Exterior d, the Local Degree, and Smoothability

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In the late forties Whitney considered cochains on R^n which were bounded linear functionals on polyhedral chains for the norm

$$|a|_{\text{Whitney}} = \inf_b \; (|a - \partial b| + |b|)$$

where $|\cdot|$ denotes mass (length, area, volume, etc.). Whitney showed 1) these cochains (on which coboundary is a bounded operator) could be identified with the integration of differential forms ω so that both ω and $d\omega$ (in the sense of generalized derivatives) have bounded measurable coefficients and 2) this class was invariant by quasiisometries, $x \to x'$ such that

$$(1/L \text{ distance}\,(x,y) \leq \text{distance}\,(x',y') \leq L \text{ distance}\,(x,y))$$

(see [1] [2] [11]).

Thus we have Whitney cochains or Whitney forms, exterior d, and wedge product on any topological manifold provided with charts so that overlap homeomorphisms are quasiisometries. The usual coordinate change calculations are possible because of the a.e. differentiability of quasiisometries and they are valid, [2] chapter 9.

Let us call a manifold equipped with quasiisometrically related charts and its Whitney forms a *Whitney manifold*[1].

We will study now two questions,

i) when does a Whitney manifold have a subsystem of smoothly related charts

ii) what characterizes the vector bundle which is the cotangent bundle for such a smoothing.

For a smooth manifold with cotangent bundle T and exterior bundle ΛT we have a natural map by integration

i) smooth sections $(\Lambda T) \xrightarrow{\gamma}$ Whitney forms (= bounded cochains for the Whitney norm)

ii) for any smooth torsion free connection ∇ on ΛT a commutative diagram

[1] We reserve the term "Lipschitz" for functions and other objects satisfying the Lipschitz condition, e.g. vector bundles over metric spaces. In section 2 we discuss Whitney spaces.

$$\begin{array}{ccccc} \text{sections } \Lambda T & \xrightarrow{\nabla} & \text{sections } \Lambda T \otimes \text{sections } T & \xrightarrow{\wedge} & \text{sections } \Lambda T \\ \downarrow \gamma & & & & \downarrow \gamma \\ \text{Whitney forms} & & \xrightarrow{d} & & \text{Whitney forms}\ . \end{array}$$

To answer questions i) and ii) we will abstract the properties i) and ii) of a smooth manifold. A new element *the local degree* appears.

For a Whitney manifold M one can consider Lipschitz vector bundles E over M, namely the overlap functions are Lipschitz mappings into $G\ell(n, R)$. We also have connections defined by Whitney 1-forms because these have restrictions a.e. to rectifiable arcs, [2] chapter 9.

Definition. A *cotangent structure* for a Whitney manifold M is a Lipschitz vector bundle E of dimension equal to dimension M and an embedding γ: Lipschitz sections of $E \xrightarrow{\gamma}$ Whitney one forms satisfying

i) γ is bounded from the Lipschitz norm to the Whitney norm, γ is a module map over the Lipschitz functions, and the induced map $\Lambda\gamma$ on sections of the top exterior powers sends a positive Lipschitz section to a bounded measurable volume form for M with an a.e. positive lower bound. (In the non orientable case we ask this last property locally on a finite covering.)

ii) near each point of M there is at least one connection ∇ on E defined by Whitney 1-forms which is *torsion free* in the sense that the following diagram commutes.

$$\begin{array}{cccc} \text{Lipschitz sections of } E & \xrightarrow{\nabla} & \begin{array}{c}\text{bounded measurable}\\ \text{sections of } E\end{array} & \otimes \text{ Whitney 1-forms} \\ & & & \downarrow \gamma \otimes \text{Identity} \\ \downarrow \gamma & & \begin{array}{c}\text{bounded measurable}\\ \text{one forms}\end{array} & \otimes \text{ Whitney 1-forms} \\ & & & \downarrow \text{wedge product} \\ \text{Whitney one forms} & \xrightarrow{\text{exterior } d} & & \text{bounded measurable 2-forms.} \end{array}$$

We have made use of the fact that the embedding γ extends to all bounded measurable sections (up to a.e. equivalence) using the action of L^∞ functions on each.

