Pergamon

Topology, Vol. 33, No. 1, pp. 85–94, 1994

0040-9383/94 $6.00 + 0.00

INFINITE CASCADES OF BRAIDS AND SMOOTH DYNAMICAL SYSTEMS

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(*Received* 20 *April* 1992; *in revised form* 16 *March* 1993)

1. FLOWS IN 3-SPACE

LET US consider the linking between orbits in a compact invariant set of a nonsingular vector field on some region in Euclidean 3-space. Let us assume in the first instance we can place a solid torus so that the invariant set is inside the torus and the flow lines are transversal to 2-disks of the solid torus (see Fig. 1). Before describing our result, we make certain remarks that are not difficult to verify.

The first point is purely topological.

(i) Two (long) finite pieces of orbits have an approximate linking number if they start and stop nearby to one another. This number is the algebraic number of times one point winds around the other when the pair of orbits is viewed after projection as a moving pair of points in one transversal 2-disk. The ambiguity in this number is one turn for all such pair of orbits, once the projection is chosen.

The second and third points are from elementary calculus.

(ii) If the flow is continuously differentiable, there is an analogous winding number for each single piece of orbit. It is defined by counting how many times a tangent vector in the disk direction turns around the orbit. Two different tangent vectors will turn the same number of turns up to an error of half a turn, because of linearity (antipodal vectors turn the same number of turns).

The following coherence between the *topological linking* of point (i) and the *infinitesimal self linking* of point (ii) is a consequence of continuous differentiability:

(iii) For each finite time T, there is an $\varepsilon = \varepsilon(T) > 0$ so that if three orbits of length T are within ε for $0 \leq t \leq T$, then the topological linking between two of them differs from the infinitesimal self linking of the third orbit by at most a turn and a half.

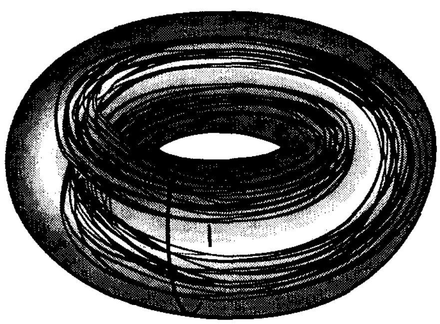

Fig. 1.

The following point is algebraic. All of these linking numbers satisfy an approximate additive property:

(iv) If we go along for T_1 and then for T_2, the numbers for T_1, T_2, and $T_1 + T_2$ are related by addition with a uniformly bounded error.

(v) In [14], using this algebraic property, D. Ruelle proved that for almost all points relative to an ergodic invariant measure, the average infinitesimal linking number has a limit as $T \to \infty$. This limit is computed as a spatial integral of the appropriate derivative.

Let us suppose the invariant set X is constructed as a limit of a cascade of periodic orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots$ where $\mathcal{O}_{n+1}$ is a connected braid in a solid torus $\mathcal{T}_n$ about $\mathcal{O}_n$ (with at least two strands to avoid triviality). We assume $\mathcal{T}_1$ is the original solid torus, and inductively the 2-disks of each new solid torus $\mathcal{T}_{n+1}$ are contained in the larger 2-disk of the preceding one $\mathcal{T}_n$, with the diameter of the 2-disks going to 0 as $n \to \infty$ (see Fig. 2). Then X is the intersection of the solid tori and there is a sequence of linking numbers of $\mathcal{O}_{n+1}$ about $\mathcal{O}_n$. Dividing this linking number by the number of times $\mathcal{O}_n$ winds around $\mathcal{T}_1$ defines the n^{th} *average linking number*, $\tilde{l}_n$.

The intersection of the invariant set X with the transversal 2-disk of $\mathcal{T}_1$ is a Cantor set and the motion there is quasi-periodic, i.e., a minimal (all orbits are dense) translation on a compact abelian group. Thus it is uniquely ergodic. From the topological point of view, it is clear any sequence of average linking numbers may appear (see Remark 6 below). Let us suppose this topological configuration is realized by a C^1 flow.

