the following well-defined virtual automorphism of G:

$$[\varrho]=[\beta_*\circ\alpha_*^{-1}\colon \alpha_*(\pi_1(\widetilde{X}))\to\beta_*(\pi_1(\widetilde{X}))].$$

Here α_* denotes the monomorphism of the fundamental group $\pi_1(\widetilde{X})$ into $\pi_1(X)=G$, and similarly β_*. We let $\mathrm{Vaut}^+(\pi_1(X))$ denote the subgroup of Vaut arising from pairs of orientation preserving coverings. The final upshot is that $\mathrm{CM}_\infty(X)$ is *isomorphic* to $\mathrm{Vaut}^+(\pi_1(X))$ and there is a natural surjective homomorphism $\pi_1(\Gamma(\mathcal{S}),X)\to \mathrm{Vaut}^+(\pi_1(X))$ whose kernel is obtained by filling in all commuting triangles in $\Gamma(\mathcal{S})$.

Acknowledgement. The concept of Vaut has arisen in group theory papers—for example [Ma], [MT]. We are grateful to Chris Odden for pointing out these references.

Remark 5.4. For the genus one object X_1 in $\mathcal{S}$, we know that the Teichmüller spaces for all unramified coverings are each a copy of the upper half-plane H. The maps $\mathcal{T}(\pi)$ are Möbius identifications of copies of the half-plane with itself, and we easily see that the pair $(\mathcal{T}_\infty(X_1),\mathrm{CM}_\infty(X_1))$ is identifiable as $(H,\mathrm{PGL}(2,\mathbf{Q}))$. In fact, $\mathrm{GL}(2,\mathbf{Q})\cong\mathrm{Vaut}(\mathbf{Z}\oplus\mathbf{Z})$, and Vaut^+ is precisely the subgroup of index 2 therein, as expected. Notice that the action has dense orbits in the genus one case.

On the other hand, if $X\in\mathrm{Ob}(\mathcal{S})$ is of any genus $g\geqslant2$, then we get an infinite-dimensional "ind-space" as $\mathcal{T}_\infty(X)$ with the action of $\mathrm{CM}_\infty(X)$ on it as described. Since the tower of coverings over X and Y (both of genus higher than 1) eventually become cofinal, it is clear that *for any choice of genus higher than one we get one isomorphism class of pairs* $(\mathcal{T}_\infty,\mathrm{CM}_\infty)$. (It is not known whether the action has dense orbits in this situation; this matter is related to some old queries on coverings of Riemann surfaces.)

We work now over the direct system of the higher genus example $(T_\infty,\mathrm{CM}_\infty)$ and obtain the main theorem. We will first explain some preliminary material on direct limits of holomorphic line bundles over a direct system of complex manifolds.

Given a direct system T_α of complex manifolds, and line bundles ξ_α over these, suppose that there are power law maps as the $\Omega(\pi)$ above, between the corresponding principal $\mathbf{C}^*$ bundles covering the mappings in the direct system of base manifolds.

Let N denote the directed set of positive integers ordered by divisibility. For each $\lambda \in N$ take a copy of $\mathbf{C}^*$, call it $(\mathbf{C}^*, \lambda)$ and form the direct system $\{(\mathbf{C}^*, \lambda)\}$, where $(\mathbf{C}^*, \lambda) \to (\mathbf{C}^*, \lambda')$ is given by the power law map: $z \to z^d$ when $\lambda' = d\lambda$. These maps are homomorphisms of groups, and the direct limit over N is canonically isomorphic to the group $\mathbf{C}^* \otimes \mathbf{Q} := \mathbf{C}^* \otimes_{\mathbf{Z}} \mathbf{Q}$. (The isomorphism maps the equivalence class of the element $(z, \lambda) \in (\mathbf{C}^*, \lambda)$ to $z \otimes 1/\lambda \in \mathbf{C}^* \otimes \mathbf{Q}$.) The direct limit object obtained from the power law connecting maps between the principal bundles associated to the DET_n^{12} system over the Teichmüller spaces will give us a $\mathbf{C}^* \otimes \mathbf{Q}$ principal bundle over the universal commensurability Teichmüller space $\mathcal{T}_\infty$, at least at the level of sets. The topological and holomorphic structure on these sets is defined for maps into these objects which factor through the direct system by imposing these properties on the factorizations.

