LINKING THE UNIVERSALITIES OF MILNOR-THURSTON FEIGENBAUM AND AHLFORS-BERS*

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Introduction

Dog-eared copies of the Milnor-Thurston unpublished preprint "On Iterated Maps of the Interval" were common sights among topologists and dynamicists for more than a decade since the earliest versions appeared around 1975. The article contained an elegant and complete exposition of the topological or combinatorial structure of 1-fold mappings of the line for example $\{x \to x^2 + a\}$, in the sufficient range $a \geq -2$.

In this quadratic family one finds that all points tend to infinity outside a unique (possibly empty) invariant interval of points. Inside this interval uncountably many different dynamical patterns can occur as a varies. These patterns are classified by the Milnor-Thurston kneading sequences which are themselves ordered systematically in a 1-dimensional continuum.

A striking feature is the theorem from "On Iterated Maps of the Interval" that any family of smooth 1-fold mappings of an interval that sweeps across the interval must also sweep through all these infinitely many Milnor-Thurston kneading sequence patterns. Moreover any pattern that occurs in any family actually occurs in the quadratic family. Nowadays one refers to this phenomenon as the Milnor-Thurston *topological universality* of the quadratic family.

* We note Myrberg, Sharkovski, Metropolis-Stein-Stein, and Smale-Williams contributed to the topological universality of Milnor-Thurston. Coullet-Tresser independently discovered the celebrated quantitative universality of Feigenbaum. Teichmüller began the work culminating in Ahlfors-Bers Universal Teichmüller Theory.

The first really interesting dynamical pattern in the natural ordering on patterns occurs after an infinite cascade of period doublings. The limiting pattern has a Cantor set on which the motion of the critical point is *quasi-periodic*—in fact conjugate to adding 1 in the 2-adic integers. Immediately after this pattern the entropy or as the famous article explains, the exponential growth rate of the number of turns in the n^{th} iterate, is positive and moves continuously as a varies to the maximum value $\log 2$. Milnor, in a beautiful exposition before the AMS in 1982 explained this theory and his interest in the ten year old question of monotonicity of the entropy as a function of a for the $x \to x^2 + a$ family.

Holomorphic methods, especially conformal mapping, entered around 1982 to resolve the entropy question for the quadratic family. This development is related, I suppose, to Milnor's longtime interest in holomorphic dynamics—see his survey "Dynamics in One Complex Variable." Perhaps it also helped delay publication until 1988 of the paper, "On Iterated Maps of the Interval" [4], since I recall Milnor being puzzled that these and other statements about the dynamics of mappings of the real interval should depend so definitely on complex methods.

Nowadays, there are many examples of this holomorphic dependence and in this note I would like to discuss one of the most recent connections between real dynamics and the theory of conformal mappings.

The first interesting pattern mentioned above, the limit of period doubling, plays a decisive role in this story because of a remarkable discovery made by Mitchell Feigenbaum in 1975, about the time the Milnor-Thurston article first appeared. Feigenbaum discovered the $1, 2, 4, 8, \ldots$ cascade of period doubling and the limiting "2^∞" Cantor set possessed self similar geometrical structure which was common or *universal* for a variety of different smooth families. In his 1976 paper *Quantitative Universality in Non linear Dynamics*, Feigenbaum described a renormalization scenario in the space of all smooth 1-fold mappings which *would* explain his numerical discoveries and their universal validity.

The numerical convergences of Feigenbaum, for example,

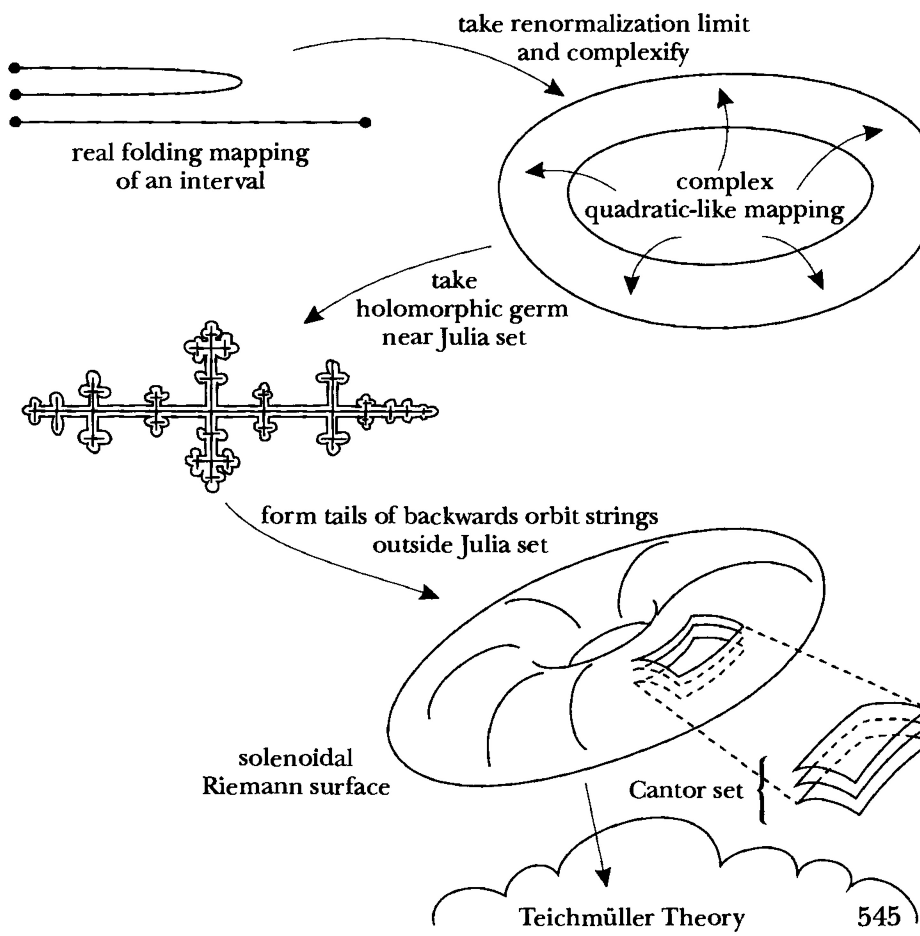
$$d_n/d_{n+1} \to 4.6692\ldots \qquad l_{n+1}/l_n \to .3995\ldots$$