THEOREM 1. (Coherence of braids cascades in a smooth flow.) *The sequence of topologically defined average linking numbers between successive orbits of the cascade must converge. The limit equals the average twisting number of the derivative.*

Conversely,

THEOREM 2. *Suppose the sequence of rational numbers* $\{\frac{l_n}{q_n}\}$ *has limit* ω, *where the sequence* $\{q_n\}$ *is strictly increasing and* $a_n = \frac{q_n}{q_{n-1}}$ *and* l_n *are coprime for each n. Then there is a cascade of iterated torus knots in a continuously differentiable flow with these rational numbers as average linking numbers converging to a quasi-periodic solenoid where the average infinitesimal twisting number is* ω.

We will prove Theorem 1 using the above remarks. Theorem 2 will be proven in the context of diffeomorphisms of the disk in the next section, after we formulate Theorems 1 and 2 in terms of maps (see Theorem 5 below).

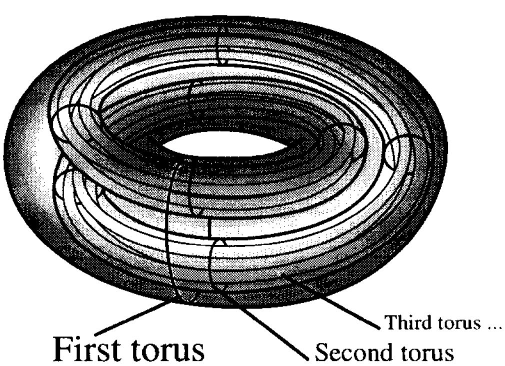

Fig. 2.

Proof of Theorem 1. Using (v), pick a point x of the Cantor set where the average infinitesimal linking number $\omega(x)$ exists. Choose $T > 0$ and N big enough for all orbits $\mathcal{O}_n$ with $n > N$ to stay within $\varepsilon(T)$ of the orbit of x for all time. Denoting by $w_t(x)$ the infinitesimal winding about the orbit of x up to time t (so that $\omega(x) = \lim_{t\to\infty}(\frac{1}{t})w_t(x)$), and by l_n the linking number of $\mathcal{O}_n$ about $\mathcal{O}_{n-1}$, we have

$$|l_n - w_{q_n}(x)| \le \frac{5}{2}\cdot\left(\frac{q_n}{T} + 1\right),$$

where we have used (iii) and the fact that the bound on the error in (iv) is 1. Thus for any accumulation point $\tilde{l}$ of the sequence $\tilde{l}_n$, we have

$$|\tilde{l} - \omega(x)| \le \frac{5}{2}\cdot\frac{1}{T}.$$

Since this holds for every $T > 0$, we get

$$\lim_{n\to\infty} \tilde{l}_n = \omega(x).$$

(Q.E.D. Theorem 1.)

Remark 3. The discussion (above and below) shows that $\omega(x)$ exists for all x in the Cantor set, and is independent of x. More generally, in the uniquely ergodic case; i.e., when there is a unique invariant measure on the invariant set, the limit of the average infinitesimal self linking number *exists* for any sequence of orbit segments whose length tends to infinity, and is independent of the sequence of orbit segments (see Proposition A3 in the Appendix; also cf [1, 3, 11, 13, 15, 16, 17, 18, 19, 21).

2. HOMEOMORPHISMS OF THE 2-DISK

Let f be an orientation preserving homeomorphism of the 2-disk $\mathbb{D}^2$. A *cascade of periodic orbits* for f is an infinite sequence of periodic orbits $\{O_n\}$ of f with periods $\{q_n\}$ such that, for each $n \ge 1$, we have:

— $q_n = a_n \cdot q_{n-1}$ with $q_0 = 1$ and $a_n > 1$,

— there exists a collection of disjoint, simple closed curves $\mathscr{C}_n^0, \ldots, \mathscr{C}_n^{q_{n-1}-1}$ bounding the disjoint disks $\mathscr{D}_n^0, \ldots, \mathscr{D}_n^{q_{n-1}-1}$, with the following properties:

— each $\mathscr{D}_n^i$ contains exactly one point of O_{n-1}, and a_n points of O_n,

— $f(\mathscr{C}_n^i)$ is isotopic to $\mathscr{C}_n^{i+1 \bmod q_{n-1}}$ in the punctured disk $\mathbb{D}^2\backslash\bigcup_{i\le n} O_i$,

— the union of the $\mathscr{D}_n^i$'s is contained in the union of the $\mathscr{D}_{n-1}^i$'s,

— the diameters of the $\mathscr{D}_n^i$'s converge uniformly to 0 with n.