We will now study the more precise

Question. When is a cotangent structure (E, γ) over a Whitney manifold M near p locally isomorphic to that of a smooth structure on M near p?

Example. Consider the map of R^2 $(r,\theta) \to (r, 2\theta)$ and pull back the standard cotangent structure of R^2. Using the module structure over Lipschitz functions on the domain one obtains a new cotangent structure over R^2. We claim this cotangent structure is not locally isomorphic to that of smooth structure near zero because the "local degree" to be defined below is 2 instead of one. Away from the origin this cotangent structure is equivalent to a smooth structure and there the "local degree" is one.

Now we discuss *the local degree of a cotangent structure.* The definition is rather easy but the possibility to make the definition depends on Reshetnyak's theory of *mappings of bounded distortion* [10].

Let us work near a point p in M and choose for each pair of points $\{x, y\}$ near p a rectifiable arc (x, y) varying in a Lipschitz way for the Whitney norm on chains. For example pull back straight line arcs in a chosen quasiisometrical chart near p. For any local Lipschitz framing $\rho = (\rho_1, \ldots, \rho_n)$ of E near p denote the corresponding Whitney 1-forms $\gamma\rho = (\gamma\rho_1, \gamma\rho_2, \ldots, \gamma\rho_n)$ and form the mapping

$$(\text{Neighbourhood of } p, p) \xrightarrow{\rho_p} (R^n, 0)$$

defined by $\rho_p(x) = \int_{(p,x)} (\gamma\rho_1, \gamma\rho_2, \ldots, \gamma\rho_n)$, where (p, x) is the arc from p to x mentioned above. We have used here the property mentioned above that Whitney 1-forms have restrictions to rectifiable arcs [2] chapter 9.

Theorem 1. *For each choice of arc systems (x, y) and local framing ρ_p of E the corresponding mapping ρ_p (Neighborhood p, p) $\xrightarrow{\rho_p}$ $(R^n, 0)$ has zero as an isolated value near p. The local degree of ρ at p is defined, belongs to $\{1, 2, 3, \ldots\}$ and, given p and the cotangent structure (E, γ), is independent of the choices.*

Proof. The proof will be given in §1. We remark here that the burden of the proof rests on a nontrivial reverse inequality (true near p)

$$\text{distance } (\rho_p(x), 0) \geq (\text{constant}) \text{ distance } (x, p)$$

which in our Lipschitz case comes from the general structure of Reshetnyak's mappings of bounded distortion [10]. The possibility to use [10] arises because we can approximate $(\rho_1, \rho_2, \ldots, \rho_n)$ in Whitney norm by closed 1-forms $(\rho'_1, \ldots, \rho'_n)$ using the connection and the Poincaré lemma in a familiar way.

The rest of the proof relies on the simple idea that if two maps of (Neighborhood of p) $- \{p\}$ into $R^n - 0$ are much closer at each point than their distance from zero they will simultaneously satisfy (or not satisfy) the reverse inequality and have the same degree.

Definition. The *local degree* $i(E, \gamma, p)$ *of a cotangent structure* (E, γ) at the point $p \in M$ is the local degree of any choice of maps ρ_p in Theorem 1. We choose orientations so that degrees are positive. In the non orientable case the local degree is the absolute value of $i(E, \gamma, x)$ for local choices of orientation.

There is a corollary to the proof of Theorem 1. Let g be a measurable Riemannian metric on M determined by γ and a Lipschitz inner product on E. For each p in M and $\varepsilon_i \to 0$ consider the sequence of metrics g_i obtained by rescaling by $1/\varepsilon_i$ the metric on ε_i balls about p. (For convenience here we define an r ball to be a ball from a coordinate system which has g-volume r^n.)

Say a measurable metric is on an open set U in R^n is *branched Euclidean* if it is obtained by pulling back the Euclidean metric by a nondegenerate branched covering – namely a Lipschitz mapping $F : U \to R^n$ so that determinant (DF) is a.e. positive with a positive lower bound.

