TEICHMÜLLER THEORY AND THE UNIVERSAL PERIOD MAPPING VIA QUANTUM CALCULUS AND THE $H^{1/2}$ SPACE ON THE CIRCLE

by

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Abstract: Quasisymmetric homeomorphisms of the circle, that arise in the Teichmüller theory of Riemann surfaces as boundary values of quasiconfomal diffeomorphisms of the disk, have fractal graphs in general and are consequently not so amenable to usual analytical or calculus procedures. In this paper we make use of the remarkable fact this group $QS(S^1)$ acts by substitution (i.e., pre-composition) as a family of bounded symplectic operators on the Hilbert space $\mathcal{H}$="$H^{1/2}$" (comprising functions mod constants on S^1 possessing a square-integrable half-order derivative). Conversely, and that is also important for our work, quasisymmetric homeomorphisms are actually *characterized* amongst homeomorphisms of S^1 by the property of preserving the space $\mathcal{H}$.

Interpreting $\mathcal{H}$ via boundary values as the square-integrable first cohomology of the disk with the cup product symplectic structure, and complex structure provided by the Hodge star, we obtain a universal form of the classical period mapping extending the map of [12] [13] from $Diff(S^1)/Mobius(S^1)$ to all of $QS(S^1)/Mobius(S^1)$ – namely to the entire universal Teichmüller space, $T(1)$. The target space for the period map is the universal Siegel space of period matrices; that is the space of all the complex structures on $\mathcal{H}$ that are compatible with the canonical symplectic structure.

Using Alain Connes' suggestion of a quantum differential $d_J^Q f = [J, f]$ – commutator of the multiplication operator with the complex structure operator – we obtain in lieu of the problematical classical calculus a quantum calculus for quasisymmetric homeomorphisms. Namely, one has operators $\{h, L\}$, $d\circ\{h, L\}$, $d\circ\{h, J\}$, corresponding to the non-linear classical objects $log(h')$, $\frac{h''}{h'}dx$, $\frac{1}{6}Schwarzian(h)dx^2$ defined when h is appropriately smooth. Any one of these objects is a quantum measure of the conformal distortion of h in analogy with the classical calculus Beltrami coefficient μ for a quasiconformal homeomorphism of the disk. Here L is the smoothing operator on the line (or the circle) with kernel $log|x-y|$, J is the Hilbert transform (which is $d \circ L$ or $L \circ d$), and $\{h, A\}$ means A conjugated by h minus A.

The period mapping and the quantum calculus are related in several ways. For example, f belongs to $\mathcal{H}$ if and only if the quantum differential is Hilbert-Schmidt. Also, the complex structures J on $\mathcal{H}$ lying on the Schottky locus (image of the period map) satisfy a quantum integrability condition $[d_J^Q, J] = 0$.

Finally, we discuss the Teichmüller space of the universal hyperbolic lamination ([20]) as a separable complex submanifold of $T(1)$. The lattice and Kähler (Weil-Petersson) metric aspect of the classical period mapping appear by focusing attention on this space.

§1 - Introduction

The Universal Teichmüller Space $T(1)$, which is a universal parameter space for all Riemann surfaces, is a complex Banach manifold that may be defined as the homogeneous space $QS\left(S^1\right)/\text{Möb}\left(S^1\right)$. Here $QS\left(S^1\right)$ denotes the group of all quasisymmetric homeomorphisms of the unit circle $\left(S^1\right)$, and $\text{Möb}\left(S^1\right)$ is the three-parameter subgroup of Möbius transformations of the unit disc (restricted to the boundary circle). There is a remarkable homogeneous Kähler complex manifold, $M = \text{Diff}\left(S^1\right) / \text{Möb}\left(S^1\right)$,– arising from applying the Kirillov-Kostant coadjoint orbit method to the C^∞-diffeomorphism group $\text{Diff}\left(S^1\right)$ of the circle ([22]) - that clearly sits embedded in $T(1)$ (since any smooth diffeomorphism is quasisymmetric).

In [15] it was proved that the canonical complex-analytic and Kähler structures on these two spaces coincide under the natural injection of M into $T(1)$. (The Kähler structure on $T(1)$ is formal – the pairing converges on precisely the $H^{3/2}$ vector fields on the circle.) The relevant complex-analytic and symplectic structures on M, (and its close relative $N = \text{Diff}\left(S^1\right) / \left(S^1\right)$), arise from the representation theory of $\text{Diff}\left(S^1\right)$; whereas on $T(1)$ the complex structure is dictated by Teichmüller theory, and the (formal) Kähler metric is Weil-Petersson. Thus, the homogeneous space M is a complex analytic submanifold (not locally closed) in $T(1)$, carrying a canonical Kähler metric.

In subsequent work ([12] [13]) it was shown that one can canonically associate infinite-dimensional period matrices to the smooth points M of $T(1)$. The crucial step in this construction was a faithful representation (Segal [18]) of $\text{Diff}\left(S^1\right)$ on the Frechet space

$$V = C^\infty \text{ Maps}\left(S^1, \mathbb{R}\right)/\mathbb{R}(\text{ the constant maps }) \tag{1}$$

$\text{Diff}\left(S^1\right)$ acts by pullback on the functions in V as a group of toplinear automorphisms that preserve a basic symplectic form that V carries.

In order to be able to extend the infinite dimensional period map to the full space $T(1)$, it is necessary to replace V by a suitable "completed" space that is invariant under quasisymmetric pullbacks. Moreover, the quasisymmetric homeomorphisms should continue to act as bounded symplectic automorphisms of this extended space. These goals are achieved in the present paper by developing the theory of the Sobolev space on the circle consisting of functions with half-order derivative. This Hilbert space $H^{1/2} = \mathcal{H}$, *which turns out to be exactly the completion of the pre-Hilbert space* V, actually *characterizes* quasisymmetric (q.s). homeomorphisms (amongst all homeomorphisms of S^1). That fact will be important for our understanding of the period mapping. The symplectic structure, S, on V extends to $\mathcal{H}$ and is preserved by the action of $QS(S^1)$, and indeed we show that this S is the *unique* symplectic structure available which is invariant under even the tiny finite-dimensional subgroup Möb $\left(S^1\right)$ $(\subset QS\left(S^1\right))$.

We utilise several different characterisations of $\mathcal{H}$ and its complexification. In particular, $\mathcal{H}$ comprises functions on S^1 which are defined quasi-everywhere (i.e., off some set of logarithmic capacity zero); alternatively, they appear as non-tangential limits of harmonic functions of finite Dirichlet energy in the disc. The last-mentioned fact allows us to interpret $\mathcal{H}$ as the first cohomology space with real coefficients of the unit disc in the Hodge-theoretic sense. That is important for our subsequent discussion of the period mapping as a theory of the variation of S-compatible complex structure on this real Hilbert space $\mathcal{H}$. The fact that quasisymmetric homeomorphisms are the *only* ones preserving $\mathcal{H}$ is necessary in our determination of the universal Schottky locus – namely the image of Π.

We present a section where we discuss quantum calculus on the line (motivated by Alain Connes), the idea being firstly to demonstrate that the $H^{1/2}$ functions have such an interpretation. That then allows us to interpret the universal Siegel space that is the target space for the period mapping as "almost complex structures on the line" and the Teichmüller points (i.e., the Schottky locus) can be interpreted as comprising precisely the subfamily of those complex structures that are *integrable*.

Notice that the fact that capacity zero sets are preserved by quasisymetric transformations – whereas merely being measure zero is not a q.s.-invariant notion – goes to exemplify how deeply quasisymmetry is connected to the properties of $\mathcal{H}$. Other characterisations found below for the complexification $\mathbb{C} \otimes \mathcal{H}$ in terms of boundary values for holomorphic and anti-holomorphic functions are of independent importance, and relate to the proof of the uniqueness of the invariant symplectic structure. That proof utilises a pair of irreducible unitary representations from the discrete series for $SL(2, \mathbb{R})$ and a version of Schur's lemma.

In universal Teichmüller space there resides the separable complex submanifold $T(H_\infty)$ – the Teichmüller space of the universal hyperbolic lamination – that is exactly the closure of the union of all the classical Teichmüller spaces of closed Riemann surfaces in $T(1)$ (see [20]). Genus-independent constructions like the universal period mapping proceed naturally to live on this completed version of the classical Teichmüller spaces. We show that $T(H_\infty)$ carries a natural convergent Weil-Petersson pairing.

We make no great claim to originality in this work. Our purpose is to survey from various different aspects the elegant role of $H^{1/2}$ in universal Teichmüller theory, the main goal being to understand the period mapping in the universal context. The Hilbert space

$\mathcal{H}$, its complexification, its symplectic form and its polarizations etc. appear so naturally in what follows that it may not be merely facetious to say that the connection of $\mathcal{H}$ with Teichmüller theory and quasiconformal mappings are not only "natural" but almost "supernatural".

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§- 2 The Hilbert space $H^{1/2}$ on the circle and the line.

Let Δ denote the open unit disc and U the upper half-plane in the plane ($\mathbb{C}$), and $S^1 = \partial\Delta$ be the unit circle.

Intuitively speaking, the real Hilbert space under concern:

$$\mathcal{H} \equiv H^{1/2}\left(S^1, \mathbb{R}\right)/\mathbb{R} \tag{2}$$

is the subspace of $L^2\left(S^1\right)$ comprising real functions of mean-value zero on S^1 which have a half-order derivative also in $L^2\left(S^1\right)$. Harmonic analysis will tell us that these functions are actually defined off some set of capacity zero (i.e., "quasi-everywhere") on the circle, and that they also appear as the boundary values of real harmonic functions of finite Dirichlet energy in Δ. Our first way (of several) to make this precise is to *identify $\mathcal{H}$ with the sequence space*

$$\ell_2^{1/2} = \{\text{complex sequences } \; u \equiv (u_1, u_2, u_3, \cdots) : \{\sqrt{n}\; u_n\} \text{ is square summable }\}. \tag{3}$$

The identification between (2) and (3) is by showing (see Proof of Theorem 2.1) that the Fourier series

$$f\left(e^{i\theta}\right)=\sum_{n=-\infty}^{\infty}u_n e^{in\theta};\quad u_{-n}=\overline{u}_n, \tag{4}$$

converges quasi-everywhere and defines a real function of the required type. The norm on $\mathcal{H}$ and on $\ell_2^{1/2}$ is, of course, the ℓ_2 norm of $\{\sqrt{n}\,u_n\}$, i.e.,

$$\|f\|_{\mathcal{H}}^2=\|u\|_{\ell_2^{1/2}}^2=2\sum_{n=1}^{\infty}n\left|u_n\right|^2. \tag{5}$$

Naturally $\ell_2^{1/2}$ and $\mathcal{H}$ are isometrically isomorphic separable Hilbert spaces. Note that $\mathcal{H}$ is a subspace of $L^2\left(S^1\right)$ because $\{\sqrt{n}\,u_n\}$ in ℓ_2 implies $\{\,u_n\}$ itself is in ℓ_2.

At the very outset let us note the fundamental fact that the space $\mathcal{H}$ is evidently closed under *Hilbert transform* ("conjugation" of Fourier series):

$$(Jf)(e^{i\theta})=-\sum_{n=-\infty}^{\infty}i\,\mathrm{sgn}(n)u_n e^{in\theta}. \tag{6}$$

In fact, $J:\mathcal{H}\to\mathcal{H}$ is an isometric isomorphism whose square is the negative identity, and thus J defines a *canonical complex structure for* $\mathcal{H}$.

Remark: In the papers [8],[12] [13] [15], we had made use of the fact that the Hilbert transform defines the almost-complex structure operator for the tangent space of the coadjoint orbit manifolds (M and N), as well as for the universal Teichmüller space $T(1)$.

Whenever convenient we will pass to a description of our Hilbert space $\mathcal{H}$ as functions on the real line, $\mathbf{R}$. This is done by simply using the Möbius transformation of the circle onto the line that is the boundary action of the Riemann mapping ("Cayley transform") of Δ onto U. We thus get an isometrically isomorphic copy, called $H^{1/2}(\mathbb{R})$, of our Hilbert space $\mathcal{H}$ on the circle defined by taking $f\in\mathcal{H}$ to correspond to $g\in H^{1/2}(\mathbb{R})$ where $g=f\circ R, R(z)=\frac{z-i}{z+i}$ being the Riemann mapping. The Hilbert transform complex structure on $\mathcal{H}$ in this version is then described by the usual singular integral operator on the real line with the "Cauchy kernel" $(x-y)^{-1}$.