Let $\{f_t\}_{t\in[0,1]}$ be an arc of homeomorphisms joining the identity map to $f = f_1$, and $\{f_t\}_{t\in\mathbb{R}}$ be the extended arc of homeomorphisms joining the identity map to all iterates of f, with $f_t = f^{[t]}\circ f_{\{t\}}$. To each cascade of periodic orbits $\{O_n\}$, we associate a *signature* $\{(\tilde{l}_n, q_n)\}$, where $\tilde{l}_n = \frac{l_n}{q_n}$ and l_n is defined as follows:

In one of the $\mathscr{D}_n^i$'s, pick the point x_{n-1} of O_{n-1}, and a point x_n of O_n; then l_n is the algebraic number of loops that the vector $\frac{f_t(x_n) - f_t(x_{n-1})}{\| f_t(x_n) - f_t(x_{n-1})\|}$ performs on the unit circle when t goes from 0 to q_n. The number l_n is independent of the choice of $\mathscr{D}_n^i$, and of the choice of the point x_n in $\mathscr{D}_n^i$.

For each signature $\{(\tilde{l}_n, q_n)\}$ with $\tilde{l}_n = \frac{l_n}{q_n}$, $\{q_n\}$ is strictly increasing and $a_n = \frac{q_n}{q_{n-1}}$ and l_n are coprime for each n.

Remark 4. When changing the isotopy $\{f_t\}_{t\in[0,1]}$, all the l_n's are changed by adding the same integer.

Figure 3 illustrates the first elements of a cascade of periodic orbits in a case when all the $\mathscr{D}_n^i$'s are geometrical disks and $f(\mathscr{C}_n^i) = \mathscr{C}_n^{i+1 \bmod q_{n-1}}$, and the braiding in Fig. 4 represents an isotopy from the identity to the same map f. The following result is a corollary of Theorems 1 and 2, in fact a reformulation of these results in terms of maps.

THEOREM 5. *If* $\{O_n\}$ *is a cascade of periodic orbits for a* C^1 *diffeomorphism* f *of the* 2*-disk, then* $\lim_{n\to\infty}(\tilde{l}_n)$ *exists. Conversely, for each signature* $\{(\tilde{l}_n, q_n)\}$ *such that* $\lim_{n\to\infty}(\tilde{l}_n)$ *exists, one can construct a* C^1 *diffeomorphism (with zero topological entropy) with a cascade of periodic orbits having this signature.*

By analogy to one dimensional dynamics, homeomorphisms with cascades of periodic orbits can be called *infinitely renormalizable* (cf. [20]).

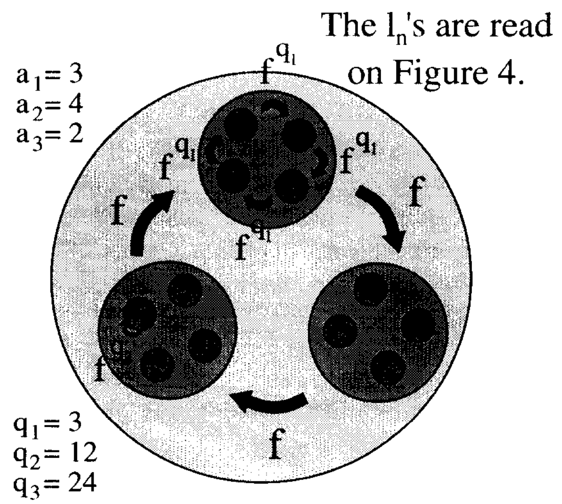

Fig. 3.