were so robust that Lanford was able to produce a rigorous computer assisted proof of Feigenbaum's renormalization scenario. Here d_n denotes the parametric distance between the n^{th} period doubling and the limiting parameter and l_n is the length of the central interval at level n in the natural presentation of the 2-adic Cantor set of the limiting parameter.

We now have a conceptual proof of Feigenbaum's renormalization scenario in a more general form which reveals quantitative universality for uncountably many of the Milnor-Thurston patterns [11]. In the last part of that proof we need to make an abstract or conceptual computation of the length of an infinitesimal change in the complex structure on a "solenoidal Riemann surface" i.e., a compact space locally homeomorphic to a (two disk) × (Cantor set) enhanced with a complex structure.

We will describe the theory of complex structures underlying this abstract computation—which is the denouement of the proof of Feigenbaum's *quantitative* universality. We note that even the statement of the quantitative universality of Feigenbaum depends on the Milnor-Thurston *qualitative* or *topological* universality for its very formulation. The theory of complex structures used is an adaptation of the Ahlfors-Bers theory of *Universal Teichmüller Space* to *solenoidal Riemann* surfaces.

We will skip part of the story suggested by the following figure:



1. Topology of solenoids and laminations

A (k-dimensional) *solenoid* is a compact space X locally homeomorphic to a (k-ball) product a totally disconnected space. A solenoid is naturally foliated or laminated by the path components which are called leaves. More generally a *foliated space* or *lamination* is a space covered by charts of the form (ball) $\times$ (transversal space) so that overlap homeomorphisms preserve the first factor. Thus for a solenoid the foliated or laminated structure is intrinsic to the topology. If a solenoid has a non-locally closed dense leaf the transversal space will be a Cantor set. This usually happens in the dynamical examples studied here.

Example 1. Take the inverse limit space X_1 of $\{\cdots \xrightarrow{d} S^1 \xrightarrow{d} S^1 \xrightarrow{d} S^1\}$ where d is a degree 2 covering map of the circle. This is the well known 2-adic solenoid. This 1-dimensional solenoid fibres over the circle with fibre the 2-adic Cantor set. The "going around" map of the fibre is equivalent to adding 1 in the 2-adic integers so every leaf is dense.

Example 2. The 1-dimensional solenoid X_1 above admits a natural self mapping $\tilde{d}$ which is the inverse limit of the degree 2 self mapping of the circle. For the second example form $X_1 \times \{y \mid y > 0\}$ with the free, properly discontinuous action of the integers generated by $(x, y) \to (\tilde{d}x, 2y)$. The orbit space L of this action is a compact 2-dimensional solenoid since we have compact fundamental domains, $\{x, y : a \le y \le 2a\}$. This is the basic solenoidal surface or lamination L required in the dynamical theory of Feigenbaum's discovery.

In this example every leaf is dense. Countably many leaves are annuli and the rest are disks. In the next sections we will study complex structures on these spaces. In this example the simply connected leaves are conformal disks and the annular leaves have finite modulus. These moduli correspond to eigenvalues at periodic points of an associated dynamical system (Appendix). These parameters insure that the Teichmüller space of homotopy classes of complex structures is infinite dimensional. It will be modeled on a complex Banach space.

Example 3. Form (a pair of pants) product (2-adic Cantor set) and glue the legs into the waist by adding the pant label as the first symbol in the infinite binary expansion. This lamination is similar to Example 2 with all dense leaves (but of infinite topology) and a countable set of special leaves (with 1 extra handle) corresponding to periodic points of an associated dynamical system.

We can build in a transversally continuous hyperbolic metric as follows. A pair of pants is determined by the waist length and the two cuff lengths. Choose waist lengths and twist parameters in an arbitrary continuous fashion along the Cantor set transversal to the waist. The length of the cuffs are determined by the gluing. The twist parameters determine the gluing.

Example 4. Form the inverse limit of the directed set of pointed isomorphism classes of all finite sheeted coverings spaces of a compact manifold. This solenoid is the same, up to homeomorphism, for any manifold chosen as base from among the finite coverings used in the construction. Thus in dimension 2 there are two non-trivial examples: one E_∞ with dense Euclidean planes, and one H_∞ with dense non Euclidean planes.