Fundamental for our set up is the dense subspace V in $\mathcal{H}$ defined by equation (1) in the introduction. At the level of Fourier series, V corresponds to those sequence $\{u_n\}$ in $\ell_2^{1/2}$ which go to zero more rapidly than n^{-k} for any $k>0$. This is so because a C^k function has Fourier ceofficients decaying at least as fast as n^{-k}. Since trigonometric polynomials are in V, it is obvious that V is norm-dense in $\mathcal{H}$. On V one has the basic symplectic form that we utilised crucially in [12], [13]:

$$S : V \times V \to \mathbb{R} \tag{7}$$

given by

$$S(f,g) = \frac{1}{2\pi} \int_{S^1} f \cdot dg. \tag{8}$$

This is essentially the signed area of the $(f(e^{i\theta}), g(e^{i\theta}))$ curve in Euclidean plane. On Fourier coefficients this bilinear form becomes

$$S(f,g) = 2 \operatorname{Im} \left(\sum_{n=1}^{\infty} n u_n \overline{v}_n \right) = -i \sum_{n=-\infty}^{\infty} n u_n v_{-n} \tag{9}$$

where $\{u_n\}$ and $\{v_n\}$ are respectively the Fourier ceofficients of the (real-valued) functions f and g, as in (4). Let us note that the Cauchy-Schwarz inequality applied to (9) shows that *this non-degenerate bilinear alternating form extends from V to the full Hilbert space $\mathcal{H}$*. We will call this extension also $S : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$. Cauchy-Schwarz asserts:

$$|S(f,g)| \leq \|f\| \cdot \|g\| . \tag{10}$$

Thus S is a jointly continuous - in fact analytic - map on $\mathcal{H} \times \mathcal{H}$.

The important interconnection between the inner product on $\mathcal{H}$, the Hilbert-transform complex structure J, and the form S is encapsulated in the identity:

$$S\,(f, Jg) = \langle f, g \rangle\,, \text{ for all } f, g \in \mathcal{H} \tag{11}$$

We thus see that V itself was naturally a pre-Hilbert space with respect to the canonical inner product arising from its symplectic form and its Hilbert-transform complex structure, and we have just established that the completion of V is nothing other than the Hilbert space $\mathcal{H}$. Whereas V carried the C^∞ theory, beacause it was diffeomorphism invariant, the completed Hilbert space $\mathcal{H}$ allows us to carry through our constructions for the full Universal Teichmüller Space because it indeed is quasisymmetrically invariant.

It will be important for us to *complexify* our spaces since we need to deal with isotropic subspaces and polarizations. Thus we set

$$\mathbb{C} \otimes V \equiv V_{\mathbb{C}} = C^\infty \text{ Maps}\left(S^1, \mathbb{C}\right)/\mathbb{C}$$

$$\mathbb{C} \otimes \mathcal{H} \equiv \mathcal{H}_{\mathbb{C}} = H^{1/2}\left(S^1, \mathbb{C}\right)/\mathbb{C} \tag{12}$$

$\mathcal{H}_{\mathbb{C}}$ is a complex Hilbert space isomorphic to $\ell_2^{1/2}(\mathbb{C})$ - the latter comprising the Fourier series

$$f\left(e^{i\theta}\right) = \sum_{n=-\infty}^{\infty} u_n e^{in\theta} \quad , \quad u_0 = 0 \tag{13}$$

with $\{\sqrt{|n|}\ u_n\}$ being square summable over $\mathbb{Z} - \{0\}$. Note that the Hermitian inner product on $\mathcal{H}_{\mathbb{C}}$ derived from (5) is given by

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} |n| u_n \overline{v}_n. \tag{14}$$

[This explains why we introduced the factor 2 in the formula (5).] The fundamental *orthogonal decomposition* of $\mathcal{H}_{\mathbb{C}}$ is given by

$$\mathcal{H}_{\mathbb{C}} = W_+ \oplus W_- \tag{15}$$

where

$$W_+ = \{f \in \mathcal{H}_{\mathbb{C}} : \text{ all negative index Fourier coefficients vanish }\}$$

and

$$\overline{W}_+ = W_- = \{f \in \mathcal{H}_{\mathbb{C}} : \text{ all positive index Fourier coefficients vanish }\}.$$

Here we denote by bar the complex anti-linear automorphism of $\mathcal{H}_{\mathbb{C}}$ given by conjugation of complex scalars.

Let us extend $\mathbb{C}$-linearly the form S and the operator J to $\mathcal{H}_{\mathbb{C}}$ (and consequently also to $V_{\mathbb{C}}$). The complexified S is still given by the right-most formula in (9). Notice that W_+ and W_- can be characterized as precisely the $-i$ *and* $+i$ *eigenspaces* (respectively) of the $\mathbb{C}$-linear extension of J, the Hilbert transform. Further, each of W_+ and W_- is *isotropic* for S, i.e., $S(f, g) = 0$, whenever both f and g are from either W_+ or W_- (see formula (9)). Moreover, W_+ and W_- are *positive isotropic* subspaces in the sense that the following identities hold:

$$\langle f_+, g_+ \rangle = iS\left(f_+, \overline{g}_+\right), \ for\ f_+, g_+ \in W_+ \tag{16}$$

and

$$\langle f_-, g_- \rangle = -iS\left(f_-, \overline{g}_-\right), \; for \; f_-, g_- \in W_-. \tag{17}$$

Remark: (16) and (17) show that we could have defined the inner product and norm on $\mathcal{H}_{\mathbb{C}}$ from the symplectic form S, by using these relations to *define* the inner products on W_+ and W_-, and declaring W_+ to be perpendicular to W_-. Thus, for general $f, g \in \mathcal{H}_{\mathbb{C}}$ one has the fundamental identity

$$\langle f, g \rangle = iS\left(f_+, \overline{g}_+\right) - iS\left(f_-, \overline{g}_-\right). \tag{18}$$

We have thus described the Hilbert space structure of $\mathcal{H}$ simply in terms of the canonical symplectic form it carries and the fundamental decomposition (15). [Here, and henceforth, we will let $f_\pm$ denote the projection of f to $W_\pm$, etc.].

In order to prove the first results of this paper, we have to rely on interpreting the functions in $H^{1/2}$ as boundary values ("traces") of functions in the disc Δ that have finite Dirichlet energy, (i.e. the first derivatives are in $L^2(\Delta)$). We start explaining this material.

Define the following "Dirichlet space" of harmonic functions:

$$\mathcal{D} = \{F : \Delta \to \mathbb{R} : F \text{ is harmonic }, F(0) = 0, \text{ and } E(F) < \infty\} \tag{19}$$

where the energy E of any (complex-valued) C^1 map on Δ is defined as the $L^2(\Delta)$ norm of grad(F) :

$$\|F\|_{\mathcal{D}}^2 = E(F) = \frac{1}{2\pi} \int\int_\Delta \left(\left| \frac{\partial F}{\partial x} \right|^2 + \left| \frac{\partial F}{\partial y} \right|^2 \right) dxdy \tag{20}$$

$\mathcal{D}$, and its complexification $\mathcal{D}_{\mathbb{C}}$, will be Hilbert spaces with respect to this energy norm.

We want to identify the space $\mathcal{D}$ as precisely the space of harmonic functions in Δ solving the Dirichlet problem for functions in $\mathcal{H}$. Indeed, the *Poisson integral representation allows us to map $P : \mathcal{H} \to \mathcal{D}$ so that P is an isometric isomorphism of Hilbert spaces.*

To see this let $f(e^{i\theta}) = \sum_{-\infty}^{\infty} u_n e^{in\theta}$ be an arbitrary member of $\mathcal{H}_{\mathbb{C}}$. Then the Dirichlet extension of f into the disc is:

$$F(z)=\sum_{n=-\infty}^{\infty} u_n r^{|n|} e^{in\theta} = \left(\sum_{n=1}^{\infty} u_n z^n\right) + \left(\sum_{m=1}^{\infty} u_{-m} \overline{z}^m\right) \tag{21}$$

where $z = re^{i\theta}$. From the above series one can directly compute the $L^2(\Delta)$ norms of F and also of $\mathrm{grad}(F) = (\partial F/\partial x, \partial F/\partial y)$. One obtains the following:

$$E(F) = \frac{1}{2\pi}\int\int_{\Delta} |\,\mathrm{grad}(F)|^2 = \sum_{-\infty}^{\infty} |n||u_n|^2 \equiv \|f\|^2_{\mathcal{H}} < \infty \tag{22}$$

We will require crucially the well-known formula of Douglas (see [2,pg. 36-38]) expressing the above energy of F as the double integral on S^1 of the square of the first differences of the boundary values f.

$$E(F) = \frac{1}{16\pi^2}\int_{S^1}\int_{S^1} [(f(e^{i\theta}) - f(e^{i\phi}))/sin((\theta-\phi)/2)]^2 d\theta d\phi \tag{23}$$

Transferring to the real line by the Möbius transform identification of $\mathcal{H}$ with $H^{1/2}(\mathbb{R})$ as explained before, the above identity becomes as simple as:

$$E(F) = \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} [(f(x) - f(y))/(x-y)]^2 dxdy = \|f\|^2 \tag{24}$$

Calculating from the series (21), the L^2-norm of F itself is:

$$\frac{1}{2\pi}\int\int_{\Delta} |F|^2 dxdy = \sum_{-\infty}^{\infty} \frac{|u_n|^2}{(|n|+1)} \leq E(F) < \infty \tag{25}$$

(22) shows that indeed *Dirichlet extension is isometric from* $\mathcal{H}$ to $\mathcal{D}$, whereas (25) shows that the functions in $\mathcal{D}$ are themselves in L^2, so that *the inclusion of* $\mathcal{D} \hookrightarrow L^2(\Delta)$ *is continuous.* (Bounding the L^2 norm of F by the L^2 norm of its derivatives is a "Poincaré inequality").

It is therefore clear that $\mathcal{D}$ is a subspace of the usual Sobolev space $H^1(\Delta)$ comprising those functions in $L^2(\Delta)$ whose first derivatness (in the sense of distributions) are also in $L^2(\Delta)$. The theory of function spaces implies (by the "trace theorems") that H^1 functions lose half a derivative in going to a boundary hyperplane. Thus it is known that the functions in $\mathcal{D}$ will indeed have boundary values in $H^{1/2}$. See [5] [7] and [21].

Moreover, the identity (24) shows that for any $F \in \mathcal{D}$, the Fourier expansion of the trace on the boundary circle is a Fourier series with $\sum |n||u_n|^2 < \infty$. But Fourier expansions with coefficients in such a weigted ℓ_2 space, as in our situation, are known to converge *quasi-everywhere* (i.e. off a set of logarithmic capacity zero) on S^1. See Zygmund [23, Vol 2, Chap. XIII]. The identification between $\mathcal{D}$ and $\mathcal{H}$ (or $\mathcal{D}_{\mathbb{C}}$ and $\mathcal{H}_{\mathbb{C}}$) is now complete.

It will be necessary for us to identify the $W_{\pm}$ polarization of $\mathcal{H}_{\mathbb{C}}$ at the level $\mathcal{D}_{\mathbb{C}}$. In fact, let us decompose the harmonic function F of (21) into its holomorphic and anti-holomorphic parts ; these are F_+ and F_-, which are (respectively) the two sums bracketed separately on the right hand side of (21). Clearly F_+ is a holomorphic function extending f_+ (the W_+ part of f), and F_- is anti-holomorphic extending f_-. We are thus led to introduce the following space of holomorphic functions whose derivatives are in $L^2(\Delta)$:

$$\mathrm{Hol}_2(\Delta) = \{H : \Delta \to \mathbb{C} : H \text{ is holomorphic }, H(0) = 0 \quad \text{and} \quad \int\int_{\Delta} |H'(z)|^2 dxdy < \infty\}. \tag{26}$$

This is a complex Hilbert space with the norm

$$\|H\|^2 = \frac{1}{2\pi} \int\int_{\Delta} |H'(z)|^2 dxdy. \tag{27}$$

If $H(z) = \sum_{n=1}^{\infty} u_n z^n$, a computation in polar coordinates (as for (21), (25)) produces

$$\|H\|^2 = \sum_{n=1}^{\infty} n \left|u_n\right|^2 . \tag{28}$$

Equations (27) and (28) show that the norm-squared is the Euclidean area of the (possibly multi-sheeted) image of the map H.