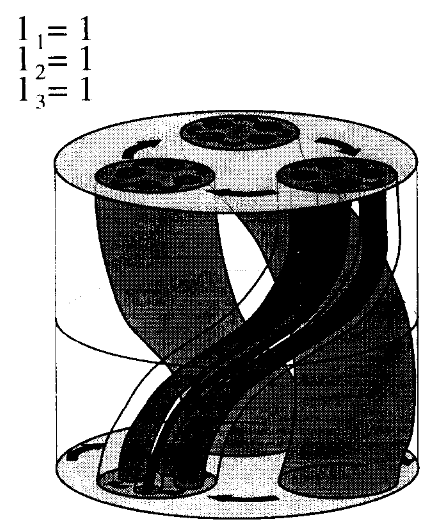

Fig. 4.

Remark 6. As noticed at the end of the proof of Theorem 2, for each signature, one can construct a homeomorphism of the 2-disk (with zero entropy) with a cascade of periodic orbits having this signature (see also [8]). Thus, Theorem 5 describes a topological obstruction to smoothability for homeomorphisms (see the questions at the end of the main text for possible follow up to this remark).

From the definition of a cascade of periodic orbits, we can deduce the following three properties:

— The set $K = \overline{\bigcup_n O_n} - \bigcup_n O_n$ of accumulation points of the orbits O_n is a Cantor set.

— The restriction of the map f to K is uniquely ergodic (i.e. $f|_K$ has a unique invariant measure): more precisely, $f|_K$ is topologically conjugate to a *generalized adding machine* (i.e., a quasi-periodic motion, obtained by adding 1 on a compact abelian group

$$\hat{\mathbb{Z}}_Q = \lim_{\leftarrow q_i} \mathbb{Z}/q_i\mathbb{Z}$$

where Q stands for a *super natural number*

$$Q = \prod_p p^{k_p} \quad \textit{where, } \forall p \textit{ prime, } 0 \leq k_p \leq \infty ,$$

and the q_i's form a sequence of divisors of Q ordered by divisibility).

— The periodic orbits O_n *converge dynamically to* K, i.e.,

$$\forall \varepsilon > 0, \ \exists n_0 > 0 \textit{ such that,}$$

$$\forall x \in K, \textit{and } \forall n > n_0, \exists x_n \in O_n \textit{ such that } \forall t > 0 \ \| f_t(x_n) - f_t(x) \| < \varepsilon.$$

This is why point (iii) of the first section applies.

Proof of Theorem 2. The following construction is strongly reminiscent of [2] (see also [12] and [6], as well as [6] for different braids) and is resumed in Figs 5 and 6. We begin by clarifying some terminology about rotation angles.

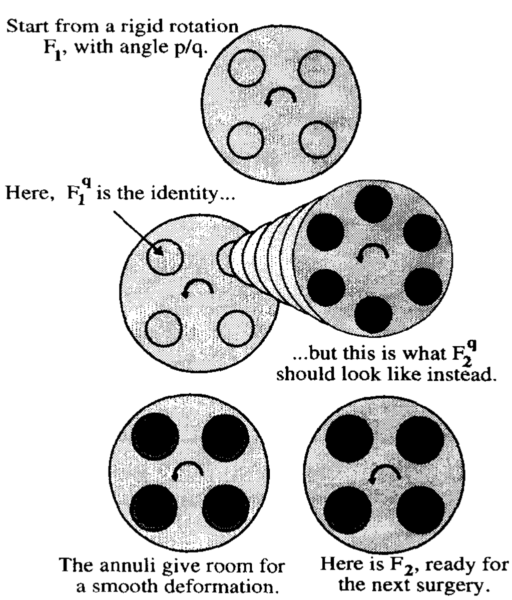

Fig. 5.