Corollary. *The sequence of rescaled metrics g_i of the ε_i balls about any p in M has limits in the sense of Gromov (see [7]) and every such limit is branched Euclidean with branching degree at p equal to $i(E, \gamma, p)$.*

Proof. The ρ' of the proof of Theorem 1 have Jacobians which are $O(\varepsilon_i)$ quasiisometries (see f) g) h)). Their $1/\varepsilon_i$ rescalings are precompact by f). The rest is definition, [7], and stability of local degree. □

The approximation ideas used in Theorem 1 draw attention to a metrical structure on the set of equivalence classes of cotangent structures cf. [6]. Choose metrics on M and E.

Definition. A cotangent structure (E', γ') is ε close to (E, γ) if there is a Lipschitz bundle isomorphism $E \xrightarrow{i} E'$ so that γ and $\gamma' \cdot i$ differ by at most ε in operator norm.

Theorem 2. *The local degree of a cotangent structure $i(E, \gamma, p)$ as a function to the positive integers is continuous and therefore constant for ε close cotangent structures. It is equal to one for p in an open dense set whose complement has topological dimension at most $(n-2)$.*

If $i(E, \gamma, p) = 1$ for all p in M, there is a sequence of cotangent structures (E_i, γ_i) converging to (E, γ) which are individually smooth. (We say that (E_i, γ_i) is smooth if there is a smooth structure α_i on M inside the Whitney quasiisometrical charts so that the standard cotangent structure associated to α_i is isomorphic to (E_i, γ_i).)

Proof. The proof is given in §1. □

Remark. We conjecture that the ε-closeness mentioned above determines an actual metric in the set of equivalence classes of cotangent structures and that $i(E, \gamma, p) \equiv 1$ implies that (E, γ) itself is smooth. The idea for the first statement is that the (lower bound for volume) part of the definition of cotangent structure should imply the image by γ of the Lipschitz sections of E is a closed subspace of Whitney forms for the Whitney norm. The idea for the second part should be that the construction of the smooth charts for (E_i, γ_i) approximating (E, γ) actually yields metrics of bounded curvature. Thus a Gromov limit can be considered as in [7].

In this metric the smooth structures would fill out certain of the uncountably many components distinguished by the different local degree functions $i(E, \gamma, p)$ and one would obtain the analogue of a Teichmüller metric on smooth structures (with equality as isomorphism), cf. [6].

Remark. The motivation for writing this note at this time was the recent activity in four dimensional smooth manifolds using the nonlinear equation of Seiberg-Witten which in turn is based on Dirac operators.

Now exterior d and thus its adjoint d^* and the associated signature operator $d+d^*$ can be constructed using algebraic topology and metrics which are locally Euclidean in the quasiisometric sense using the work of Whitney [2] for d and the work of Teleman [13] to see that d and d^* have a common dense domain in L^2 and that the signature operator $d + d^*$ is essentially self adjoint. Yang Mills theory and Donaldson invariants can be constructed as well for Whitney manifolds, [3]. In fact the known Donaldson invariants of smooth manifolds are actually invariants of the local quasiisometry or Whitney structure.

It was noticed long ago that these Lipschitz or Whitney manifolds cannot have a "Dirac package – spinors, Dirac operator, index formula,..." because there is a Whitney M^8 which is homologically like the quaternionic plane (and so the second Stiefel Whitney class is zero) but where the $\widehat{A}$-genus is not an integer.

We conjecture that Whitney manifolds with a "full-Dirac package" (to be defined) are actually smooth. A corollary conjecture is then that the new gauge theory for 4-manifolds requires the underlying smooth structure.

A further speculation is that the new gauge theory produces smooth invariants which are not biLipschitz invariants in dimension 4.

The idea for the conjecture and the definition of "full-Dirac package" would be the following. The only known construction of Dirac operators (as opposed to d or the signature operator) uses a connection on an abstract vector bundle (the spinor bundle) and an action of the forms on that bundle (Clifford multiplication). These two elements of structure constitute a refinement of the pair (E, γ) in the cotangent structure above.

Namely choose an orthogonal structure on E. Then choose an orthogonal connection on E which is torsion free in the sense of part ii) of the definition of cotangent structure. (This is done by projecting torsion free connections to skew symmetric connections.) Construct the Clifford algebra of sections of ΛE and over any open set where $\omega_2(E) = 0$ a spinor module S over Clifford, the associated connection ∇_S and the associated Dirac operator, $\Gamma(S) \overset{\nabla_S}{\to} \Gamma(S) \otimes$

$\Omega^1 \underset{\text{multiplication}}{\overset{\text{Clifford}}{\rightarrow}} \Gamma(S)$, cf. [9].