We let $\overline{\mathrm{Hol}_2}(\Delta)$ denote the Hilbert space of antiholomorphic functions conjugate to those in $\mathrm{Hol}_2(\Delta)$. The norm is defined by stipulating that the anti-linear isomorphism of Hol_2 on $\overline{\mathrm{Hol}_2}$ given by conjugation should be an isometry. The Cauchy-Riemann equation for F_+ and $\overline{F}_-$ imply that

$$\left|\mathrm{grad}(F)\right|^2 = 2\left\{\left|F_+'\right|^2 - \left|F_-'\right|^2\right\} \tag{29}$$

and hence

$$\|F_+\|^2 + \|F_-\|^2 = \|f\|^2_{\mathcal{H}_{\mathbb{C}}} . \tag{30}$$

Now, the relation between $\mathcal{D}$ (harmonic functions in $H^1(\Delta)$) and $\mathrm{Hol}_2(\Delta)$ is transparent, so that the holomorphic functions in Hol_2 will have non-tangential limits quasi-everywhere on S^1 - defining a function W_+. We thus collect together, for the record, the various representations of our basic Hilbert space:

THEOREM 2.1: There are *canonical isometric isomorphisms* between the following complex Hilbert spaces:

(1) $\mathcal{H}_{\mathbb{C}} = H^{1/2}\left(S^1, \mathbb{C}\right)/\mathbb{C} = \mathbb{C} \otimes H^{1/2}(\mathbb{R}) = W_+ \oplus W_-$;

(2) The sequence space $\ell_2^{1/2}(\mathbb{C})$ (constituting the Fourier coefficients of the above quasi-everywhere defined functions);

(3) $\mathcal{D}_{\mathbb{C}}$, comprising normalized finite-energy harmonic functions (either on Δ or on the half-plane U); [the norm-squared being given by (20) or (22) or (23) or (24)];

(4) $\mathrm{Hol}_2(\Delta) \oplus \overline{\mathrm{Hol}_2}(\Delta)$.

Under the canonical identifications, W_+ maps to $\mathrm{Hol}_2(\Delta)$ and W_- onto $\overline{\mathrm{Hol}_2}(\Delta)$. ∎

Remark: One advantage of introducing the full Sobolev space $H^1(\Delta)$ (rather than only its harmonic subspace $\mathcal{D}$) is that we may use *Dirichlet's principle* to rewrite the norm on $\mathcal{H}$ as

$$\|f\|^2_{\mathcal{H}} = \inf\{E(F) : F \text{ ranges over all extensions to } \Delta \text{ of } f\} \tag{31}$$

By Dirichlet principle, the infimum is realized by the harmonic extension $P(f) = F$ of (23). In connection with this it is worth pointing out still another formula for the norm :

$$\|f\|^2_{\mathcal{H}} = \int_{S^1} F \cdot \frac{\partial F}{\partial n} ds \tag{32}$$

where F is the harmonic extension to Δ of f. This follows from the well-known Green's identity:

$$\int\int_{\Delta} |\mathrm{grad}\, F|^2 = \int_{\Delta}\int F(\Delta F) + \int_{S^1} F \cdot \frac{\partial F}{\partial n} \cdot ds \tag{33}$$

The first term on the right drops out since F is harmonic. Hence (32) follows. The close relation of formula (32) with the symplectic pairing formula (8) should be noted.

Remark: Since the isomorphisms of the Theorem are all isometric, and because the norm arose from the canonical symplectic structure, (formulas (16), (17), (18)), it is instructive to work out the formulas for the symplectic form S on $\mathcal{D}_{\mathbb{C}}$ and on $\mathrm{Hol}_2(\Delta)$. This is left to the interested reader.

§**3 - Quasisymmetric invariance.**

Quasiconformal (q.c.) self-homeomorphisms of the disc Δ (of the upper half-plane) U are known to extend continuously to the boundary. The action on the boundary circle (respectively, on the real line $\mathbb{R}$) is called a *quasisymmetric* (q.s.) homeomorphism. By [4], $\varphi : \mathbb{R} \to \mathbb{R}$ is quasisymmetric if and only if, for all $x \in \mathbb{R}$ and all $t > 0$, there exists some $K > 0$ such that

$$\frac{1}{K} \leq \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \leq K \tag{34}$$

On the circle this condition for $\varphi : S^1 \to S^1$ means that $|\varphi(2I)|/|\varphi(I)| \leq K$, where I is any interval on S^1 of length less that π, $2I$ denotes the interval obtained by doubling I keeping the same mid-point, and $|\bullet|$ denotes Lebesgue measure on S^1. See [3] [11] and [14] as general references.

Given any orientation preserving homeomorphism $\varphi : S^1 \to S^1$, we use it to pullback functions in $\mathcal{H}$ by precomposition:

$$V_\varphi(f) = \varphi^*(f) = f \circ \varphi - \frac{1}{2\pi} \int_{S^1} (f \circ \varphi). \tag{35}$$

[We subtract off the mean value in order that the resulting function also possess zero mean. Since constants pullback to themselves, the operation is well-defined on $\mathcal{H}$.]

We prove:

THEOREM 3.1: V_φ maps $\mathcal{H}$ to itself (i.e., the space $\mathcal{H} \circ \varphi$ is $\mathcal{H}$) if and only if φ is quasisymmetric. The operator norm of $V_\varphi \leq \sqrt{K + K^{-1}}$, whenever φ allows a K-quasiconformal extension into the disc.

COROLLARY 3.2: The group of all quasisymmetric homeomorphism on $S^1, QS\left(S^1\right)$, acts faithfully by bounded toplinear automorphisms on the Hilbert space $\mathcal{H}$ (and therefore also on $\mathcal{H}_{\mathbb{C}}$).

Proof of sufficiency

Assume φ is q.s. on S^1, and let $\Phi : \Delta \to \Delta$ be any quasiconformal extension. Let $f \in \mathcal{H}$ and suppose $P(f) = F \in \mathcal{D}$ is its unique harmonic extension into Δ. Clearly $G \equiv F \circ \Phi$ has boundary values $f \circ \varphi$, the latter being (like f) also a continuous function on S^1 defined quasi-everywhere. [Here we recall that q.s. homeomorphisms carry capacity zero sets to again such sets, although measure zero sets can become positive measure.] To prove that $f \circ \varphi$ minus its mean value is in $\mathcal{H}$, it is enough to prove that the Poisson integral of $f \circ \varphi$ again has finite Dirichelet energy. Indeed we will show

$$E(\text{harmonic extension of } \varphi^*(f)) \leq 2\left(\frac{1+k^2}{1-k^2}\right) E(F). \tag{36}$$

Here $0 \leq k < 1$ is the q.c. constant for Φ, i.e.,

$$|\Phi_{\overline{z}}| \leq k\,|\Phi_z| \quad \text{a.e. in } \Delta. \tag{37}$$

The operator norm of V_φ is thus no more that $2^{1/2}\left(\frac{1+k^2}{1-k^2}\right)^{1/2}$. The last expression is equal to the bound quoted in the Theorem, where, as usual, $K = (1+k)/(1-k)$.

Towards establishing (36) we prove that the inequality holds with the left side being the energy of the map $G = F \circ \Phi$. Since G is therefore also a finite energy extension of $f \circ \varphi$ to Δ, Dirichlet's principle (see (31) above) implies the required inequality. (Note that since Dirichlet integral is insensitive to adding a constant to a function, the energy of G is the same as the energy of $G - G(0)$.)

To compute $E(G)$ we note that

$$\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2 \leq 2\left[\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2\right]\left[|\Phi_z|^2 + |\Phi_{\overline{z}}|^2\right]. \tag{38}$$

We have writen $\Phi(x,y) = u(x,y) + iv(x,y)$ and $F = F(u,v)$. (38) follows by straight computation using the chain rule. But notice that the Jacobian of Φ is

$$\operatorname{Jac}(\Phi) = |\Phi_z|^2 - |\Phi_{\overline{z}}|^2. \tag{39}$$

By the quasiconformality (37) we therefore get from (38):

$$\left[G_x^2 + G_y^2\right] \leq 2\left(\frac{1+k^2}{1-k^2}\right)\left[F_u^2 + F_v^2\right]\operatorname{Jac}(\Phi) \tag{40}$$

Using change of variables in the Dirichlet integral we therefore derive

$$E(G) \leq 2\left(\frac{1+k^2}{1-k^2}\right) E(F) \tag{41}$$

as desired. ∎

Remark: Since the Dirichlet integral in two dimensions is invariant under conformal mappings, it is not too surprising that it is quasi-invariant under quasiconformal transformations. Such quasi-invariance has been noted before and is applied, for example, in [1] and [16].

Proof of necessity: As we mentioned before, the idea of this proof is taken from the notes of M. Zinsmeister.

Since two-dimensional Dirichlet integrals are conformally invariant, we will pass to the upper half-plane U and its boundary line $\mathbb{R}$ to aid our presentation. As explained earlier, using the Cayley transform we transfer everything over to the half-plane; the traces on the boundary constitute the space of quasi-everywhere defined functions called $H^{1/2}(\mathbb{R})$.

From the Douglas identity, equation (24), we recall that an *equivalent way of expressing the Hilbert space norm on* $H^{1/2}(\mathbb{R})$ is

$$\|g\|^2 = \frac{1}{4\pi^2} \int\int_{\mathbb{R}^2} \left[\frac{g(x) - g(y)}{x - y}\right]^2 dxdy, \quad g \in H^{1/2}(\mathbb{R}). \tag{42}$$

Equation (42) immediately shows that $\|g\| = \|\widetilde{g}\|$ where $\widetilde{g}(x) = g(ax + b)$ for any real $a(\neq 0)$ and b. This will be important.

Assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is an orientation preserving homeomorphism such that $V_{\varphi^{-1}} : H^{1/2}(\mathbb{R}) \to H^{1/2}(\mathbb{R})$ is a bounded automorphism. Let us say that the norm of this operator is M.

Fix a bump function $f \in C_0^\infty(\mathbb{R})$ such that $f \equiv 1$ on $[-1, 1] f \equiv 0$ outside $[-2, 2]$ and $0 \leq f \leq 1$ everywhere. Choose any $c \in \mathbb{R}$ and any positive t. Denote $I_1 = [x - t, x]$ and $I_2 = [x, x + t]$. Set $g(u) = f(au + b)$, choosing a and b so that g is identically 1 on I_1 and zero on $[x + t, \infty)$.

By assumption, $g \circ \varphi^{-1}$ is in $H^{1/2}(\mathbb{R})$ and $\left\|g \circ \varphi^{-1}\right\| \leq M \|g\| = M \|f\|$. We have

$$\begin{aligned} M \|f\| &\geq \int\int_{\mathbb{R}^2} \left[\frac{g \circ \varphi^{-1}(u) - g \circ \varphi^{-1}(v)}{u - v}\right]^2 dudv \\ &\geq \int_{v=\varphi(x-t)}^{v=\varphi(x)} \int_{u=\varphi(x+t)}^{u=\infty} \frac{1}{(u-v)^2} dudv \\ &= \log\left(1 + \frac{\varphi(x) - \varphi(x-t)}{\varphi(x+t) - \varphi(x)}\right). \end{aligned} \tag{43}$$

[We utilise the elementary integration $\displaystyle\int_\gamma^\infty \int_\alpha^\beta \frac{1}{(u-v)^2} dudv = \log\left(1 + \frac{\beta - \alpha}{\gamma - \beta}\right)$, for $\alpha < \beta < \gamma$.] We thus obtain the result that

$$\frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \geq \frac{1}{e^{M\|f\|} - 1} \tag{44}$$

for arbitrary real x and positive t. By utilising symmetry, namely by shifting the bump to be 1 over I_2 and 0 for $u \leq x - t$, we get the opposite inequality:

$$\frac{\varphi(x+t)-\varphi(x)}{\varphi(x)-\varphi(x-t)} \leq e^{M\|f\|}-1. \tag{45}$$

The Beurling-Ahlfors condition on φ is verified, and we are through. Both the theorem and its corollary are proved. ∎

§4 - The invariant symplectic structure.