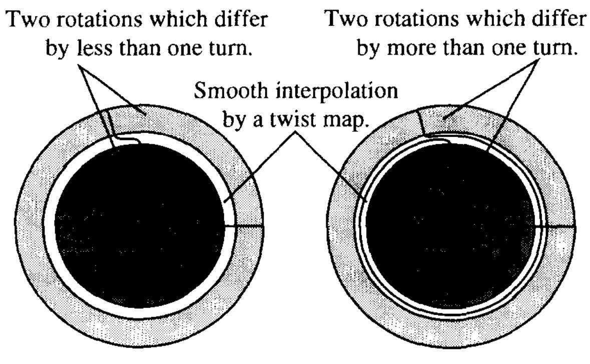

Fig. 6.

When considering a single rotation, its angle is defined mod 1, hence can legitimately be considered as a rational number in [0, 1). However, when considering a continuous family of rotations, only one of them can have its angle chosen to be in a given interval of length 1. In the following construction, all rotation angles will be meant as real number, the only freedom being to choose one of them, say the first one. Continuous families of rotations will occur in the form of *simple twist maps*, i.e. maps which preserve the foliation by circles of an annulus, and rotate these circles by an angle which varies monotonically along the radius (see Fig. 6).

Let F_1 be a rigid rotation of $\mathbb{D}^2$, with rational angle $\frac{p_1}{a_1}$, where (p_1 and a_1 are coprime. Choose a_1 disjoint closed disks $D_1^{(i)}$, with $0 \le i \le a_1 - 1$, which are cyclicly permuted by F_1. If we identify the disks by a rigid translation, the tangent map to F_1 restricted to any of these disks is a rigid rotation with angle $\frac{p_1}{a_1}$.

We can now define a second map, F_2, as follows:

In each of the disks $D_1^{(i)}$, we post compose F_1 with the rigid rotation R_{θ_1} and we write $\frac{p_1}{a_1} + \theta_1 = \frac{p_2}{q_2}$, where θ_1 is chosen so that $a_2 = \frac{q_2}{q_1} > 1$ and $(p_2, a_2) = 1$.

We use a simple twist map acting on a small annulus around each $D_1^{(i)}$ for two purposes:

— to interpolate between F_2 on the small disks and F_1 outside the small disks, hence allowing F_2 to be arbitrarily smooth,

— to give a meaning to the integer part of θ_1.

The restriction of $F_2^{q_1}$ to any $D_1^{(i)}$ is a rigid rotation with angle $\frac{p_2}{a_2} = a_1 \cdot \frac{p_2}{q_2}$, and thus $F_2^{q_1}$ permutes cyclicly a_2 disjoint smaller closed disks since $(p_2, a_2) = 1$. Otherwise speaking, F_2 permutes (cyclicly) $q_2 = a_2 \cdot a_1$ small disks $D_2^{(j)}$. Let us again identify these disks by a rigid translation. With this identification, the tangent map to F_2 restricted to any $D_2^{(j)}$ is a rigid rotation with angle $\frac{p_2}{q_2}$.

The third map F_3 is defined by post composing F_2 in the disks $D_2^{(j)}$ with the rigid rotation R_{θ_2}, and we write $\frac{p_2}{q_2} + \theta_2 = \frac{p_3}{q_3}$, where θ_2 is chosen so that $a_3 = \frac{q_3}{q_2} > 1$ and $(p_3, a_3) = 1$. Simple twist maps in annuli around the disks $D_2^{(j)}$ are again used to guaranty the smoothness of F_3 and give a meaning to $[\theta_2]$.

Generally, if at the m^{th} stage of the construction there are $q_m = a_m \cdot \ldots \cdot a_2 \cdot a_1$ disjoint disks $D_m^{(k)}$, so that after proper identification, F_m restricted to $D_m^{(k)}$ is a rotation with angle $\frac{p_m}{q_m}$, F_{m+1} is obtained from F_m by post-composition in these q_m disks by an angle θ_m so that $\frac{p_m}{q_m} + \theta_m = \frac{p_{m+1}}{q_{m+1}}$, where θ_m is chosen so that $a_{m+1} = \frac{q_{m+1}}{q_m} > 1$ and $(p_{m+1}, a_{m+1}) = 1$.