Example 5. There are many examples of 1-dimensional solenoids: mapping tori for homeomorphisms of Cantor sets, geodesic laminations in surfaces, general inverse limits of expanding mappings on branched 1 manifolds [23]. It seems interesting to study their classification up to homeomorphism. For example, for a Markov homeomorphism of the Cantor set determined by a matrix A of nonnegative integers, the integer $\det(I - A)$ is an invariant of the homeomorphism type of the associated solenoid [16] which is a complete invariant when it is square free [17].

2. The Teichmüller set of classes of complex structures and its "metric"

A *complex structure* on a lamination L is a maximal covering of L by lamination charts (disk) $\times$ (transversal) so that overlap homeomorphisms are complex analytic in the disk direction. Two complex structures are *Teichmüller equivalent* if they are related by a homeomorphism which is homotopic to the identity through leaf preserving continuous mappings of L. The set of classes is called the *Teichmüller set* $T(L)$.

THEOREM. (a) Each complex structure on L determines a smooth structure on L (see definition below).

(b) If L is a solenoidal surface then

(i) L has smooth structures.

(ii) Any two smooth structures are conjugate by a homeomorphism homotopic to the identity.

COROLLARY. Any (oriented) solenoidal surface L has complex structures and the Teichmüller set $T(L)$ can be represented by the smooth conformal structures on L relative to a chosen background smooth structure modulo the equivalence relation generated by diffeomorphism homotopic to the identity.

DEFINITION (Smooth category for laminations). The only subtlety is the idea that the objects should be smooth in the disk direction and continuous in the transverse direction for the smooth topology on the objects. Thus to define a smooth structure the overlap homeomorphisms are C^∞ diffeomorphisms on the horizontal disks and the transversal variation is continuous for the C^∞ topology on C^∞ diffeomorphisms. Riemannian metrics, conformal structures, and tensors are treated similarly [13].

PROOF OF THEOREM. Part (a) follows from the fact that C^0 convergence on a disk of holomorphic homeomorphisms implies these C^∞ diffeomorphisms also converge as C^∞ diffeomorphisms on proper subdisks, say using the Cauchy formulae for the derivatives.

Part (b) follows by considering a local relative approximation proof of the corresponding result for surfaces and observing a choice made for one example can be extended continuously for nearby examples. Using the totally disconnected nature of the transversal now allows this result to go over to solenoids.

Problem Assignment. Prove part (b) for general laminations using a more elaborate families approximation result for surfaces.

PROOF OF COROLLARY. By part (b) there are smooth structures. Choose one. Then choose a smooth conformal structure. By Ahlfors-Bers [6] we can integrate these to get a complex structure. By (a) and (b) any complex structure on the topological solenoid is Teichmüller equivalent to one constructed this way.

CONVENTION. We now fix smooth structures on each solenoid (or lamination if possible) denote by $\mathscr{C}(L)$ the smooth conformal structures (corresponding precisely by Ahlfors-Bers to those complex structures with the given underlying smooth structure) and consider $\mathscr{C}(L)/\sim$, the space of equivalence classes of these smooth complex structures modulo homeomorphisms (which become diffeomorphisms because the underlying smooth structure is the same) homotopic to the identity.

We continue the discussion with this changed presentation of $T(L)$. For solenoids we have the same set. For laminations we have a set depending on a chosen smooth structure. Once the problem assignment is carried out for laminations the two sets will be the same. Otherwise the subsequent discussion applies to Teichmüller theory of a general lamination based on the choice of an underlying smooth structure.

DEFINITION. The *Poincaré distance on* $\mathscr{C}(L)$ the set of smooth conformal structures, is the sup over points of L of the Poincaré distance between the conformal structures on the tangent spaces. (Recall the set of positive definite inner products up to scale on a 2-dimensional vector space V is a homogeneous space of $GL(V)$ with isotropy the similarity group of the inner product structure. Thus it is naturally a model of non-Euclidean geometry and carries a natural Poincaré metric.)

The "Teichmüller metric" on $T(L)$ is defined by the Poincaré distance in $\mathscr{C}(L)$ between the equivalence classes of smooth complex structures defined above. We will see below (sections 3 and 6) this defines a metric on $T(L)$ in case L is *hyperbolic*—every leaf is conformally covered by the disk.

DEFINITION (TLC, the transversally locally constant theory). For a solenoid, one can define an interesting subtheory *TLC* defined by objects which are continuous and locally constant in the transverse direction. Thus we have *TLC* complex structures defined by charts whose overlap homeomorphisms preserve both factors, *TLC* smooth structures, *TLC* conformal structures, *TLC* tensors, etc.

Let us fix a *TLC* smooth structure and define $\mathscr{C}(L)'$ and $T(L)'$ using *TLC* objects as above.

THEOREM. If L is a solenoid, $\mathscr{C}(L)' \subset \mathscr{C}(L)$ is dense. The Teichmüller equivalence relation in $\mathscr{C}(L)' \times \mathscr{C}(L)'$ is dense in the Teichmüller relation in $\mathscr{C}(L) \times \mathscr{C}(L)$. The Poincaré distance in $\mathscr{C}(L)$ restricted to $\mathscr{C}(L)'$ equals the Poincaré distance in $\mathscr{C}(L)'$ (by definition) so the natural map $T(L)' \to T(L)$ is distance preserving and has dense image.

Remark. We know $T(L)' \to T(L)$ is an injection for the lamination in Example 2 because the conformal moduli are a determining set of invariants for elements of $T(L)'$ (Appendix). But at present we don't have a general argument showing injectivity of $T(L)' \to T(L)$.