The quasisymmetric homeomorphism group, $QS\left(S^1\right)$, acts on $\mathcal{H}$ by precomposition (equation (35)) as bounded operators, *preserving the canonical symplectic form* $S : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ (introduced in (8), (9), (10)). This is the central fact which we will analyse in this section. It is the crux on which the extension of the period mapping to all of $T(1)$ hinges:

PROPOSITION 4.1: For every $\varphi \in QS\left(S^1\right)$, and all $f, g \in \mathcal{H}$,

$$S\left(\varphi^*(f), \varphi^*(g)\right) = S(f,g). \tag{46}$$

Considering the complex linear extension of the action to $\mathcal{H}_{\mathbb{C}}$, one can assert that the only quasisymmetrics which preserve the subspace $W_+ = \mathrm{Hol}_2(\Delta)$ are the Möbius transformations. Then Möb $\left(S^1\right)$ acts as unitary operators on W_+ (and W_-).

Before proving the proposition we would like to point out that this canonical symplectic form enjoys a far stronger invariance property:

LEMMA 4.2: If $\varphi : S^1 \to S^1$ is any (say C^1) map of winding number (= degree) k, then

$$S(f\circ\varphi, g\circ\varphi) = kS(f,g) \tag{47}$$

for arbitrary choice of (C^1) functions f and g on the circle. In particular, S is invariant under pullback by all degree one mappings.

Proof: The proof of (47), starting from (8), is an exercise in calculus. Lift φ to the universal cover to obtain $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$; the degree of φ being $k (\in \mathbb{Z})$ implies that $\widetilde{\varphi}(t+2\pi) = \widetilde{\varphi}(t) + 2k\pi$. Breaking up $[0, 2\pi]$ into pieces on which $\widetilde{\varphi}$ is monotone, and applying the change of variables formula in each piece, produces the result. ∎

Proof of Proposition 4.1: The Lemma shows that (46) is true whenever the quasisymmetric homeomorphism φ is at least C^1. By Lehto-Virtanen [11, Chapter II, Section 7.4] we know that for arbitrary q.s φ, there exist real analytic q.s. homeomorphisms φ_m (with the same quasisymmetry constant as φ) that converge uniformly to φ. An approximation argument, as below, then proves the required result.

Let us denote the n^{th} Fourier coefficient of a function f on S^1 by $F_n(f)$. Recall from equation (9) that

$$S(f,g) = -i \sum_{n=-\infty}^{\infty} nF_n(f)F_{-n}(g) \tag{48}$$

for all f, g in $\mathcal{H}_{\mathbb{C}}$. Now since S is continuous it is enough to check (46) on the dense subspace V of smooth functions f and g. Therefore assume f and g to be smooth.

Since $\varphi_m \to \varphi$ uniformy it follows that $F_n(f \circ \varphi_m) \to F_n(f \circ \varphi)$ as $m \to \infty$ (for each fixed n). Applying the dominated convergence theorem to the sums (48) we immediately see that as $m \to \infty$,

$$S\left(\varphi_m^*(f), \varphi_m^*(g)\right) \to S\left(\varphi^*(f), \varphi^*(g)\right). \tag{49}$$

Lemma 4.1 says that for each m, $S\left(\varphi_m^*(f), \varphi_m^*(g)\right) = S(f,g)$. We are through.

If the action of φ on $\mathcal{H}_{\mathbb{C}}$ preserves W_+ it is easy to see that φ must be the boundary values of some holomorphic map $\Phi : \Delta \to \Delta$. Since φ is a homeomorphism one can see that Φ is a holomorphic homeomorphism (as explained also in [12, Lemma of Section 1]) - hence a Möbius transformation. Since every φ preserves S, and since S induces the inner product on W_+ and W_- by (16) (17), we note that such a symplectic transformation preserving W_+ must necessarily act unitarily. ∎

Remark: The remarkable invariance property (47) leads us to ask a question that may shed light on the structure of degree k maps of S^1 onto itself. Given a vector space V equipped with a bilinear form S, one may fix some constant $k(\neq 0)$ and study the family of linear maps A in $\mathrm{Hom}(V,V)$ such that

$$S\left(A\left(v_1\right), A\left(v_2\right)\right) = kS\left(v_1, v_2\right) \tag{50}$$

holds for all v_1, v_2 in V. Of course, the trivial multiplication (by $\sqrt{k}$) will be such a map, but we have in Lemma 4.1 a situation where the interesting family of linear maps obtained by degree k pullbacks provide a profusion of examples - precisely when k is an integer.

Furthermore, in the situation at hand, we may take V as the space of C^∞ (real or complex) functions on the circle. Then V also carries *algebra* structure by pointwise multiplication. The pullbacks by degree k mappings clearly preserve this multiplicative structure (whereas dilatations do not). It is interesting to question whether the only linear maps that preserve the algebra structure and also satisfy the relation (50), (for integer k), must necessarily arise from some degree k mapping of S^1 on itself.

Theorem 3.1 and Proposition 4.1 enable us to consider $QS\left(S^1\right)$ as a subgroup of the bounded symplectic operators on $\mathcal{H}$. Since the heart of the matter in extending the period mapping from Witten's homogeneous space M (as in [12], [13]) to $T(1)$ lies in the property of preserving this symplectic form on $\mathcal{H}$, we prove below that S is indeed the *unique* symplectic form that is $\mathrm{Diff}\left(S^1\right)$ or $QS\left(S^1\right)$ invariant. It is all the more surprising that the form S is canonically specified by requiring its invariance under simply the 3-parameter subgroup $\text{Möb}\left(S^1\right)$ ($\hookrightarrow \mathrm{Diff}\left(S^1\right) \hookrightarrow QS\left(S^1\right)$).

THEOREM 4.3: Let $S \equiv S_1$ be the canonical symplectic form on $\mathcal{H}$. Suppose $S_2 : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is any other continuous bilinear form such that $S_2(\varphi^*(f), \varphi^*(g)) = S_2(f, g)$, for all f, g in $\mathcal{H}$ whenever φ is in $\text{Möb}\left(S^1\right)$. Then S_2 is necessarily a real multiple of S. Thus every form on $\mathcal{H}$ that is $\text{Möb}\left(S^1\right) \equiv PSL(2, \mathbb{R})$ invariant is necessarily non-degenerate (if not identically zero) and remains invariant under the action of the whole of $QS\left(S^1\right)$. (Also, it automatically satisfies the even stronger invariance property (47)).

The proof requires some representation theory. Since this paper is written with complex analysts in mind, we have presented some detail. We start with:

LEMMA 4.4: The duality induced by canonical form S_1 is (the negative of) the Hilbert transform (equation (6)). Thus the map Σ_1 (induced by S_1) from $\mathcal{H}$ to $\mathcal{H}^*$ is an invertible isomorphism.

Proof: Given the continuous bilinear pairing $S_i : \mathcal{H} \times \mathcal{H} \to \mathbb{R} (i = 1, 2)$ we are considering the induced "duality" maps

$$\Sigma_i : \mathcal{H} \to \mathcal{H}^* \qquad (i = 1, 2) \tag{51}$$

which are bounded linear operators defined by:

$$\Sigma_i(g) = S_i(\bullet, g)\ ,\ g \in \mathcal{H}. \tag{52}$$

since $\mathcal{H}$ is a Hilbert space, the dual $\mathcal{H}^*$ is canonically isomorphic to $\mathcal{H}$ via:

$$\lambda(g) = \langle \bullet, g \rangle\,. \tag{53}$$

Here $\lambda(g)$ is the linear functional represented by $g \in \mathcal{H}$. Equation (11) says $S_1(f, Jg) = \langle f, g \rangle$ and therefore that

$$\Sigma_1(g) = \lambda(-Jg) \tag{54}$$

as required. ∎

The basic tool in proving the Theorem 4.3 is to consider the *"interwining operator"*

$$M = \Sigma_1^{-1} \circ \Sigma_2 : \mathcal{H} \to \mathcal{H} \tag{55}$$

which is a bounded linear operator on $\mathcal{H}$ by the above Lemma.

LEMMA 4.5: M commutes with every invertible linear operator on $\mathcal{H}$ that preserves both the forms S_1 and S_2.

Proof: M is defined by the identity $S_1(v, Mw) = S_2(v, w)$. If T preserves boths forms then one has the string of equalities:

$$S_1(Tv, TMw) = S_1(v, Mw) = S_2(v, w) = S_2(Tv, Tw) = S_1(Tv, MTw)$$

Since T is assumed invertible, this is the same as saying

$$S_1(v, TMw) = S_1(v, MTw), \text{ for all } v, w \in \mathcal{H} \tag{56}$$

But S_1 is non-degenerate, namely Σ_1 was an isomorphism. Therefore (56) implies that $TM \equiv MT$, as desired. ∎

It is clear that to prove S_2 is a real multiple of S_1 means that the intertwining operator M has to be simply multiplication by a scalar. This can now be deduced by looking at the complexified representation of Möb $\left(S^1\right)$ on $\mathcal{H}_{\mathbb{C}}$, which is unitary, and applying Schur's Lemma.

LEMMA 4.6: The unitary representation of $SL(2,\mathbb{R})$ on $\mathcal{H}_{\mathbb{C}}$ decomposes into precisely two irreducible pieces - namely on W_+ and W_-. In fact these two representations correspond to the two lowest (conjugate) members in the discrete series for $SL(2,\mathbb{R})$.

Proof: We refer to [10] or [19] for the list of *irreducible unitary representations* of $SL(2,\mathbb{R})$ that constitute what is called its "discrete series". Each of these representations is indexed by an integer $m = \pm 2, \pm 3, \pm 4, \cdots$. For $m \geq 2$ one can write this representation on the L^2 space of holomorphic functions in Δ with the following weighted Poincaré measure:

$$d\nu_m = (1 - |z|^2)^m \frac{dxdy}{(1 - |z|^2)^2} \ , \ |z| < 1. \tag{57}$$

On the Hilbert space $L^2_{\mathrm{Hol}}(\Delta, d\nu_m)$ the discrete series representation of $SL(2,\mathbb{R})$ corresponding to this chosen value of m is given by $\pi_m : SL(2,\mathbb{R}) \to \mathrm{Aut}\left(L^2_{\mathrm{Hol}}(\Delta, d\nu_m)\right)$, where

$$\pi_m(\gamma)(f(z)) = f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-m}. \tag{58}$$

Here, of course, $\gamma \in SL(2,\mathbb{R})$ corresponds to the $(PSU(1,1))$ Möbius transformation $\dfrac{az+b}{cz+d}$ on the disc obained as usual by conjugating the $SL(2,\mathbb{R})$ matrix by the Möbius isomorphism (Cayley transform) of the upper half-plane onto the disc.

For $m \leq -2$ the anti-holomorphic functions conjugate to those in the above Hilbert spaces need only be used.

We claim that the representation given by the operators V_φ on W_+ (equation (35)), $\varphi \in$ Möb $\left(S^1\right)$, can be indentified with the $m = 2$ case of the above discrete series of representations of $SL(2,\mathbb{R})$. Note, Möb $\left(S^1\right) \equiv PSU(1,1) \cong SL(2,\mathbb{R})/(\pm I)$. Recall from Theorem 2.1 that W_+ is identifiable as $\mathrm{Hol}_2(\Delta)$. The action of φ is given on Hol_2 by :

$$V_\varphi(F) = F \circ \varphi - F \circ \varphi(0), \quad F \in \mathrm{Hol}_2(\Delta). \tag{59}$$

But Hol_2 consists of normalized $(F(0) = 0)$ holomorphic functions in Δ whose *derivative is in* $L^2(\Delta,$ Euclidean measure). From (59), by the chain rule,

$$\frac{d}{dz}V_\varphi(F) = \left(\frac{dF}{dz} \circ \varphi\right)\varphi' \tag{60}$$

So we can rewrite the representation on the derivatives of the functions in Hol_2 by the formula (60) - which coincides with formula (58) for $m = 2$. Indeed $d\nu_2$ is, by (57), simply the Euclidean (Lebesgue) measure on the disc and thus $L^2_{\mathrm{Hol}}(\Delta, d\nu_2) \cong \mathrm{Hol}_2(\Delta)$. (This last isomorphism being given by sending $F \in \mathrm{Hol}_2(\Delta)$ to its derivative.) Our claim is proved. ∎

It is clear that the representation on the conjugate space will correspond to the $m = -2$ (highest weight vector of weight -2) case of the discrete series. In particular, the representations we obtain of Möb $\left(S^1\right)$ by unitary operators of W_+ and W_- are both *irreducible.*

Proof of Theorem 4.3: By Lemma 4.5, (the $\mathbb{C}$-linear extension of) the intertwining operator M commutes with every one of the unitary operators $V_\varphi : \mathcal{H}_\mathbb{C} \to \mathcal{H}_\mathbb{C}$ as φ varies over Möb $\left(S^1\right)$. Since W_+ and W_- are the only two invariant subspaces for all the V_φ, as proved above, it follows that M must map W_+ either to W_+ or to W_-. Let us first assume the former case. Then M commutes with all the unitary operators V_φ on W_+, which we know to be an irreducible representation. Schur's Lemma says that a unitary representation will be irreducible if and only if the only operators that commute with all the operators in the representation are simply the scalars (see [19, page 11]). Since M was a real operator to start with, the scalar must be real.