By putting appropriate twist maps in the annuli which are used to preserve the smoothness and to represent the integer part of the angles θ_m at each step of the construction, we get that the C^1 distance from F_m to F_{m+n+1} is bounded by a fixed multiple of $\theta_m + \cdots + \theta_{m+n}$ if the diameter of the $D_m^{(k)}$ goes to zero as $m \to \infty$. More precisely, the C^0 distance from F_m to F_{m+n+1} goes to zero as a direct consequence of the diameter condition, and the contribution of the derivatives to the C^1 distance from F_r to F_{r+1} is proportional to the angle difference θ_r (see Figs 5 and 6). Hence, within the class of maps we have described, a necessary and sufficient condition for a sequence $\{F_i\}$ to form a Cauchy sequence of diffeomorphisms in the C^1 topology is that $\{\theta_m\}$ be a convergent series, which is equivalent to say that the sequence $\{\tilde{l}_m\} = \{\frac{p_m}{q_m}\}$ converges.

We notice that the condition that the diameters of $D_m^{(k)}$ goes to zero as $m \to \infty$ is sufficient to allow $\{F_i\}$ to form a Cauchy sequence of homeomorphisms in the C^0 topology, so that the limit homeomorphism always exists, for any signature $\{(\tilde{l}_n, q_n)\}$. This limit homeomorphism has infinitely many nested periodic orbits which accumulate on a unique invariant Cantor set, and has zero topological entropy. At each step of the construction, the complement of the q_n small disks is invariant and do not carry any entropy. Positive topological entropy thus could only be provided by the restriction of the limit homeomorphism to its invariant Cantor set, which does not happen since this restricted map is topologically conjugate to a generalized adding machine.

(Q.E.D. Theorem 2.)

Remark 7. The above construction could be modified to produce C^2 infinitely renormalizable diffeomorphisms when

$$\Delta_n = \left| \frac{l_{n-1}}{a_{n-1}} - \frac{l_n}{a_{n-1} \cdot a_n} \right|$$

is bounded, or does not grow too fast: it would be necessary to keep the disks $(D_{n-1}^{(k)})$ large enough for the second derivative to not blow up in the estimate of $\| F_{n-1} - F_n \|_2$. Each time some $a_i = 2$, one would need to use a trick from [5], consisting in incorporating two successive orbits at a time. Details can be found in [7] in the case when $l_n = 1$ for each n, which are easily adapted to the generality mentioned here.

Remark 8. Lots of C^∞ examples of cascades of periodic orbits occur in generic Hamiltonian diffeomorphisms.

Remark 9. Combining the geometric ideas in [9] and the extension to two dimensions of the renormalization theory in dimension one described in [4] with the rigidity results in [20] about all cascades with bounded a_n for quadratic-like maps, yields infinitely many C^∞ examples of cascades of periodic orbits which occur in the process of formation of a horseshoe.

Remark 10. Among the examples in Remark 9, only the case $a_n = 2$ is known to be C^∞-realizable with zero topological entropy [9], because only the cascades with $a_n = 2$ can be realized with zero entropy by an endomorphism of the interval.

Remark 11. One parameter families of endomorphisms of the interval are the simplest (but degenerate) model of creation of a horseshoe map by an isotopy to a contraction. These maps preserve orientation on some sub-intervals and reverse it on others. Knowing the number n of points of a periodic orbit of period p in the orientation reversing intervals, gives a direct way to compute the l_n's.

Remark 12. A series of simple C^∞ zero entropy examples is obtained by taking l_n bounded and $q_n \to \infty$ fast enough (cf. [15] and [7]).

Questions.

I. Among the C^1-realizable cascades of periodic orbits:
   (i) which do occur in C^∞ maps? (cf. Remarks 7 and 8).
   (ii) which do occur in zero-entropy C^∞ maps? (cf. Remarks 9 and 11).

II. Recall that J. Harrison [10] proved the existence for each k of a C^k diffeomorphism which is not topologically conjugate to any C^{k+1} diffeomorphism. Can a similar classification of cascades be carried out?
III. Does the rigidity phenomenon manifest itself for finite smoothness, at least when q_n is bounded? Namely, after some critical smoothness is present, the geometric structure of the Cantor set at finite scale is rigid (cf. Remark 9).