PROOF. (i) Using a partition of unity and the fact that continuous functions on a totally disconnected space can be approximated by locally continuous constant functions we see that in a *TLC* smooth structure conformal structures can be approximated by *TLC* conformal structures. Note also vector fields can be approximated by *TLC* vector fields.

(ii) This approximation also works for diffeomorphisms because families of the close C^∞ diffeomorphisms can be averaged. Thus the *TLC* smooth structure on L is unique up to isotopy because the smooth structure on L is unique up to isotopy (section 2). This means the two ways, as discussed in the convention above, of defining $T(L)'$ are equivalent.

Observation (i) means $\mathcal{C}(L)' \subset \mathcal{C}(L)$ is dense. Observation (ii) means that the equivalence relation in $\mathcal{C}(L)' \times \mathcal{C}(L)'$ is dense in the equivalence relation of $\mathcal{C}(L) \times \mathcal{C}(L)$.

CONVENTION TLC. The point of the smooth convention assumed above for general laminations is that transversal pointwise convergence a.e. is needed to develop several steps in the theory. Fixing the smooth structure is a convenient way to make sense of this transversal continuity.

In the *TLC* context this subtlety disappears because objects are transversally locally constant. Thus we can work also with non-smooth objects where appropriate. For example, leafwise bounded measurable conformal structures which are *TLC* work fine for the Ahlfors-Bers integration of a conformal structure to obtain *TLC* complex structures.

We will make use of this extra freedom in the dynamical application of section 10. We note that [25] generalizes to transverse continuity of measurable conformal structures in the sup norm topology, which makes sense in the presence of *TLC* quasiconformal charts.

3. The injective map from $T(L)$ to $T(\widetilde{\text{leaf}})$ for hyperbolic laminations

If each leaf in a lamination L with complex structure is conformally covered by the 2-disk we say L is hyperbolic. Then by Candel's thesis [12] the Poincaré metric on each leaf defines a smooth metric on L.

Problem. Give a *topological* definition or characterization of the hyperbolic property of compact laminations.

A leaf preserving homotopy of the identity on a hyperbolic lamination L is bounded in the Poincaré metric so it becomes the identity on the ideal boundary of the disk covering of any leaf. This means we have a well-defined restriction mapping $T(L) \to T(\tilde{l})$ from homotopy classes of conformal structures on L to the universal Teichmüller space of bounded homotopy classes of bounded smooth conformal structures on the universal cover $\tilde{l}$ of any leaf l.

THEOREM. If l is a dense leaf then the restriction map $T(L) \to T(\tilde{l})$ is an injection.

PROOF. Recall that if a conformal map of the disk $\tilde{l}$ moves points a bounded amount in the Poincaré metric it must be the identity. If an almost conformal map of $\tilde{l}$ moves points a bounded amount it must be close to the identity. The latter statement works if the map is defined on a large enough Poincaré disk. Namely, it must be close to the identity on a ball of half the radius, say, by a limit contradiction argument.

If two conformal structures c_1 and c_2 on L when restricted to l become equivalent in $T(\tilde{l})$ this means they are related in $\tilde{l}$ by a map π which moves points a bounded distance.

If π were continuous on L it would transport c_1 to c_2 on every leaf because c_1 and c_2 are transversally continuous for almost everywhere pointwise convergence.

Claim. π determines a continuous map of L.

Proof of the claim. For simplicity suppose l is simply connected. Take a large Poincaré disk in l and move it transversally by an almost conformal map I_1 to a large disk in a very nearby passage of l (first conformal structure). Apply π to these disks. Since π only moves a bounded amount in the Poincaré metric large subdisks of the image disks in the second structure are related by an almost conformal map I_2 using a very small transversal motion. The composition $I_1^{-1}\pi^{-1}I_2\pi$ is almost conformal (structure one) on a large Poincaré disk and moves points a bounded amount so it must be close to the identity on a large subball. This proves the claim when l is simply connected.

When l is not simply connected, the same argument can be applied to deduce that π commutes with the covering group of $\tilde{l} \to l$. This proves the claim and shows π on L is homotopic to the identity using geodesics between x and $\pi(x)$, constructed first in $\tilde{l}$ then everywhere by density.

COROLLARY OF PROOF. If l_α is a collection of leaves whose union is dense, then $T(L) \to \prod_\alpha T(\tilde{l}_\alpha)$ is an injection.

PROOF. Take the π_α and use the same argument to show $\bigcup \pi_\alpha$ extends continuously to L and is homotopic to the identity.

4. A continuity property of the Ahlfors-Bers construction on the upper half plane

There is a map constructed by Ahlfors-Bers $\beta: \mathscr{C} \to B$ from bounded smooth conformal structures on the upper half plane to holomorphic quadratic differentials in the lower half plane whose pointwise norm is bounded by 6 in the Poincaré metric [6]. If $\tilde{c}$ is the conformal structure on the sphere which is c on the upper half plane and standard on the lower half plane, then $\beta(c)$ is the Schwarzian of $w(c)$ on the lower half plane where $w(c)$ is any quasi conformal homeomorphism which carries $\tilde{c}$ to the standard conformal structure on the sphere. The bound 6 above is Nehari's Inequality for univalent mappings.

PROPOSITION. The value of $\beta(c)$ at a point z in the lower half plane depends continuously on c for bounded a.e. pointwise a.e. convergence in the upper half plane.