The alternative assumption that M maps W_+ to W_- is untenable. In fact, if that were so we could replace M by M followed with complex conjugation. This new M will map W_+ to itself and will again commute with all the V_φ, hence it must be a scalar. Since the original M arose from a real operator this scalar can be seen to be real. But scalar multiplication preserves W_+ - hence the intertwining operator must map W_+ (and W_-) to itself.$\blacksquare$

Our proof is complete. The absolute naturality of the symplectic form thus established will be utilised in understanding the $H^{1/2}$ space as a *Hilbertian* space, – namely a space possessing a fixed symplectic structure but a large family of compatible complex structures. See the following sections.

§5- The $H^{1/2}$ space as first cohomology:

The Hilbert space $H^{1/2}$, that is the hero of our tale, can be interpreted as the first cohomology space with real coefficients of the "universal Riemann surface" – namely the unit disc – in a Hodge-theoretic sense. That will be fundamental for us in explaining the properties of the period mapping on the universal Teichmüller space.

In fact, in the classical theory of the period mapping, the vector space $H^1(X, \mathbb{R})$ plays a basic role, X being a closed orientable topological surface of genus g to start with. This real vector space comes equipped with a canonical symplectic structure given by the cup-product pairing, S. Now, whenever X has a complex manifold structure, this real space $H^1(X, \mathbb{R})$ of dimension $2g$ gets endowed with *a complex structure J that is compatible with the cup-pairing S*. This happens as follows: When X is a Riemann surface, the cohomology space above is precisely the vector space of real harmonic 1-forms on X, by the Hodge theorem. Then the *complex structure J is the Hodge star operator on the harmonic 1-forms*. The compatibility with the cup form is encoded in the relationships (61) and (62):

$$S(J\alpha, J\beta) = S(\alpha, \beta), \quad \text{for all} \;\; \alpha, \beta \in H^1(X, \mathbb{R}) \tag{61}$$

and that, intertwining S and J exactly as in equation (11),

$$S(\alpha, J\beta) = inner \;\; product(\alpha, \beta) \tag{62}$$

should define a positive definite inner product on $H^1(X, \mathbb{R})$. [In fact, as we will further describe in Section 7, the Siegel disc of period matrices for genus g is precisely the space of all the S-compatible complex structures J.] Consequently, the period mapping can be thought of as the variation of the Hodge-star complex structure on the topologically determined symplectic vector space $H^1(X, \mathbb{R})$. See Sections 7 and 8 below.

Remark: Whenever X has a complex structure, one gets an isomorphism between the real vector space $H^1(X, \mathbb{R})$ and the g dimensional complex vector space $H^1(X, \mathcal{O})$, where $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions. That is so because $\mathbb{R}$ can be considered as a subsheaf of $\mathcal{O}$ and hence there is an induced map on cohomology. It is interesting to check that this natural map is an isomorphism, and that the complex structure so induced on $H^1(X, \mathbb{R})$ is the same as that given above by the Hodge star.

For our purposes it therefore becomes relevant to consider, for an *arbitrary* Riemann Surface X, *the Hodge-theoretic first cohomology vector space as the space of L^2 (square-integrable) real harmonic 1-forms on X.* This real Hilbert space will be denoted $\mathcal{H}(X)$. Once again, in complete generality, this Hilbert space has a non-degenerate symplectic form S given by the cup (= wedge) product:

$$S(\phi_1, \phi_2) = \int\int_X \phi_1 \wedge \phi_2 \tag{63}$$

and the Hodge star is the complex structure J of $\mathcal{H}(X)$ which is again compatible with S as per (61) and (62). In fact, one verifies that the L^2 inner product on $\mathcal{H}(X)$ is given by the triality relationship (62) – which is the same as (11).

Since in the universal Teichmüller theory we deal with the "universal Riemann surface" – namely the unit disc Δ – (being the universal cover of all Riemann surfaces), we require the

PROPOSITION 5.1: For the disc Δ, the Hilbert space $\mathcal{H}(\Delta)$ is isometrically isomorphic to the real Hilbert space $\mathcal{H}$ of Section 2. Under the canonical identification the cup-wedge pairing is the canonical symplectic form S and the Hodge star becomes the Hilbert-transform on $\mathcal{H}$.

Proof: For every $\phi \in \mathcal{H}(\Delta)$ there exists a unique real harmonic function F on the disc with $F(0) = 0$ and $dF = \phi$. Clearly then, $\mathcal{H}(\Delta)$ is isometrically isomorphic to the Dirichlet space $\mathcal{D}$ of normalized real harmonic functions having finite energy. But in Section 2 we saw that this space is isometrically isomorphic to $\mathcal{H}$ by passing to the boundary values of F on S^1.

If $\phi_1 = dF_1$ and $\phi_2 = dF_2$, then integrating $\phi_1 \wedge \phi_2$ on the disc amounts to, by Stokes' theorem,

$$\int\int_\Delta dF_1 \wedge dF_2 = \int_{S^1} F_1 dF_2 = S(F_1, F_2)$$

as desired.

Finally, let $\phi = udx + vdy$ be a L^2 harmonic 1-form with $\phi = dF$. Suppose G is the harmonic conjugate of F with $G(0) = 0$. Then $dF + idG$ is a holomorphic 1-form on Δ with real part ϕ. It follows that the Hodge star maps ϕ to dG; hence, under the above canonical identification of $\mathcal{H}(\Delta)$ with $\mathcal{H}$, we see that the star operator becomes the Hilbert transform. ▮

Remark on the generalised Jacobi variety: The complex torus that is the Jacobi variety of a closed genus g Riemann surface X can be described as the complex vector space $(H^1(X,\mathbb{R}), \text{Hodge star})$ modulo the lattice $H_1(X,\mathbb{Z})$. Indeed, the integral homology group acts on $H^1(X,\mathbb{R})$ as linear functionals by integration of 1-forms on cycles, and since $H^1(X,\mathbb{R})$ is a Hilbert space, we may canonically identify the dual space with itself. Thus $H_1(X,\mathbb{Z})$ appears embedded inside $H^1(X,\mathbb{R})$, and the quotient is the complex torus that is the classical Jacobi variety of the Riemann surface.

But for the same reasons as above, for an *arbitrary* Riemann surface X, $H_1(X,\mathbb{Z})$ does sit inside the Hodge-theoretic first cohomology Hilbert space $\mathcal{H}(X)$. And this last space carries, as we know, the Hodge star complex structure. Thus it makes sense to try to define the generalised Jacobi variety of X as the quotient of this complex Hilbert space by the "discrete subgroup" $H^1(X,\mathbb{Z})$. For certain classes of open Riemann surfaces that quotient is a reasonable object, and we will report on these matters in future articles. For the unit disc itself then, the generalised Jacobian *is* the Hilbert space $H^{1/2} = \mathcal{H}$ equipped with the Hilbert transform complex structure.

§6- Quantum calculus and $H^{1/2}$:

A.Connes has proposed (see, for example, [6] and Connes' book "Geometrie Non-Commutatif") a "quantum calculus" that associates to a function f an operator that should be considered its quantum derivative – so that the operator theoretic properties of this $d^Q(f)$ capture the smoothness properties of the function. One advantage is that this operator can undergo all the operations of the functional calculus. The fundamental definition in one real dimension is

$$d^Q(f) = [J, M_f] \tag{64}$$

where J is the Hilbert transform in one dimension explained in Section 2, and M_f stands for (the generally unbounded) operator given by multiplication by f. One can think of the quantum derivative as operating (possibly unboundedly) on the Hilbert space $L^2(S^1)$ or on other appropriate function spaces.

Note: We will also allow quantum derivatives to be taken with respect to other Hilbert-transform like operators; in particular, the Hilbert transform can be replaced by some conjugate of itself by a suitable automorphism of the Hilbert space under concern. In that case we will make explicit the J by writing $d_J^Q(f)$ for the quantum derivative. See Section 8 for applications.

As sample results relating the properties of the quantum derivative with the nature of f, we quote: $d^Q(f)$ is a bounded operator on $L^2(S^1)$ *if and only if* the function f is of bounded mean oscillation. In fact, the operator norm of the quantum derivative is equivalent to the BMO norm of f. Again, $d^Q(f)$ is a compact operator on $L^2(S^1)$ *if and only if* f is in $L^\infty(S^1)$ and has vanishing mean oscillation. Also, if f is Hölder, (namely in some Hölder class), then the quantum derivative acts as a compact operator on Hölder. See [6], [6b]. (Note that the union of all the Hölder classes is both quasisymmetrically invariant and Hilbert-transform stable. Moreover, functions that are of bounded variation and Hölder form a quasisymmetrically invariant subspace of $H^{1/2}$.) Similarly, the requirement that f is a member of certain Besov spaces can be encoded in properties of the quantum derivative.

Our Hilbert space $H^{1/2}(\mathbb{R})$ has a very simple interpretation in these terms:

PROPOSITION 6.1: $f \in H^{1/2}(\mathbb{R})$ if and only if the operator $d^Q(f)$ is Hilbert-Schmidt on $L^2(\mathbb{R})$ [or on $H^{1/2}(\mathbb{R})$]. The Hilbert-Schmidt norm of the quantum derivative *coincides* with the $H^{1/2}$ norm of f.

Proof: Recall that the Hilbert transform on the real line is given as a singular integral operator with integration kernel $(x-y)^{-1}$. A formal calculation therefore shows that

$$(d^Q(f))(g)(x) = \int_{\mathbb{R}} \frac{f(x)-f(y)}{x-y} g(y)dy \tag{65}$$

But the above is an integral operator with kernel $K(x,y) = (f(x)-f(y))/(x-y)$, and such an operator is Hilbert-Schmidt if and only if the kernel is square-integrable over $\mathbb{R}^2$. Utilising now the Douglas identity – equations (24) or (42) – we are through. ∎

Since the Hilbert transform, J, is the standard complex structure on the $H^{1/2}$ Hilbert space, and since this last space was shown to allow an action by the quasisymmetric group, $QS(\mathbb{R})$, some further considerations become relevant. Introduce the operator L on 1-forms on the line to functions on the line by:

$$(L\varphi)(x) = \int_{\mathbb{R}} [log|x-y|]\varphi(y)dy \tag{66}$$

One may think of the Hilbert transform J as operating on either the space of functions or on the space of 1-forms (by integrating against the kernel $dx/(x-y)$). Let d as usual denote total derivative (from functions to 1-forms). Then notice that L above is an operator that is essentially a smoothing inverse of the exterior derivative. In fact, L and d are connected to J via the relationships:

$$d \circ L = J_{1-forms}; \quad L \circ d = J_{functions} \tag{67}$$

The Quasisymmetrically deformed operators: Given any q.s. homeomorphism $h \in QS(\mathbb{R})$ we think of it as producing a q.s. change of structure on the line, and hence we define the corresponding transformed operators, L^h and J^h by $L^h = h \circ L \circ h^{-1}$ and $J^h = h \circ J \circ h^{-1}$. ($J$ is being considered on functions in $\mathcal{H} = H^{1/2}(\mathbb{R})$, as usual.) The q.s homeomorphism (assumed to be say C^1 for the deformation on L), operates standardly on functions and forms by pullback. Therefore, J^h *simply stands for the Hilbert transform conjugated by the symplectomorphism* T_h *of* $\mathcal{H}$ *achieved by pre-composing by the q.s. homeomorphism* h. J^h is thus a new complex structure on $\mathcal{H}$.