APPENDIX

We give here a proof (in terms of maps) that the unique ergodicity hypothesis implies the infinitesimal self linking numbers along orbits in the Cantor set converge.

Let us consider a C^1 orientation preserving embedding f of the closed 2-disk $\mathbb{D}^2$ and an isotopy f_t from the identity map $f_0 = Id_{\mathbb{D}^2}$ to $f_1 = f$, that we extend to a one real parameter isotopy by setting $f_t = f^{[t]} \circ f_{\{t\}}$.

Denoting by S_x^1 the unit tangent bundle at x, we can associate, to each $(x, t) \in \mathbb{D}^2 \times \mathbb{R}$, a C^0 map $\phi_x^t: S_x^1 \to S_{f_t(x)}^1$ defined by:

$$u \mapsto \frac{Df_t(x)[u]}{\| Df_t(x)[u] \|}.$$

where we identify $T_{f_t(x)}\mathbb{R}^2$ to $\mathbb{R}^2$ in order to use the Euclidean norm $\|\cdot\|$.

We choose the lift Φ_x^t of ϕ_x^t: $\mathbb{R} \to \mathbb{R}$, so that:

i. The map $(x, t) \to \Phi_x^t$ from $\mathbb{D}^2 \times \mathbb{R}$ to $C^0(\mathbb{R})$ is continuous.

ii. $\Phi_{(\cdot)}^0 = Id_{\mathbb{R}}$.

With these conventions the map $\Phi_x = \Phi_x^1$ is uniquely defined by the isotopy.

Remark A1. The map Φ: $x \mapsto \Phi_x$ depends on the isotopy but, by changing the isotopy, we get a new map Φ' which differs from Φ by an integer.

Remark A2. Because of the symmetry $Df_t(x)[-u] = -Df_t(x)[u]$, the map $\Phi_x^t - Id_{\mathbb{R}}$ is $\frac{1}{2}$-periodic.

Notice that ϕ_x^t is a homeomorphism from S_x^1 to $S_{f_t(x)}^1$. Consequently, using Remark A2, for any x and t, $\Phi_x^t(\theta) - \theta$, is independent of θ, except for a bound error of $\frac{1}{2}$.

In the following we shall denote by $\Phi_{(n,x)}$ the composition $\Phi_{f^{n-1}(x)} \circ \ldots \circ \Phi_x$.

In [14], Ruelle gave an ergodic theorem for 2×2 matrice valued functions. Then, for a diffeomorphism f in dimension two preserving some measure μ and isotopic to the identity on the support of μ, he would define μ-a.e. a *rotation number* for f, and correspond to the infinitesimal self linking number of a suspension of f, as defined in §1. The following result tells us that we can avoid the a.e. aspect of this theory when dealing with a uniquely ergodic measure μ, i.e., when the support of μ carries a single f-invariant measure.

PROPOSITION A3. *Let f be a C^1 orientation preserving embedding of the 2-disk $\mathbb{D}^2$, and H be a closed invariant subset of f such that f restricted to H is uniquely ergodic. Then for all x in H and all θ in $\mathbb{R}$ the limit when $n \to \infty$, of $\frac{1}{n}\Phi_{(n,x)}(\theta)$ exists and is independent of x in H and θ in $\mathbb{R}$.*

Proof of Proposition A3. Let us define $\Psi_{(n,x)} = sup_{\theta \in \mathbb{R}}(\Phi_{(n,x)}(\theta) - \theta)$. As the maps $\Phi_{(n,x)} - Id_{\mathbb{R}}$ are continuous periodic maps with period $\frac{1}{2}$, this *sup* is attained, and depends continuously on x.