PROOF. One knows $w(c)$ for the topology of uniform convergence on the sphere depends continuously on c in the above sense. On the lower half plane $w(c)$ is holomorphic so uniform convergence on the sphere implies uniform convergence of the derivatives on compact subsets of the lower half plane.

Remark. Ahlfors-Bers and Ahlfors-Weill showed that β is holomorphic and covers the ball of radius 2 [6], [7], [8]. In the next section we adapt this to hyperbolic laminations.

5. The Ahlfors-Bers construction for hyperbolic laminations

Let τ be the unit tangent bundle of L. Then for each t in τ we can construct a canonical covering isometry of the upper half plane U onto the leaf containing t, $U \xrightarrow{\pi_t} L$. Namely, send a fixed unit vector in U to t and extend isometrically to preserve orientations.

The map $\tau \times U \to L$ defined by the $\bigcup \pi_t$ is the quotient map of an action of $PGL(2, R)$, the isometry group of U, on $\tau \times U$. Namely, if we identify τ

with the leafwise isometric maps of U into L, $\mathrm{Isom}(U, L)$, then the natural action on $\mathrm{Isom}(U, L) \times U$ is $(I, u) \to (I \cdot g^{-1}, gu)$ and $\tau \times U \to L$ becomes $(I, u) \to I(u)$ which is constant along the orbits of the action. Thus objects on L are in one-to-one correspondence with invariant objects on $\tau \times U$.

For example, given a second complex structure c on L we pull back c to U using π_t to obtain c_t, a complex structure on the upper half plane depending on t and invariant by the action. Then let φ_t denote the quadratic differential $\beta(c_t)$ transported by symmetry to the upper half plane. The collection $\{\varphi_t\}$ is invariant for the action. In the transversal direction $t \to \varphi_t$ is continuous by the Proposition of section 4.

In the reverse direction there is the Ahlfors-Weill section $\varphi \to \mu_\varphi$, where $\mu_\varphi = \overline{\varphi}(\text{Poincaré metric})^{-2}$ of the Ahlfors-Bers construction defined on the ball of radius 2. In both directions we use C^0 convergence of holomorphic objects implies convergence in the C^∞ topology.

THEOREM. For each complex structure c on L, the Ahlfors-Bers construction defines a map $\beta\colon \mathscr{C}(L) \to B(c)$ from smooth conformal structures on the lamination to the transversally continuous holomorphic quadratic differentials on L. The map β is holomorphic, the image contains the ball of radius 2, and the fibres of β are precisely the Teichmüller equivalence classes.

PROOF. The first parts are proved above. We prove the last part. If β maps c_1 and c_2 to the same holomorphic quadratic differential then for every t the φ_t's are also the same. Thus for every leaf l the restrictions to $T(\tilde{l})$ are the same and conversely. Now apply the theorem of section 3 and its corollary.

Remark. The Ahlfors-Bers construction works for a *TLC* complex structure and each bounded measurable *TLC* conformal structure. The image holomorphic quadratic differential is not *TLC* but only transversally continuous.

6. The metric of $T(L)$ is locally Banach*

From section 5 we have for each conformal structure c_0 on L a composition $\beta(c_0)$

$$\mathscr{C}(L) \to \mathscr{C}(L)/\sim \to B(c_0)$$

* This section is based on joint work with Frederick Gardiner [25] [26].

induced by the Ahlfors-Bers construction on the upper half plane where $\sim$ is the Teichmüller equivalence ($c_1 \sim c_2$ if there is a homeomorphism of L homotopic to the identity, carrying c_1 to c_2.) We denote as usual $\mathcal{C}(L)/\sim$ by $T(L)$, the Teichmüller space of L.

THEOREM. The Poincaré metric on smooth conformal structures $\mathcal{C}(L)$ induces a *metric* on homotopy classes $T(L)$, called the Teichmüller metric, which enjoys the following property: for each point (c_0) of $T(L)$ there is a quasi isometry with universal constants between a neighborhood of (c_0) in $T(L)$ with the Teichmüller metric and the unit ball in the Banach space $B(c_0)$ so that (c_0) is mapped to the origin.

PROOF. By Schwarz's lemma the distance between fibres is at least the distance between their images in the Poincaré metric $P(6)$ on the ball of radius 6 which contains image $\beta(c_0)$.

Using the Ahlfors-Weill section, section 5, we see the Poincaré metric $P(2)$ on the ball of radius 2, by Schwarz's lemma again, is greater than the $\mathcal{C}(L)$ distance between the fibres cut by the section.

Now $P(6)$ and $P(2)$ are equivalent to the Banach metric on the ball of radius 1 with universal constants.

COROLLARY. As c ranges over complex structures on L all the Banach spaces $B(c)$ are locally bi-Lipschitz equivalent and $T(L)$ is a Lipschitz manifold modeled on this one isomorphism class.

PROOF. Since $\mathcal{C}(L)$ is path connected by geodesics $T(L)$ is connected by rectifiable curves. We cover this curve by a finite union of "unit ball" neighborhoods as in the theorem. The overlap homeomorphisms between intersecting balls are locally bi-Lipschitz.