Note: The complex structures on $\mathcal{H}$ of type J^h are exactly those that constitute the image of $T(1)$ by the universal period mapping. (See Section 8.) The target manifold, the universal Siegel space, can be thought of as a space of S-compatible complex structures on $\mathcal{H}$.

Let us write the perturbation achieved by h on these operators as the "quantum brackets":

$$\{h, L\} = L^h - L; \quad \{h, J\} = J^h - J. \tag{68}$$

Now, for instance, the operator $d \circ \{h, J\}$ is represented by the kernel $(h \times h)^* m - m$ where $m = dxdy/(x-y)^2$. For h suitably smooth this is simply $d_y d_x(log[(h(x) - h(y))/(x-y)])$. It is well known that $(h \times h)^* m = m$ when h is a Möbius transformation. Interestingly, therefore, on the diagonal ($x = y$), this kernel becomes (1/6 times) the Schwarzian derivative of h (as a quadratic differential on the line). For the other operators in the table below the kernel computations are even easier.

Set $K(x, y) = log[(h(x) - h(y))/(x - y)]$ for convenience. We have the following table of quantum calculus formulas:

Operator	**Kernel**	**On diagonal**	**Cocycle on** $QS(\mathbb{R})$
$\{h, L\}$	$K(x, y)$	$log(h')$	$function - valued$
$d \circ \{h, L\}$	$d_x K(x, y)$	$\frac{h''}{h'} dx$	$1 - form - valued$
$d \circ \{h, J\}$	$d_y d_x K(x, y)$	$\frac{1}{6} Schwarzian(h) dx^2$	$quadratic - form - valued$

(69)

The point here is that these operators make sense when h is merely quasisymmetric. If h happens to be appropriately smooth, we can restrict the kernels to the diagonal to obtain the respective nonlinear classical derivatives (affine Schwarzian, Schwarzian, etc.) as listed in the table above.

Remark: It is worth pointing out that the central extensions associated to the three cocycles in the horizontal lines of the table above respectively correspond to the subgroups: *(i)Translations, (ii)Affine transformations,* and *(iii)Projective (Möbius) transformations.*

§7- The universal period mapping on $T(1)$:

Having now all the necessary background results behind us, we are finally set to move into the theory of the universal period (or polarisations) map itself.

The Frechet Lie group, $Diff(S^1)$ operating by pullback (= pre-composition) on smooth functions, had a faithful representation by bounded symplectic operators on the symplectic vector space V of equation (1). That induced the natural map Π of the homogeneous space $M = Diff(S^1)/M\ddot{o}b(S^1)$ into Segal's version of the Siegel space of period matrices. In [12] [13] we had shown that this map:

$$\Pi : Diff(S^1)/M\ddot{o}b(S^1) \to Sp_0(V)/U \tag{70}$$

is *equivariant, holomorphic, Kähler isometric immersion*, and moreover that it qualifies as a *generalised period matrix map* (remembering ([15]) that the domain is a complex submanifold of the universal space of Riemann surfaces $T(1)$).

From the results of Sections 2, 3, and 4, we know that the full quasisymmetric group, $QS(S^1)$ operates as bounded symplectic operators on the Hilbert space $\mathcal{H}$ that is the completion of the pre-Hilbert space V. The same proof as offered in the articles quoted demonstrates that the subgroup of QS acting unitarily is the Möbius subgroup. Clearly then we have obtained the *extension of* Π (also called Π to save on nomenclature) *to the entire universal Teichmüller space*:

$$\Pi : T(1) \to Sp(\mathcal{H})/U \tag{71}$$

Let us first exhibit the nature of the complex Banach manifold that is the target space of the period map (71). This space, which is the universal Siegel period matrix space, denoted $\mathcal{S}_\infty$, has several interesting descriptions:

(a): $\mathcal{S}_\infty$= the space of positive polarizations of the symplectic Hilbert space $\mathcal{H}$. Recall ([12], [13], [18]) that a positive polarization signifies the choice of a closed complex subspace W in $\mathcal{H}_{\mathbb{C}}$ such that (i) $\mathcal{H}_{\mathbb{C}} = W \oplus \overline{W}$; (ii) W is S-isotropic, namely S vanishes on arbitrary pairs from W; and (iii) $iS(w, \overline{w})$ defines the square of a norm on $w \in W$.

(b): $\mathcal{S}_\infty$= the space of S-compatible complex structure operators on $\mathcal{H}$. That consists of bounded invertible operators J of $\mathcal{H}$ onto itself whose square is the negative identity and J is compatible with S in the sense that requirements (61) and (62) are valid. Alternatively, these are the complex structure operators J on $\mathcal{H}$ such that $H(f,g) = S(f,Jg) + iS(f,g)$ is a positive definite Hermitian form having S as its imaginary part.

(c): $\mathcal{S}_\infty$= the space of bounded operators Z from W_+ to W_- that satisfy the condition of S-symmetry: $S(Z\alpha,\beta) = S(Z\beta,\alpha)$ and are in the unit disc in the sense that $(I - Z\overline{Z})$ is positive definite. The matrix for Z is the "period matrix" of the classical theory.

(d): $\mathcal{S}_\infty$= the homogeneous space $Sp(\mathcal{H})/U$; here $Sp(\mathcal{H})$ denotes all bounded symplectic automorphisms of $\mathcal{H}$, and U is the unitary subgroup defined as those symplectomorphisms that keep the subspace W_+ (setwise) invariant.

Introduce the *Grassmannian* $Gr(W_+,\mathcal{H}_\mathbb{C})$ of subspaces of type W_+ in $\mathcal{H}_\mathbb{C}$, which is obviously a complex Banach manifold modelled on the Banach space of all bounded operators from W_+ to W_-. Clearly, $\mathcal{S}_\infty$ is embedded in Gr as a complex submanifold. The connections between the above descriptions of the Siegel universal space are transparent:

(a:b) the positive polarizing subspace W is the $-i$-eigenspace of the complex structure operator J (extended to $\mathcal{H}_\mathbb{C}$ by complex linearity).

(a:c) the positive polarizing subspace W is the graph of the operator Z.

(a:d) $Sp(\mathcal{H})$ acts transitively on the set of positive polarizing subspaces. W_+ is a polarizing subspace, and the isotropy (stabilizer) subgroup thereat is exactly U.

$\mathcal{H}$ **as a Hilbertian space:** Note that the method (b) above describes the universal Siegel space as a space of Hilbert space structures on the fixed underlying symplectic vector space $\mathcal{H}$. By the result of Section 4 we know that the symplectic structure on $\mathcal{H}$ is completely canonical, whereas each choice of J above gives a Hilbert space inner product on the space by intertwining S and J by the fundamental relationship (11) (or (62)). Thus $\mathcal{H}$ is a *"Hilbertian space"*, which signifies a complete topological vector space with a canonical symplectic structure but lots of compatible inner products turning it into a Hilbert space in many ways.

We come to one of our *Main Theorems:*
THEOREM 7.1: The universal period mapping Π is an injective, equivariant, holomorphic immersion between complex Banach manifolds.
Proof: From our earlier papers [12] [13] we know these facts for Π restricted to M. The proof of equivariance is the same (and simple). The map is an injection because if we know the subspace W_+ pulled back by w_μ, then we can recover the q.s. homeomorphism w_μ. In

fact, inside the given subspace look at those functions which map S^1 homeomorphically on itself. One sees easily that these must be precisely the Möbius transformations of the circle pre-composed by w_μ. The injectivity (global Torelli theorem) follows.

Let us write down the matrix for the symplectomorphism T on $\mathcal{H}_{\mathbb{C}}$ obtained by pre-composition by w_μ. We will write in the standard orthonormal basis $e^{ik\theta}/k^{1/2}$, $k = 1, 2, 3..$ for W_+, and the complex conjugates as o.n. basis for W_-.

In $\mathcal{H}_{\mathbb{C}} = W_+ \oplus W_-$ block form, T is given by maps: $A : W_+ \to W_+$, $B : W_- \to W_+$. The conjugates of A and B map W_- to W_- and W_+ to W_-, respectively. The matrix entries for $A = ((a_{pq}))$ and $B = ((b_{rs}))$ turn out to be:

$$a_{pq} = (2\pi)^{-1} p^{1/2} q^{-1/2} \int_0^{2\pi} (w_\mu(e^{i\theta}))^q e^{-ip\theta} d\theta, \quad p, q \geq 1$$

$$b_{rs} = (2\pi)^{-1} r^{1/2} s^{-1/2} \int_0^{2\pi} (w_\mu(e^{i\theta}))^{-s} e^{-ir\theta} d\theta, \quad r, s \geq 1$$

Recalling the standard action of symplectomorphisms on the Siegel disc (model **(c)** above), we see that the corresponding operator [=period matrix] Z appearing from the Teichmüller point $[\mu]$ is given by:

$$\Pi[\mu] = \overline{B} A^{-1}$$

The usual proof of finite dimensions shows that for any symplectomorphism A must be invertible – hence the above explicit formula makes sense.

Since the Fourier coefficients appearing in A and B vary only *real-analytically* with μ, it may be somewhat surprising that Π is actually *holomorphic*. In fact, a computation of the first variation of Π at the origin of $T(1)$ (i.e., the derivative map) in the Beltrami direction ν shows that the following *Rauch variational formula* subsists:

$$(d\Pi([\nu]))_{rs} = \pi^{-1} (rs)^{1/2} \int \int_\Delta \nu(z) z^{r+s-2} dx dy$$

The proof of this formula is as shown for Π on the smooth points submanifold M in our earlier papers. The manifest complex linearity of the derivative, i.e., the validity of the Cauchy-Riemann equations, combined with equivariance, demonstrates that Π is complex analytic on $T(1)$, as desired. ∎

Interpretation of Π as period map: Let us take a moment to recall why the map Π qualifies as a universal version of the classical genus g period maps. As we had explained in our previous papers, in the light of P.Griffiths' ideas, the classical period map may be thought of as associating to a Teichmüller point a positive polarizing subspace of the first cohomology $H^1(X,\mathbb{R})$. The point is that when X has a complex structure, then the complexified first cohomology decomposes as:

$$H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \tag{72}$$

The period map associates the subspace $H^{1,0}(X)$ – which is positive polarizing with respect to the cup-product symplectic form – to the given complex structure on X. Of course, $H^{1,0}(X)$ represents the holomorphic 1-forms on X, and that is why this is nothing but the usual period mapping.

But that is precisely what Π *is doing in the universal Teichmüller space.* Indeed, by the results of Section 5, $\mathcal{H}$ is the Hodge-theoretic real first cohomology of the disc, with S being the cup-product.

The standard complex structure on the unit disc has holomorphic 1-forms that are of the form dF where F is a holomorphic function on Δ with $F(0) = 0$. Thus the boundary values of F will have only positive index Fourier modes – corresponding therefore to the polarizing subspace W_+. Now, an arbitrary point of $T(1)$ is described by the choice of a Beltrami differential μ on Δ perturbing the complex structure. We are now asking for the holomorphic 1-forms on Δ_μ. Solving the Beltrami equation on Δ provides us with the μ-conformal quasiconformal self-homeomorphism w_μ of the disc. This w_μ is a holomorphic uniformising coordinate for the disc with the μ complex structure. The holomorphic 1-forms subspace, $H^{1,0}(\Delta_\mu)$, should therefore comprise those functions on S^1 that are the W_+ functions *precomposed with the boundary values of the q.c. map* w_μ. That is exactly the action of Π on the Teichmüller class of μ. This explains in some detail why Π behaves as an infinite dimensional period mapping.

Remark: On Segal's C^∞ version of the Siegel space – constructed using Hilbert-Schmidt operators Z, there existed the universal *Siegel symplectic metric*, which we studied in [12] [13] and showed to be the same as the Kirillov-Kostant (= Weil-Petersson) metric on $Diff(S^1)/Mob(S^1)$. For the bigger Banach manifold $\mathcal{S}_\infty$ above, that pairing fails to converge on arbitrary pairs of tangent vectors because the relevant operators are not any more trace-class in general. The difficulties associated with this matter will be addressed in Section 9 below, and in further work that is in progress.