On Lipschitz sections with compact support such a Dirac operator is formally self-adjoint, namely it satisfies $(Df, g) = (f, Dg)$. However, it was shown by Chou [14] that there is a defect in the essential self adjointness for branched Euclidean metrics (in the polyhedral setting). If in the example above we use $(r, \theta) \to (r, \ell\,\theta)$ the spin structure extends correctly if ℓ is odd but the Dirac operator sees that ℓ is not 1 in its spectral properties, and these spectral properties obstruct self adjointness.

Thus we are lead to the following

Conjecture. *Over open sets where the cotangent structure (E, γ) admits spin structures the associated Dirac operators should be essentially selfadjoint rel boundary (a concept which can be formulated locally) and then the manifold is smoothable with cotangent structure (E, γ).*

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1 The Proofs

Preparation. 1) Divide R^n in the standard way into congruent cubes and thicken these slightly and congruently into a cover. If we multiply the picture by $\varepsilon_i \to 0$ we obtain a family U_i' of finer and finer covers of R^n with locally constant shape and geometry.

2) Choose a finite cover U_α of M by quasiisometrically related contractible charts so that for chosen compact subsets $K_\alpha \subset U_\alpha$ the interiors of K_α also cover. For i sufficiently large all the little (thickened) cubes of U_i which intersect image K_α are contained in image U_α. Thus they lift back to M and define there a system of fine covers $U_{i,\alpha}$ of M.

3) Choose a Lipschitz inner product on E and a Lipschitz trivialization of E over the original cover U_α.

4) Choose a connection in E over U_α satisfying ii) of the definition of cotangent structure. This may logically entail rechoosing the U_α at the start, but such connections may be added using a partition of unity to make them global. These connections don't need to be orthogonal.

Proof of Theorem 1

a) For an open set B_j of the cover at level i push the framing at a central point out along rays using the connection. (The connection is defined by a matrix θ_j of Whitney 1-forms for the original framing and Whitney 1-forms have well defined restrictions to rectifiable arcs a.e. for arc length measure [2] chapter 9.) This new framing will differ from the old one by a gauge transformation σ_j of the form Identity $+0(\varepsilon_i)$. The original curvature matrices $\Omega_j = d\theta_j + \theta_j \wedge \theta_j$ are Whitney bounded and are conjugated by $\sigma_j = I + 0(\varepsilon_i)$ to obtain the new curvature matrices.

Lemma. *The new connection matrices* $\theta'_j = \sigma_j^{-1}\, d\sigma_j + \sigma_j^{-1}\, \theta_j\, \sigma_j$ *are small in* L^∞ *norm, on the order of* ε_i.

Proof. The holonomy along a very short polygonal arc a of size a in the coordinate system can be estimated by considering the very narrow triangle obtained by coning a to the center.

We break the triangle into pieces of area at most a^2 and observe the holonomy around each is on the order $I + 0(a^2)$ since curvature is bounded, cf. [2] chapter 5. These are conjugated by bounded transformations and then multiplied to get the holonomy around the triangle. Altogether the holonomy around the triangle is estimated by O(area of triangle) $= 0(\varepsilon_i \cdot a)$. Since this equals the holonomy along a by construction of our frame which is parallel along the long sides of the triangle we deduce our estimate of the L^∞ norm of θ'_j on a polygonal arc is $O(\varepsilon_i)$. This completes the proof of the lemma. □

Note we are not claiming the Whitney norm of θ'_j is $O(\varepsilon_i)$. In fact this is not true since $\Omega'_j = d\theta'_j + \theta'_j \wedge \theta'_j = d\theta'_j + 0(\varepsilon_i^2)$ and Ω'_j could well be of order 1 in L^∞ norm so $d\theta'_j$ is expected to be of order 1 in L^∞ norm. The conjugation step in a discussion like the above is what foils such an $O(\varepsilon_i)$ estimate for the Whitney norm.