7. The tangent, cotangent, and manifold structure of $T(L)$*

First some definitions. A tangent vector to $\mathcal{C}(L)$ at a point c is a tensor ν of the form $\nu(z,\lambda)\,d\bar{z}/dz$ and is called a *Beltrami vector.* Equivalence classes of those will be the *tangent vectors* to $T(L)$. A cotangent vector to $T(L)$ will be an object in a chart of the form $\varphi(z,\lambda)\,dz^2\,d\lambda$ where $\varphi(z,\lambda)$ is holomorphic

* The theorem, the remark and the second corollary of this section are needed for the dynamical application of section 10.

in z and $|\varphi|\,|dz|^2\,d\lambda$ is a finite measure on L. Here $d\lambda$ is a measure on the chart transversal. Such an object is called a *cotangent holomorphic quadratic differential.* The following theorem characterizes "Teichmüller trivial" Beltrami vectors.

THEOREM. Let ν be a smooth Beltrami vector, a tensor of the form $\nu(z,\lambda)\,d\bar{z}/dz$. Then $\nu = \bar{\partial} V$ where V is a smooth vector field on L if and only if for all cotangent holomorphic quadratic differentials φ, $\int_L \nu\varphi = 0$.

PROOF. Assume the test integrals vanish. If we lift the problem to the universal cover of a leaf, thought of as a disk D we see that V would have to vanish on the boundary and so by the Pompeiu formula

$$(*) \qquad V(z) = \frac{1}{2\pi i}\int_D \frac{\nu(\xi)\,d\xi\,d\bar{\xi}}{z-\xi}.$$

Now on the disk it is known V defined by $(*)$ is the unique solution of $\bar{\partial} V = \nu$ and $V = 0$ on ∂D if and only if $\int_D \nu\varphi$ vanishes for every holomorphic integrable quadratic differential on the disk [19]. We can push the test quadratic differentials on the disk forward to get test quadratic differentials on L [18]. Then by our hypothesis we know the integrals are zero, and we can form V on the disk cover of each leaf using $(*)$.

The approximate value of V at a point z of L only depends on ν on a large Poincaré disk about z. Since the Poincaré metric on L is transversally continuous and ν is transversally continuous in the C^∞ topology we see that the unique V by formula $(*)$ is also.

The other direction is a Stokes theorem calculation.

Remark. If ν is a bounded measurable *TLC* Beltrami vector whose test integrals vanish the proof yields a continuous vector field V so that $\bar{\partial} V = \nu$, and conversely. This V is not *TLC*, which is why we develop the more general theory.

COROLLARY. The kernel of the tangent map of the Ahlfors-Bers construction $\beta\colon \mathcal{C}(L) \to B(c)$ at the complex structure c consists precisely of those Beltrami vectors ν which annihilate all cotangent holomorphic quadratic differentials.

PROOF. The tangent map of β applied to ν is the Schwarzian (i.e., third derivative) of a holomorphic vector field on the lower half plane whose restriction to the boundary agrees with the restriction of V to the boundary where $\bar\partial V = \nu$. But the latter restriction is zero because V is bounded in the Poincaré metric on leaves. This argument also works in reverse.

COROLLARY. The Banach space $B(c)$ of transversally continuous holomorphic quadratic differentials at c is isomorphic to smooth Beltrami vectors ν modulo Teichmüller trivial ones, those of the form $\bar\partial V$, V a smooth vector field on L.

Remark. We refer to elements in $B(c)$ as *tangent holomorphic quadratic differentials.*

COROLLARY. For each complex structure on L the two infinitesimal norms inf over V of ess. sup $|\nu + \bar\partial V|$ where V ranges over smooth vector fields and sup over φ of $\int_L \nu\varphi$, where φ ranges over mass 1 cotangent holomorphic quadratic differentials, are equal.

PROOF. This is an abstract duality result. Under the natural dual pairing, $\int \varphi\nu$, between smooth forms, the total mass norms and ess. sup norms are in duality. The image of $\bar\partial$ is the annihilator of the holomorphic subspace by the previous theorem. It follows the induced norm in the quotient by the image of $\bar\partial$ is the restricted norm on the holomorphic subspace.

If we work with C^r conformal structures $\mathscr{C}^r(L)$ for $0 < r < \infty$ and not an integer, then by Ahlfors-Bers integration we have precisely the complex structures with a given underlying C^{1+r} structure. This uses the fact that the infinitesimal step in setting up the Ahlfors-Bers integration uses a Hilbert type transform which preserves C^r when and only when r is not an integer. The Teichmüller set and its metric $T(L)$ defined this way is the same for all such r or ∞. Now $\mathscr{C}^r(L)$ is a complex Banach manifold and $\beta^r : \mathscr{C}^r(L) \to B(c)$ is again holomorphic, covers the ball of radius 2 and has the Ahlfors-Weill section. Thus locally β^r is a holomorphic submersion and the fibres, the Teichmüller equivalence classes, by Banach differential topology, are the holomorphic leaves of a holomorphic foliation.

COROLLARY. The quotient map $\mathscr{C}^r(L) \to \mathscr{C}^r(L)/\sim$ defines a complex Banach manifold structure in $T(L)$. The tangent space is identified with the tangent holomorphic quadratic differentials or the Beltrami vectors modulo Teichmüller trivial ones.

PROOF. For the first part we use the foliation charts to define charts in $T(L)$. The point is the leaves do not reenter small enough foliation charts because of the metric information of section 6. The rest of the corollary puts together statements achieved above.

8. $T(L)$ has many almost geodesic Poincaré disks*

Here are some definitions beyond section 7. A tangent vector or Beltrami vector at a point c in $\mathcal{C}(L)$ generates a complex geodesic disk $D(\nu)$ in $\mathcal{C}(L)$ called a *Beltrami disk.* $D(\nu)$ consists of the conformal structures related to c by the Beltrami coefficients $\lambda \cdot \nu$ where sup norm$(\lambda \cdot \nu) < 1$.