§8- The universal Schottky locus and quantum calculus:

Our object is to exhibit the image of Π in $\mathcal{S}_\infty$. The result (equation (73)) can be recognized to be a quantum "integrability condition" for complex structures on the circle or the line.

PROPOSITION 8.1: If a positive polarizing subspace W is in the "universal Schottky locus", namely if W is in the image of $T(1)$ under the universal period mapping Π, then W

possesses a dense subspace which is *multiplication-closed* (i.e., an "algebra" under pointwise multiplication modulo subtraction of mean-value.) In quantum calculus terminology, this means that

$$[d^Q_J, J] = 0 \tag{73}$$

where J denotes the S-compatible complex structure of $\mathcal{H}$ whose $-i$-eigenspace is W. (Recall the various descriptions of $\mathcal{S}_\infty$ spelled out in the last section.)

Multiplication-closed polarizing subspace: The notion of being multiplication-closed is well-defined for the relevant subspaces in $\mathcal{H}_{\mathbb{C}}$. Let us note that the original polarizing subspace W_+ contains the dense subspace of holomorphic trigonometric polynomials (with mean zero) which constitute an algebra. Indeed, the identity map of S^1 is a member of W_+, call it j, and positive integral powers of j clearly generate W_+ – since polynomials in j form a dense subspace therein. Now if W is any other positive polarizing subspace, we know that it is the image of W_+ under some $T \in Sp(\mathcal{H})$. Thus, W will be multiplication-closed precisely when the image of j by T generates W, in the sense that its positive integral powers (minus the mean values) also lie in W (and hence span a dense subspace of W).

In other words, we are considering W ($\in \mathcal{S}_\infty$ [description (a)]) to be multiplication-closed provided that the pointwise products of functions from W (minus their mean values) that happen to be $H^{1/2}$ functions actually land up in the subspace W again. Multiplying f and g modulo arbitrary additive constants demonstrates that this notion is well-defined when applied to a subspace.

Quantum calculus and equation (73): We suggest a quantum version of complex structures in one real dimension, and note that the integrable ones correspond to the universal Schottky locus under study.

In the spirit of algebraic geometry one takes the real Hilbert space of functions $\mathcal{H} = H^{1/2}(\mathbb{R})$ as the "coordinate ring" of the real line. Consequently, a complex structure on $\mathbb{R}$ will be considered to be a complex structure on this Hilbert space. Since $\mathcal{S}_\infty$ was a space of (symplectically-compatible) complex structures on $\mathcal{H}$, we are interpreting $\mathcal{S}_\infty$ as a space of quantum complex structures on the line (or circle).

Amongst the points of the universal Siegel space, those that can be interpreted as the holomorphic function algebra for some complex structure on the circle qualify as the "integrable" ones. But $T(1)$ parametrises all the quasisymmetrically related circles, and for each one, the map Π associates to that structure the holomorphic function algebra corresponding to it; see the interpretation we provided for Π in the last section. It is clear therefore that $\Pi(T(1)$ should be the integrable complex structures. The point is that taking the standard circle as having integrable complex structure, all the other integrable complex structures arise from this one by a QS change of coordinates on the underlying circle. These are the complex structures J^h introduced in Section 6 on quantum calculus. The $-i$-eigenspace for J^h is interpreted as the algebra of analytic functions on the quantum real line with the h-structure. We will see in the proof that (73) encodes just this condition.

Proof of Proposition 8.1: For a point of $T(1)$ represented by a q.s. homeomorphism ϕ, the period map sends it to the polarizing subspace $W_\phi = W_+ \circ \phi$. But W_+ was a multiplication-closed subspace, generated by just the identity map j on S^1, to start with. Clearly then, $\Pi(\phi) = W_\phi$ is also multiplication-closed in the sense explained, and is generated by the image of the generator of W_+ – namely by the q.s homeomorphism ϕ (as a member of $\mathcal{H}_\mathbb{C}$). ∎

We suspect that the converse is also true: that the $T(W_+)$ is such an "algebra" subspace for a symplectomorphism T in $Sp(\mathcal{H})$ only when T arises as pullback by a quasisymmetric homeomorphism of the circle. This converse assertion is reminescent of standard theorems in Banach algebras where one proves, for example, that every (conjugation-preserving) algebra automorphism of the algebra $C(X)$ (comprising continuous functions on a compact Hausdorff space X) arises from homeomorphisms of X. [Remark of Ambar Sengupta.] Owing to the technical hitch that $H^{1/2}$ functions are not in general everywhere defined on the circle, we are as yet unable to find a rigorous proof of this converse.

Here is the sketch of an idea for proving the converse. Thus, suppose we are given a subspace E that is multiplication-closed in the sense explained. Now, $Sp(\mathcal{H})$ acts transitively on the set of positive polarizing subspaces. We consider a $T \in Sp(\mathcal{H})$ that maps W_+ to E preserving the algebra structure (modulo subtracting off mean values as usual). Denote by j the identity function on S^1 and let $T(j) = w$ be its image in E.

Since j is a homeomorphism and T is an invertible real symplectomorphism, one expects that w is also a homeomorphism on S^1. (Recall the signed area interpretation of the canonical form (8).) It then follows that the T is nothing other that precomposition by this w. That is because:

$$T(j^m) = T(j)^m - \text{mean value} = (w(e^{i\theta}))^m - \text{mean value} = j^m \circ w - \text{mean value}.$$

Knowing T to be so on powers of j is sufficient, as polynomials in j are dense in W_+.

Again, since T is the complexification of a real symplectomorphism, seeing the action of T on W_+ tells us T on all of $\mathcal{H}_\mathbb{C}$; namely, T is everywhere precomposition by that homeomorphism w of S^1. *By the necessity part of Theorem 3.1 we see that w must be quasisymmetric*, and hence that the given subspace E is the image under Π of the Teichmüller point determined by w (i.e., the coset of w in $QS(S^1)/M\ddot{o}b(S^1)$).

Proof of equation (73): Let J be *any* S-compatible complex structure on $\mathcal{H}$, namely J is an arbitrary point of $\mathcal{S}_\infty$ (description (b) of Section 7). Let J_0 denote the Hilbert transform itself, which is the reference point in the universal Siegel space; therefore $J = TJ_0T^{-1}$ for some symplectomorphism T in $Sp(\mathcal{H})$. The $-i$-eigenspace for J_0 is, of course, the reference polarizing subspace W_+, and the subspace W corresponding to J consists of the functions $(f + i(Jf))$ for all f in $\mathcal{H}$. Now, the pointwise product of two such typical elements of W gives:

$$(f + i(Jf))(g + i(Jg)) = [fg - (Jf)(Jg)] + i[f(Jg) + g(Jf)]$$

In order for W to be multiplication closed the function on the right hand side must also be of the form $(h + i(Jh))$. Namely, for all relevant f and g in the real Hilbert space $\mathcal{H}$ we must have:

$$J[fg - (Jf)(Jg)] = [f(Jg) + g(Jf)] \tag{74}$$

Now recall from the concepts introduced in Section 6 that one can associate to functions f their quantum derivative operators $d^Q_J(f)$ which is the commutator of J with the multiplication operator M_f defined by f. The quantum derivative is being taken with respect to any Hilbert-transform-like operator J as explained above. But now a short computation demonstrates that equation (74) is the same as saying that:

$$J \circ d^Q_J(f) = -d^Q_J(f)$$

operating by J on both sides shows that this is the same as (73). That is as desired. ∎

Remark: For the classical period mapping on the Teichmüller spaces T_g there is a way of understanding the Schottky locus in terms of Jacobian theta functions satisfying the nonlinear K-P equations. In a subsequent paper we hope to relate the finite dimensional Schottky solution with the universal solution given above.

Remark: For the extended period-polarizations mapping Π, the Rauch variational formula that was exhibited in [12], [13], [13a], and here in the proof of Theorem 7.1, continues to hold.

§9- The Teichmüller space of the universal lamination and Weil-Petersson:

The Universal Teichmüller Space, $T(1)$=$T(\Delta)$, is a *non-separable* complex Banach manifold that contains, as properly embedded complex submanifolds, all the Teichmüller spaces, T_g, of the classical compact Riemann surfaces of every genus g (≥ 2). T_g is $3g - 3$ dimensional and appears (in multiple copies) within $T(\Delta)$ as the Teichmüller space $T(G)$ of the Fuchsian group G whenever Δ/G is of genus g. The closure of the union of a family of these embedded T_g in $T(\Delta)$ turns out to be a separable complex submanifold of $T(\Delta)$ (modelled on a separable complex Banach space). That submanifold can be identified as being itself the Teichmüller space of the "universal hyperbolic lamination" H_∞. We will show that $T(H_\infty)$ carries a canonical, genus-independent version of the Weil-Petersson metric, thus bringing back into play the Kähler structure-preserving aspect of the period mapping theory.

The universal laminated surfaces: Let us proceed to explain the nature of the (two possible) "universal laminations" and the complex structures on these. Starting from any closed topological surface, X, equipped with a base point, consider the inverse (directed) system of all finite sheeted unbranched covering spaces of X by other closed pointed surfaces. The covering projections are all required to be base point preserving, and isomorphic covering spaces are identified. The *inverse limit space* of such an inverse system is the "lamination" – which is the focus of our interest.

The lamination E_∞: Thus, if X has genus one, then, of course, all coverings are also tori, and one obtains as the inverse limit of the tower a certain compact topological space – every path component of which (the laminating leaves) – is identifiable with the complex plane. This space E_∞ (to be thought of as the "universal Euclidean lamination") is therefore a fiber space over the original torus X with the fiber being a Cantor set. The Cantor set corresponds to all the possible backward strings in the tower with the initial element being the base point of X. The total space is compact since it is a closed subset of the product of all the compact objects appearing in the tower.

The lamination H_∞: Starting with an arbitrary X of higher genus clearly produces the *same* inverse limit space, denoted H_∞, independent of the initial genus. That is because given any two surfaces of genus greater than one, there is always a common covering surface of higher genus. H_∞ is our universal hyperbolic lamination, whose Teichmüller theory we will consider in this section. For the same reasons as in the case of E_∞, this new lamination is also a compact topological space fibering over the base surface X with fiber again a Cantor set. (It is easy to see that in either case the space of backward strings starting from any point in X is an uncountable, compact, perfect, totally-disconnected space – hence homeomorphic to the Cantor set.) The fibration restricted to each individual leaf (i.e., path component of the lamination) is a universal covering projection. Indeed, notice that the leaves of H_∞ (as well as of E_∞) must all be simply connected – since any non-trivial loop on a surface can be unwrapped in a finite cover. [That corresponds to the residual finiteness of the fundamental group of a closed surface.] Indeed, group-theoretically speaking, covering spaces correspond to the subgroups of the fundamental group. Utilising only normal subgroups (namely the regular coverings) would give a cofinal inverse system and therefore the inverse limit would still continue to be the H_∞ lamination. This way of interpreting things allows us to see that the transverse Cantor-set fiber actually has a group structure. In fact it is the pro-finite group that is the inverse limit of all the deck-transformation groups corresponding to these normal coverings.

Complex structures : Let us concentrate on the universal hyperbolic lamination H_∞ from now on. For any complex structure on X there is clearly a complex structure induced by pullback on each surface of the inverse system, and therefore H_∞ itself inherits a complex structure on each leaf, so that now biholomorphically each leaf is the Poincare hyperbolic plane. If we think of a reference complex structure on X, then any new complex structure is recorded by a Beltrami coefficient on X, and one obtains by pullback a complex structure on the inverse limit in the sense that each leaf now has a complex structure and the Beltrami coefficients vary continuously from leaf to leaf in the Cantor-set direction. Indeed, the complex structures obtained in the above fashion by pulling back to the inverse limit from a complex structure on any closed surface in the inverse tower, have the special property that the Beltrami coefficients on the leaves are locally constant in the transverse (Cantor) direction. These "locally constant" families of Beltrami coefficients on H_∞ comprise the *transversely locally constant* (written "TLC") complex structures on the lamination. The generic complex structure on H_∞, where all continuously varying Beltrami coefficients in the Cantor-fiber direction are admissible, will be a limit of the TLC subfamily of complex structures.