b) For this radially parallel frame on B_j consider the corresponding 1-forms $(\rho_1, \rho_2, \ldots, \rho_n) = \rho$ using the embedding γ of property i) of cotangent tructure. By property ii) $(d\rho_1, d\rho_2, \ldots, d\rho_n) = d\rho = \theta_j \wedge \rho$. Now $d\rho$ is closed and by a) has L^∞ norm $O(\varepsilon_i)$ when restricted to any planar triangle thus the Whitney norm of $d\rho$ is $O(\varepsilon_i)$. (Whitney norm and L^∞ norm on planar triangles are equivalent for closed 2-forms by definition [2] chapter 5.)

c) Now apply the cone Poincaré lemma operator of [2] chapter 7 Lemma 10b to construct a Whitney form $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ of Whitney norm $O(\varepsilon_i)$ so that $d\eta = d\rho$. This is possible because the Whitney norm of $d\rho$ is $O(\varepsilon_i)$ by b). Now we consider the closed form $\rho' = (\rho'_1, \rho'_2, \ldots, \rho'_n) = \rho - \eta$.

d) Now we consider the mapping $\rho'(x) = \int_{(p,x)}(\rho'_1, \rho'_2, \ldots, \rho'_n)$. This is a Lipschitz mapping because we have Whitney forms and the triangle estimate of a) applies (in the more trivial abelian form). Its differential is determined by the closed 1-forms $(\rho'_1, \rho'_2, \dots, \rho'_n)$. By the third property of part i) of the definition of cotangent structure the wedge product $(\rho'_1 \wedge \rho'_2 \wedge \dots \wedge \rho'_n)$ has a definite positive lower bound. Since ρ' is a $O(\varepsilon_i)$ perturbation of ρ and since ρ is bounded from above (by the first part of property i)) it follows that $\rho'_1 \wedge \rho'_2 \wedge \dots \wedge \rho'_n$ has essentially the same positive lower bound for ε_i sufficiently small. It follows that the mapping $x \to \rho'(x)$ is a *mapping of bounded distortion* in sense of [10] chapter 1 §4.

e) By developing some results for certain nonlinear elliptic PDE with bounded measurable coefficients and the ideas of conformal capacity geometric information about mappings f of bounded distortion is deduced in [10]. For example

i) the value $f(q)$ at an interior point q of the domain is taken on uniquely near q, Theorem 6.3 [10].

ii) on a small ball centered at q the image of small concentric spheres about q is contained in spherical shells centered about $f(q)$ with ratio of radii controlled by the derivative data which in our case is L^n/d. Here L is the Lipschitz constant, n is the dimension, and d is the lower bound on the Jacobian determinant. Theorem 7.2 [10].

iii) the local degree at q, well defined by i), is a positive integer and equals 1 precisely when f is a local homeomorphism on a neighborhood of q, Theorem 6.1 and Theorem 6.6 of [10]. (We have arranged orientations so Jacobian determinants are positive.)

f) Let us return to slightly simpler case of ρ', which is our Lipschitz mapping of bounded distortion. Applying e) we see that ρ' maps a small ball of radius r about p with positive degree so that the boundary stays in a controlled shape spherical shell of radius r'. By applying the coarea formula (cf. Theorem 2.2 of [10]) we obtain $r^n \sim$ (local degree at p) $(r')^n$ where the constants are controlled by L^n and d of e). By Lemma 4.8 of [10] and Arzela Ascoli the set of mappings with derivative data controlled by L and d as defined in e) and with $\rho'(p)$ bounded is compact. Thus if we had a sequence of such mappings $\rho^1, \rho^2, \dots$ with L, d control and with the local degree tending to infinity we could rescale domain and range by the same amount to keep r constant. This doesn't change L or d. But then r' would have to tend to zero and the limit map would be constant. This contradicts the fact that d stays positive in the limit by Lemma 4.8 of [10] alluded to above. We conclude the local degree at p is between 1 and a positive integer controlled by L and d.