A Beltrami vector ν at c is $(1-\varepsilon)$-*coherent* with a cotangent holomorphic quadratic differential φ of total mass 1 at c if and only if outside a set of $|\varphi|$ mass ε the pointwise norm of ν varies in ratio by at most $1+\varepsilon$ and the angle between the major axis of ν and the horizontal trajectory of φ (defined by $\varphi > 0$) is at most ε.

It turns out that a *Beltrami disk $D(\nu)$ in $\mathcal{C}(L)$ projected into $T(L)$ is almost geodesic if and only if ν is very coherent with some cotangent holomorphic quadratic differential φ.* More precisely,

THEOREM (Almost Geodesic Characterization). Let $R, \varepsilon, \delta > 0$. Then there are universal positive functions $\varepsilon' = \varepsilon'(\delta, R)$ and $\delta' = \delta'(\varepsilon, R)$ so that for R fixed $\varepsilon' \to 0$ as $\delta \to 0$ and $\delta' \to 0$ as $\varepsilon \to 0$ and

(i) If for some c' in the Beltrami disk $D(\nu)$ at c, Teichmüller distance $(c, c') \geq (1-\delta)r$ where r is the Poincaré distance between c and c' in $D(\nu)$, $0 < r \leq R$, and $\delta < \delta'(\varepsilon, R)$ then ν is $(1-\varepsilon)$-coherent with some cotangent holomorphic quadratic differential φ.

(ii) If for some φ, a cotangent quadratic differential, ν is $(1-\varepsilon)$-coherent with φ and $\varepsilon < \varepsilon'(\delta, R)$ then for all c' in the R-disk about c in the Poincaré metric on the Beltrami disk $D(\nu)$, Teichmüller distance $(c, c') \geq (1-\delta)$ Poincaré distance (c, c').

PROOF. First we sketch the proof of (i).

(a) If ν is not $(1-\varepsilon)$-coherent then $\sup\left(\int_L \nu\varphi\right)$ over φ of mass 1 is strictly less than $(1-\varepsilon_1)$(sup norm ν) for a universal ε_1 depending on R and ε. This is a reinterpretation of the elementary statement that if an average of complex

* This section is the main point for the dynamical application in section 10.

numbers has modulus close to the sup then most of them are close to this average.

(b) By the second corollary of section 7 we can find ν_1 so that $\mu = \nu - \nu_1$ is of the form $\bar{\partial} V$ for a vector field on L, where sup norm ν_1 is less than sup norm ν by a definite amount ε_2.

(c) Now we apply the Bers mapping to the Beltrami disk $D(\mu)$ in $\mathscr{C}(L)$ generated by μ to get a holomorphic map π into the 6-ball of $B(c)$ sending the center of $D(\mu)$ to the origin of $B(c)$. The derivative at the origin is the Schwarzian of a holomorphic vector field on the lower half plane which on the boundary agrees with V pulled back to the upper half plane and restricted to the boundary. Since this restriction is zero, because V is bounded in the Poincaré metric, π has derivative zero at zero. Thus $|\pi(\lambda\mu)| = O(|\lambda|^2)$ and Teichmüller distance $(c, c+\lambda\mu) = O(|\lambda|^2)$ using the theorem of section 6 with universal constants on the ball of radius R which is fixed. (See Chapter 6, [19].)

(d) If c' of the hypothesis of (i) is written $c + \lambda\nu$, then move on the Teichmüller fibre of $\beta\colon \mathscr{C}(L) \to B(c)$ containing c to a structure c_0 which is $O(|t\lambda\mu|^2)$ from $c + t\lambda\mu$ using paragraph (c) above. Up to second order terms in t the ratio of the Poincaré distance between ($c_0(t)$ and c') and (c and c') is at most $1 - O(t)$ with a universal constant. This is a computation in the Poincaré disk of conformal structures at each point where the sup norm is almost achieved. But $c_0(t)$ and c are Teichmüller equivalent so we can choose t to contradict (i) if δ is small enough relative to ε. The radius R affects the nonlinear terms as well so R influences the universal function $\delta(\varepsilon)$.

Part (i) was first proved for Riemann surfaces $\varepsilon = \delta = 0$ by Hamilton [14]. Part (ii) was proved for Riemann surfaces $\varepsilon = \delta = 0$ by Reich-Strebel [15].

Now we sketch the proof of part (ii). A cotangent holomorphic quadratic differential defines a foliation of horizontal trajectories with discrete singularities and a finite measure on L. When we condition the measure on trajectories we get well-defined measures up to local constants, i.e., affine structures on each trajectory. An abstract Poincaré recurrence argument shows almost all trajectories are R or R/Z in this affine structure. (Finite intervals have midpoints. The other kind of affine circles occur discretely on leaves. Right half lines allow a construction of a measure preserving map moving to the left. See [18].)

We also get from the cotangent holomorphic quadratic differential a Teichmüller metric (up to scale) on the universal covering of each leaf which has non-positive curvature, and the trajectories are geodesics in this structure. Now a homotopy can only make long geodesic segments longer, up to an error related to how the endpoints move. This idea combined with recurrence and the Grötzsch-Teichmüller argument yields the result. See [11] and [18] for more details. This argument for Riemann surfaces with $\varepsilon = \delta = 0$ was begun by Teichmüller and vigorously developed by Reich and Strebel [5], [15].