LEMMA A4. *Under the same assumptions as in Proposition* 1, *for all x in H the limit, when $n \to \infty$ of $\frac{1}{n}\Psi_{(n,x)}$ exists and is independent of x in H.*

Proof of Lemma A4. Part of this proof will follow an argument of Ruelle [14]. First notice that for all x in $\mathbb{D}^2$ and all positive integers n and m, we have:

$$(**) \quad \Psi_{(n+m,x)} \leq \Psi_{(n,x)} + \Psi_{(m,f^n(x))} \leq \Psi_{(n+m,x)} + 1.$$

Consider the Euclidian division of n by m: $n = k \cdot m + r$ with $0 \leq r \leq m-1$. Using $(**)$ we get

$$\Psi_{(n,x)} \leq \Psi_{(k\cdot m,x)} + \Psi_{(r,f^{k\cdot m}(x))} \leq \Psi_{(n,x)} + 1.$$

Using $(**)$ again we get:

$$\Psi_{(k\cdot m,x)} \leq \Psi_{(j,x)} + \sum_{i=0}^{k-2} (\Psi_{(m,f^{i\cdot m+j}(x))} + \Psi_{(m-j,f^{(k-1)\cdot m+j}(x))}) \leq \Psi_{(k\cdot m,x)} + k,$$

for $j = 0, 1, \ldots, m-1$. By adding up all these inequalities we obtain:

$$\Psi_{(k\cdot m,x)} \leq \left(\frac{1}{m}\right)\cdot\left(\sum_{j=0}^{m-1} (\Psi_{(j,x)} + \Psi_{(m-j,f^{(k-1)\cdot m+j}(x))}) + \sum_{i=0}^{(k-1)\cdot m-1} \Psi_{(m,f^i(x))}\right) \leq \Psi_{(k\cdot m,x)} + k.$$

This yields:

$$\left|\frac{\Psi_{(n,x)}}{n} - \left(\frac{1}{m}\right)\cdot\left(\frac{1}{n}\right)\cdot\sum_{i=0}^{n-1} \Psi_{(m,f^i(x))}\right| \leq \left(\frac{1}{n}\right)\cdot(k + 1 + A(\Psi, x, m) + |\Psi_{(r,f^{k\cdot m}(x))}|)$$

where

$$A(\Psi, x, m) = \left(\frac{1}{m}\right)\cdot\left|\sum_{j=0}^{m-1} (\Psi_{(j,x)} + \Psi_{(m-j,f^{(k-1)\cdot m+j}(x))}) + \sum_{j=0}^{m+r-1} \Psi_{(m,f^{(k-1)\cdot m+j}(x))}\right|.$$

Notice that there exists a uniform bound $B(m)$ such that $A(\Psi, x, m) + |\Psi_{(r,f^{k\cdot m}(x))}| \leq B(m)$, thus:

$$\left|\frac{\Psi_{(n,x)}}{n} - \left(\frac{1}{m}\right)\cdot\left(\frac{1}{n}\right)\cdot\sum_{i=0}^{n-1} \Psi_{(m,f^i(x))}\right| \leq \left(\frac{1}{n}\right)\cdot(k + 1 + B(m)).$$

The continuity of the function $\Psi_{(m,.)}$ and the unique ergodicity hypothesis for $f_{|H}$ now insure that the limit of $(\frac{1}{n})\cdot\sum_{i=0}^{n-1} \Psi_{(m,f^i(x))}$ exists and is independent of $x \in H$. We denote by $c(m)$ this limit. For any accumulation point l of the sequence $\frac{\Psi_{(n,x)}}{n}$, we necessarily have:

$$\left|l - \frac{c(m)}{m}\right| \leq \frac{1}{m}.$$

Since this must be true for m, the limit of $\frac{\Psi_{(n,x)}}{n}$ exists and is independent of x.

(Q.E.D. Lemma A4.)

The proof of Proposition A3 follows easily. It is enough to notice that for all x in $\mathbb{D}^2$ and θ in $\mathbb{R}$, it follows from Remark A2 that:

$$|\Psi_{(n,x)} - \Phi_{(n,x)}(\theta)| \leq \frac{1}{2}.$$

(Q.E.D. Proposition A3.)

Using Remark A1, the rotation number depends on the isotopy to identity we have chosen, but two different isotopies yield two rotation numbers which differ by an integer.

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