Let us consider the direct limit bundles obtained from a family of such bundles ξ_α, and from the family obtained by raising each ξ_α to the tensor power d. These are two $\mathbf{C}^* \otimes \mathbf{Q}$ bundles over the direct limit of the bases which may be thought to have the same total spaces (as sets) but the $\mathbf{C}^* \otimes \mathbf{Q}$ action on the second one is obtained by precomposing the original action by the automorphism of $\mathbf{C}^* \otimes \mathbf{Q}$ obtained from the homomorphism $z \mapsto z^d$ on $\mathbf{C}^*$.

THEOREM 5.5. *Fix any integer n. Starting from any base surface $X \in \mathrm{Ob}(\mathcal{S})$, we obtain a direct system of principal $\mathbf{C}^*$ bundles $\mathcal{L}_n(Y) := \mathrm{DET}_{n,g(Y)}^{\otimes 12}$ over the Teichmüller spaces $\mathcal{T}(Y)$ with holomorphic homomorphisms $\Omega(\pi)$ (see Theorem 4.3) between the total spaces; here $Y \xrightarrow{\pi} X$ is an arbitrary morphism of $\mathcal{S}$ with target X.*

By passing to the direct limit, one therefore obtains over the universal commensurability Teichmüller space, $\mathcal{T}_\infty(X)$, a principal $\mathbf{C}^* \otimes \mathbf{Q}$ bundle:

$$\mathcal{L}_{n,\infty}(X) = \operatorname{ind\,lim} \mathcal{L}_n(Y).$$

Since the maps $\Omega(\pi)$ preserved the Quillen unit circles, the limit object also inherits such a Quillen "Hermitian" structure.

The construction is functorial with respect to change of the base X in the obvious sense that the directed systems and their limits are compatible with the biholomorphic identifications $\mathcal{T}_\infty(\pi)$ of equation (5.2). *In particular, the commensurability modular group action $\mathrm{CM}_\infty(X)$ on $\mathcal{T}_\infty(X)$ has a natural lifting to $\mathcal{L}_{n,\infty}(X)$—acting by unitary automorphisms.*

Finally, the Mumford isomorphisms persist:

$$\mathcal{L}_{n,\infty}(X) = \mathcal{L}_{0,\infty}(X)^{\otimes(6n^2-6n+1)}.$$

Namely, if we change the action of $\mathbf{C}^* \otimes \mathbf{Q}$ on the "Hodge" bundle $\mathcal{L}_{0,\infty}$ by the "raising to the $(6n^2-6n+1)$-th power" automorphism of $\mathbf{C}^* \otimes \mathbf{Q}$, then the principal $\mathbf{C}^* \otimes \mathbf{Q}$ bundles are canonically isomorphic.

Remark 5.6. In other words, the Mumford isomorphism in the above theorem means that $\mathcal{L}_{n,\infty}$ and $\mathcal{L}_{0,\infty}$ are equivariantly isomorphic relative to the automorphism of $\mathbf{C}^*\otimes\mathbf{Q}$ induced by the homomorphism of $\mathbf{C}^*$ that raises to the power exhibited. Also, we could have used the Quillen Hermitian structure to reduce the structure group from $\mathbf{C}^*$ to $U(1)$, and thus obtain direct systems of $U(1)$ bundles over the Teichmüller spaces. Passing to the direct limit would then produce $U(1)\otimes\mathbf{Q}$:= *"tiny circle"* bundles over $\mathcal{T}_\infty$, which can be tested for maps into these objects as above.