(Note this upper bound on degree doesn't hold in the K-quasiconformal context for $n = 2$ (consider $z \to z^N$) and may not be known for $n > 2$, to the best of my knowledge.)

g) A corollary to the (L^n, d) upper bound on local degree in f) is that r and r' are of the same size with constants depending only on L^n and d. This shows that the mapping ρ' of f) satisfies a reverse inequality

$$|\rho'(x) - \rho'(p)| \geq \text{constant}(L^n, d)\,|x - p|$$

on a small ball about p (whose size cannot be estimated – see the example of introduction).

h) For ε_i small compared to the constant in g) the mapping ρ of Theorem 1 also satisfies the reverse inequality for essentially the same constant because $|\rho(x) - \rho'(x)|$ is $O(\varepsilon_i$ distance $(x,p))$. (Recall ρ' depends on ε_i as $\varepsilon_i \to 0$ while ρ is defined once and for all by integrating along the rays from p.) Thus the local degrees are the same. One last remark is that a constant (positive determinant) linear change of the framing of E doesn't alter the local degree. This completes the proof of Theorem 1. □

Proof of Theorem 2

a) The continuity of the local degree under approximation follows immediately from the principle used in h) of the proof of Theorem 1 that two maps which are closer than their distance from the zero have the same local degree.
b) The second statement is due to Chernavskii see Theorem 6.7 chapter 2 of [10] together with the observation that since the forms defining ρ' are closed we can use ρ' to compute the index at a point near p as well.

c) Here is the proof of the third statement. By Theorem 6.6 [10] if the local degree is 1 then the ρ' of d) e) f) ... of the proof of Theorem 1 is a local homeomorphism. In fact this will be true for ρ' which are perturbations of ρ coming from the orthogonal trivializations of the bundle E from the preparation.

Such ρ' will provide immersions of the cover at level i which will be $(I + 0(\varepsilon_i))$ quasiisometries for the measurable metric on M associated to the orthogonal structure on E – i.e. the derivatives will be $O(\varepsilon_i)$ perturbations of the canonical isomorphism between the orthogonal trivialization of E and the canonical basis in R^n. Thus the overlap homeomorphisms for these charts will have $O(\varepsilon_i)$ almost constant, $O(\varepsilon_i)$ almost isometric Jacobians between open sets in Euclidean space. By proposition 1 p.76 of [6] smooth mappings which are regularizations of these will be local diffeomorphisms. Thus one can choose fine handle decompositions and work inductively and relatively with standard averaging procedures (see [6] part II for details in the averaging procedure) to smooth these overlap homeomorphisms.

This constructs a smooth structure α_i whose cotangent structure is $O(\varepsilon_i)$ close to (E, γ), and completes the proof of Theorem 2. □

2 Whitney spaces

The existence of forms as above with d and wedge was also produced by Whitney in [2] on metric spaces which are locally quasiisometric to polyhedra. The construction is so elegant one can begin it *for any metric space.* The steps are the following

i) define the mass $|\ |$ of one-chains, two-chains, etc.

ii) introduce the Whitney norm $|\ |_\omega$ as above as $|a|_\omega = \inf_b\ (|a - \partial b| + |b|)$

iii) the "Whitney forms" are the continuous linear functionals on this space

iv) prove or assume a Whitney bounded Poincaré lemma (see Chapter VI of [2]). Gromov has suggested that this should be true for algebraic varieties with the induced metric using semialgebraic triangulations.

v) prove or assume the cup product formulae converge under subdivision to a well defined product (by definition the wedge).

Definition. *A* **Whitney space** *is a metric space where the above steps can be carried out* (compare [11]).

Remark. It is known that all manifolds outside dimension 4 have Whitney structures which are related by isotopies close to the identity, [4]. In dimension 4 this fails because of the Donaldson-Freedman theory and its extension in [3].

It is also known that Whitney structures exist on all open 4-manifolds [5] because there are always smooth structures. For example compact 4-manifolds less one point have (non-unique) Whitney structures. The ambiguity can be made to be countable [8].

Remark. Suppose a topological 4-manifold has a metric making it into a *Whitney space.* We can consider vector bundles with connections over such a space. To develop Donaldson invariants one basic further ingredient is required. We need to be able to pick out from the 2-forms ω a class of positive forms so that the pointwise norm of ω is estimated by $(\omega \wedge \omega)^{1/2}$.

Given a Freedman topological 4-manifold which by Donaldson theory admits no smooth (and therefore no Whitney structure as in the introduction) it is interesting to wonder which axiom in the above chain (§2 only) fails.

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