To be precise, a *complex structure* on a lamination L is a covering of L bylamination charts (disc) $\times$ (transversal) so that the overlap homeomorphisms are complex analytic on the disc direction. Two complex structures are *Teichmüller equivalent* whenever they are related to each other by a homeomorphism that is homotopic to the identity through leaf-preserving continuous mappings of L. For us L is, of course, H_∞. Thus we have defined the set $T(H_\infty)$.

Note that there is a distinguished leaf in our lamination, namely the path component of the point which is the string of all the base points. Call this leaf l. Note that all leaves are dense in H_∞, in particular l is dense. With respect to the base complex structure the leaf l gets a canonical identification with the hyperbolic unit disc Δ. Hence we have the natural "restriction to l" mapping of the Teichmüller space of H_∞ into the Universal Teichmüller space $T(l) = T(1)$. Since the leaf is dense, the complex structure on it records the entire complex structure of the lamination. The above restriction map is therefore actually *injective* (see [20]) and therefore describes $T(H_\infty)$ as an embedded complex analytic submanifold in $T(1)$.

Indeed, as we will explain in detail below, $T(H_\infty)$ embeds as precisely the closure in $T(1)$ of the union of the Teichmüller spaces $T(G)$ as G varies over all finite-index subgroups of a fixed cocompact Fuchsian group. These finite dimensional classical Teichmüller spaces lying within the separable, infinite-dimensional $T(H_\infty)$, comprise the TLC points of $T(H_\infty)$.

Alternatively, one may understand the set-up at hand by looking at the direct system of maps between Teichmüller spaces that is obviously induced by our inverse system of covering maps. Indeed, each covering map provides an immersion of the Teichmüller space of the covered surface into the Teichmüller space of the covering surface induced by the standard pullback of complex structure. These immersions are Teichmüller metric preserving, and provide a direct system whose direct limit, when completed in the Teichmüller metric, gives produces again $T(H_\infty)$. The direct limit already contains the classical Teichmüller spaces of closed Riemann surfaces, and the completion corresponds to taking the closure in $T(1)$.

Let us elaborate somewhat more on these various possible embeddings of $T(H_\infty)$ [which is to be thought of as the *universal Teichmüller space of* **compact** *Riemann surfaces*] within the classical universal Teichmüller space $T(\Delta)$.

Explicit realizations of $T(H_\infty)$ within the universal Teichmüller space: Start with any cocompact (say torsion-free) Fuchsian group G operating on the unit disc Δ, such that the quotient is a Riemann surface X of *arbitrary* genus g greater than one. Considering the inverse limit of the directed system of all unbranched finite-sheeted pointed covering spaces over X gives us a copy of the universal laminated space H_∞ equipped with a complex structure induced from that on X. Every such choice of G allows us to embed the separable Teichmüller space $T(H_\infty)$ holomorphically in the Bers universal Teichmüller space $T(\Delta)$.

To fix ideas, let us think of the universal Teichmüller space as: $T(\Delta) = T(1) = QS(S^1)/Mob(S^1)$ as usual.

For any Fuchsian group Γ define:

$$QS(\Gamma) = \{w \in QS(S^1) : w\Gamma w^{-1} \textit{ is again a Mobius group.}\}$$

We say that the quasisymmetric homeomorphisms in $QS(\Gamma)$ are those that are *compatible* with Γ. Then the Teichmüller space $T(\Gamma)$ is $QS(\Gamma)/Mob(S^1)$ clearly sits embedded within $T(1)$. [We always think of points of $T(1)$ as left-cosets of the form $Mob(S^1) \circ w = [w]$ for arbitrary quasisymmetric homeomorphism w of the circle.]

Having fixed the cocompact Fuchsian group G, the Teichmüller space $T(H_\infty)$ is now the closure in $T(1)$ of the direct limit of all the Teichmüller spaces $T(H)$ *as H runs over all the finite-index subgroups of the initial cocompact Fuchsian group G*. Since each $T(H)$ is actually embedded injectively within the universal Teichmüller space, and since the connecting maps in the directed system are all inclusion maps, we see that the direct limit (which is in general a quotient of the disjoint union) in this situation is nothing other that just the set-theoretic *union of all the embedded $T(H)$ as H varies over all finite index subgroups of G* . This union in $T(1)$ constitutes the dense "TLC" (transversely locally constant) subset of $T(H_\infty)$. Therefore, *the TLC subset of this embedded copy of $T(H_\infty)$*

comprises the Möb-classes of all those QS-homeomorphisms that are compatible with some finite index subgroup in G.

We may call the above realization of $T(H_\infty)$ as *" the G-tagged embedding" of* $T(H_\infty)$ in $T(1)$.

Remark: We see above, that just as the Teichmüller space of Riemann surfaces of any genus p have lots of realizations within the universal Teichmüller space (corresponding to choices of reference cocompact Fuchsian groups of genus p), the Teichmüller space of the lamination H_∞ also has many different realizations within $T(1)$.

Therefore, in the Bers embedding of $T(1)$, this realization of $T(H_\infty)$ is the intersection of the domain $T(1)$ in the Bers-Nehari Banach space $B(1)$ with the separable Banach subspace that is the inductive (direct) limit of the subspaces $B(H)$ as H varies over all finite index subgroups of the Fuchsian group G. (The inductive limt topology will give a complete (Banach) space; see, e.g., Bourbaki's "Topological Vector Spaces".) It is relevant to recall that $B(H)$ comprises the bounded holomorphic quadratic forms for the group H. By Tukia's results, the Teichmüller space of H is exactly the intersection of the universal Teichmüller space with $B(H)$.

Remark: Indeed one expects that the various G-tagged embeddings of $T(H_\infty$ must be sitting in general discretely separated from each other in the Universal Teichmüller space. There is a result to this effect for the various copies of $T(\Gamma)$, as the base group is varied, due to K.Matsuzaki (preprint – to appear in Annales Acad. Scient. Fennicae). That should imply a similar discreteness for the family of embeddings of $T(H_\infty)$ in $T(\Delta)$.

It is not hard now to see how many different copies of the Teichmüller space of genus p Riemann surfaces appear embedded within the G-tagged embedding of $T(H_\infty)$. That corresponds to non-conjugate (in G) subgroups of G that are of index $(p-1)/(g-1)$ in G. This last is a purely topological question regarding the fundamental group of genus g surfaces.

Modular group: One may look at those elements of the full universal modular group $Mod(1)$ [quasisymmetric homeomorphism acting by right translation (i.e., pre-composition) on $T(1)$] that preserve setwise the G-tagged embedding of $T(H_\infty)$. Since the modular group $Mod(\Gamma)$ on $T(\Gamma)$ is induced by right translations by those QS-homeomorphisms that are in the normaliser of Γ:

$$N_{qs}(\Gamma) = \{t \in QS(\Gamma) : t\Gamma t^{-1} = \Gamma\}$$

it is not hard to see that only the elements of $Mod(G)$ itself will manage to preserve the G-tagged embedding of $T(H_\infty)$. [Query: Can one envisage some limit of the modular groups of the embedded Teichmüller spaces as acting on $T(H_\infty)$?]

The Weil-Petersson pairing: In [20], it has been shown that the tangent (and the cotangent) space at any point of $T(H_\infty)$ consist of certain holomorphic quadratic differentials on the universal lamination H_∞. In fact, the Banach space $B(c)$ of tangent holomorphic quadratic differentials at the Teichmüller point represented by the complex structure c on the lamination, consists of holomorphic quadratic differentials on the leaves that vary continuously in the transverse Cantor-fiber direction. Thus locally, in a chart, these objects look like $\varphi(z,\lambda)dz^2$ in self-evident notation; (λ represents the fiber coordinate). The lamination H_∞ also comes equipped with an invariant transverse measure on the Cantor-fibers (invariant with respect to the holonomy action of following the leaves). Call that measure (fixed up to a scale) $d\lambda$. [That measure appears as the limit of (normalized) measures on the fibers above the base point that assign (at each finite Galois covering stage) uniform weights to the points in the fiber.] From [20] we have directly therefore our present goal:

PROPOSITION 9.1: The Teichmüller space $T(H_\infty)$ is a separable complex Banach manifold in $T(1)$ containing the direct limit of the classical Teichmüller spaces as a dense subset. The Weil-Petersson metrics on the classical T_g, normalized by a factor depending on the genus, fit together and extend to a finite Weil-Petersson inner product on $T(H_\infty)$ that is defined by the formula:

$$\int_{H_\infty} \varphi_1\varphi_2(Poin)^{-2}dz\wedge d\bar{z}d\lambda \tag{75}$$

where $(Poin)$ denotes the Poincare conformal factor for the Poincare metric on the leaves (appearing as usual for all Weil-Petersson formulas). ∎

Remark on Mostow rigidity for $T(H_\infty)$: The quasisymmetric homeomorphism classes comprising this Teichmüller space are again very non-smooth, since they appear as limits of the fractal q.s. boundary homeomorphisms corresponding to deformations of co-compact Fuchsian groups. Thus, the transversality proved in [15, Part II] of the finite dimensional Teichmüller spaces with the coadjoint orbit homogeneous space M continues to hold for $T(H_\infty)$. As explained there, that transversality is a form of the Mostow rigidity phenomenon. The formal Weil-Petersson converged on M and coincided with the Kirillov-Kostant metric, but that formal metric fails to give a finite pairing on the tangent spaces to the finite dimensional T_g. Hence the interest in the above Proposition.

§10-The Universal Period mapping and the Krichever map:

We make some remarks on the relationship of Π with the Krichever mapping on a certain family of Krichever data. This could be useful in developing infinite-dimensional theta functions that go hand-in-hand with our infinite dimensional period matrices.

The positive polarizing subspace, $T_\mu(W_+)$, that is assigned by the period mapping Π to a point $[\mu]$ of the universal Teichmüller space has a close relationship with the Krichever subspace of $L^2(S^1)$ that is determined by the Krichever map on certain Krichever data, when $[\mu]$ varies in (say) the Teichmüller space of a compact Riemann surface with one puncture (distinguished point). I am grateful to Robert Penner for discussing this matter with me.

Recall that in the Krichever mapping one takes a compact Riemann surface X, a point $p \in X$, and a local holomorphic coordinate around p to start with (i.e., a member of the "dressed moduli space"). One also chooses a holomorphic line bundle L over X and a particular trivialization of L over the given (z) coordinate patch around p. We assume that the z coordinate contains the closed unit disc in the z-plane. To such data, the Krichever mapping associates the subspace of $L^2(S^1)$ [here S^1 is the unit circle in the z coordinate] comprising functions which are restrictions to that circle of holomorphic sections of L over the punctured surface $X - \{p\}$.

If we select to work in a Teichmüller space $T(g, 1)$ of pointed Riemann surfaces of genus g, then one may choose z canonically as a certain horocyclic coordinate around the point p. Fix L to be the canonical line bundle $T^*(X)$ over X (the compact Riemann surface). This has a corresponding trivialization via "dz". The Krichever image of this data can be considered as a subspace living on the unit horocycle around p. That horocycle can be mapped over to the boundary circle of the universal covering disc for $X - \{p\}$ by mapping out by the natural pencil of Poincare geodesics having one endpoint at a parabolic cusp corresponding to p.

We may now see how to recover the Krichever subspace (for this restricted domain of Krichever data) from the subspace in $H^{1/2}_{\mathbb{C}}(S^1)$ *associated to* (X, p) *by* Π. Recall that the functions appearing in the Π subspace are the boundary values of the Dirichlet-finite harmonic functions whose derivatives give the holomorphic Abelian differentials of the Riemann surface. Hence, to get Krichever from Π one takes Poisson integrals of the functions in the Π image, then takes their total derivative in the universal covering disc, and restricts these to the horocycle around p that is sitting inside the universal cover (as a circle tangent to the boundary circle of the Poincare disc).

Since Krichever data allows one to create the tau-functions of the KP -hierarchy by the well-known theory of the Sato school (and the Russian school), one may now use the tau-function from the Krichever data to associate a tau (or theta) function to such points of our universal Schottky locus. The search for natural theta functions associated to points of the universal Siegel space $\mathcal{S}$, and their possible use in clarifying the relationship between the universal and classical Schottky problems, is a matter of interest that we are pursuing.

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