Rational line bundles on ind-spaces. A line bundle on the inductive limit of an inductive system of varieties or spaces, is, by definition ([Sh]), a collection of line bundles on each stratum (i.e., each member of the inductive system of spaces) together with compatible line bundle (linear on fibers) morphisms. Such a direct system of line bundles determines an element of the inverse limit of the Picard groups of the stratifying spaces. See [KNR], [Sh]. (Recall: For any complex space T, $\mathrm{Pic}(T)$:= the group (under $\otimes$) of isomorphism classes of line bundles on T. In the case of the Teichmüller spaces, we refer to the modular-group invariant bundles as constituting the relevant Picard group—see [BN].)

Now, utilising the GRR morphisms $\Gamma(\pi)$ themselves (without involving the power law maps), we know from §3 that the "dth root" of the bundle $\mathrm{DET}_{n,\tilde{g}}$ fits together with the bundle $\mathrm{DET}_{n,g}$ ($d=(\tilde{g}-1)/(g-1)$). A "rational" line bundle over the inductive limit is defined to be an element of the inverse limit of the $\mathrm{Pic}_{\mathbf{Q}}=\mathrm{Pic}\otimes\mathbf{Q}$. Therefore we may also state a result about the existence of canonical elements of the inverse limit, $\varprojlim \mathrm{Pic}(\mathcal{T}_{g_i})_{\mathbf{Q}}$, by our construction. Indeed, in the notation of §3, by using the morphisms $\Gamma(\pi)\otimes 1/\deg(\pi)$ between $\mathrm{DET}_{n,g}$ and $\mathrm{DET}_{n,\tilde{g}}\otimes 1/\deg(\pi)$ to create a directed system, we obtain canonical elements representing the Hodge and higher DET_n bundles, with respective Quillen metrics:

$$\Lambda_m \in \varprojlim \mathrm{Pic}(\mathcal{T}_{g_i})_{\mathbf{Q}}, \quad m=0,1,2,\ldots . \tag{5.7}$$

The pullback (i.e., restriction) of Λ_m to each of the stratifying Teichmüller spaces $\mathcal{T}_{g_i}$ is $(n_i)^{-1}$ times the corresponding DET_m bundle (with $(n_i)^{-1}$ times its Quillen metric) over $\mathcal{T}_{g_i}$. Here n_i is the degree of the covering of the surface of genus g_i over the base surface. As rational Hermitian line bundles the Mumford isomorphisms persist:

$$\Lambda_m = \Lambda_0^{\otimes(6m^2-6m+1)} \tag{5.8}$$

as desired. This statement is different from that of the theorem. For further details see [BN].

Polyakov measure on $\mathcal{M}_g$ and our constructions. In his study of bosonic string theory, Polyakov constructed a measure on the moduli space $\mathcal{M}_g$ of curves of genus g ($\geqslant 2$).

Details can be found, for example, in [A], [Ne]. Subsequently, Belavin and Knizhnik, [BK], showed that the Polyakov measure has the following elegant mathematical description. First note that a Hermitian metric on the canonical bundle of a complex space gives a measure on that space. Fixing a volume form (up to scale) on a space therefore amounts to fixing a fiber metric (up to scale) on the canonical line bundle, K, over that space. But the Hodge bundle λ has its natural Hodge metric (arising from the L^2 pairing of holomorphic 1-forms on Riemann surfaces). Therefore we may transport the corresponding metric on λ^{13} to K by Mumford's isomorphism (as we know the choice of this isomorphism is unique up to scalar)—thereby obtaining a volume form on $\mathcal{M}_g$. [BK] showed that this is none other than the Polyakov volume. Therefore, the presence of Mumford isomorphisms over the moduli space of genus g Riemann surfaces describes the Polyakov measure structure thereon.

Above we have succeeded in fitting together the Hodge and higher DET bundles over the ind-space $\mathcal{T}_\infty$, together with the relating Mumford isomorphisms. We thus have from our results a structure on $\mathcal{T}_\infty$ that suggests a genus-independent, universal, version of the Polyakov structure.

We remark that since the genus is considered the perturbation parameter in the above formulation of the standard perturbative bosonic Polyakov string theory, our work can be considered as a contribution towards a *non-perturbative* formulation of that theory.

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