Remark. (1) Steps (a), (b), (c) and (d) work for bounded measurable ν and $D(\nu)$ in the *TLC* context. Thus (i) is true also. (2) Part (ii) is true for all bounded $|\varphi|$ measurable ν on the general lamination.

9. The Teichmüller metric is the integral of the Teichmüller norm on tangent vectors*

Gromov calls a metric space a *length space* if the distance between points is the infimum of lengths of rectifiable paths between them. $\mathscr{C}(L)$ with its Poincaré metric is a length space by definition. Now $T(L)$ is a quotient of $\mathscr{C}(L)$ with the induced metric. Thus by definition $T(L)$ is also a length space. What are these path lengths in $T(L)$?

THEOREM. The Teichmüller length of a rectifiable path in $T(L)$ is the integral of the infinitesimal norm (section 7) of its a.e. defined tangent vector field for any Lipschitz parameterization.

PROOF. As a corollary to the proof of the theorem in section 8 we get the Teichmüller length of the path $c + t\nu$ is $|\nu| \cdot t + O(t^2)$ where $|\nu|$ is the infinitesimal norm of section 7.

Now given a Lipschitz parameterized rectifiable path γ in $T(L)$ we can locally lift it to a $\mathscr{C}(L)$. It will have a tangent field $\nu(t)$ a.e. Now $\nu(t)$ determines a tangent vector field to γ in $T(L)$ which satisfies Teichmüller norm $= |\nu|$ by definition (Corollary, section 7). So by elementary calculus we have computed the length as promised [28].

10. The dynamical applications of $T(L)$

We apply the above theory to the 2 dimensional solenoid L of Example 2, section 1. The germ quasiconformal conjugacy class plus a *TLC* complex

* This section is not needed for the dynamical application in section 10.

structure on L up to Teichmüller equivalence is precisely the holomorphic conjugacy class of a germ of quadratic-like mapping on a neighborhood of the Julia set (assumed to be connected, non-separating, and to support no invariant measurable line field.) This statement is a reinterpretation of [20] using the Appendix, that such an object is determined by the quasiconformal information plus the real analytic conjugacy class of a degree 2 expanding mapping of the circle. One knows the latter is determined by the eigenvalues of the periodic points which are moduli of the annuli of L (Appendix). Thus the point in the *TLC* theory is determined by its image in $T(L)$. Thus $T(L)'$ injects in $T(L)$ for this L.

Renormalization is an operator defined on $T(L)'$ for each renormalizable pair of quasiconformal mapping germs [11]. For the real quadratic examples we know from the real bounds, the sector theorem, and the factoring theorem of [11] that iterated bounded type renormalization creates a definite annulus for some representative in $T(L)'$. By the pull back argument we deduce as in [11] the following:

THEOREM. Consider iterated bounded type renormalization of symmetric (about the real axis) germ conjugacy classes of quadratic like holomorphic mappings. When viewed as an iteration on the real part of $T(L)'$ this iteration preserves bounded measurable transversally locally constant Beltrami lines and brings any two points to within a definite distance, say d, of each other. Moreover, the number of renormalizations depends on the bound on the return time of each renormalization and the initial annuli of the representatives.

COROLLARY. Each real tangent vector to $T(L)'$ is contracted a definite amount by iterated bounded type renormalization. Each pair of points in the real part of $T(L)'$ converges together in the Teichmüller metric under bounded type renormalization.

PROOF. (1) We consider a *TLC* Beltrami vector symmetric about the real axis whose sup norm is not much bigger than its infinitesimal Teichmüller norm. A long piece of Beltrami path renormalizes to distance $< d$ by the theorem, so the renormalized Beltrami vector must have bigger sup norm than its infinitesimal norm by part (ii) of the Theorem of section 8. This proves the first part.

(2) Given two points in the real part of $T(L)'$ choose an efficient *TLC* Beltrami path between them. The Beltrami vector is efficient by part (i) of the Theorem of section 8. After renormalization the points come definitely

closer because otherwise using (i) again the Beltrami vector would not have been contracted a definite amount which was just proven above.

Remark. This convergence in the Teichmüller metric implies directly the universal geometric structure of the critical orbit Cantor sets [11].

Appendix. Dynamical solenoids and laminations

Let $d\colon S^1 \to S^1$ be the standard degree 2 self-mapping of the circle and let $\tilde{d}\colon \tilde{S} \to \tilde{S}$ be the inverse limit self mapping of the inverse limit solenoid. Let L denote the lamination fibring over the circle with fibre $\tilde{S}$ and going around mapping $\tilde{d}$. This L is Example 2 of section 1.

A collection $\{\varphi\}$ of local homeomorphisms of the real line is uniformly asymptotically affine (uaa) if for $\varepsilon > 0$ there is $\delta > 0$ so that, whenever defined,

$$\left| \log \frac{\varphi(x) - \varphi(x-\delta)}{\varphi(x+\delta) - \varphi(x)} \right| < \varepsilon.$$

A (uaa) structure on the circle is defined by a maximal covering whose overlap homeomorphisms belong to the pseudogroup generated by homeomorphisms individually (uaa). Compare [21].

A (uaa) structure for d is a (uaa) structure on S^1 so that the collection of local branches of d^{-n} $n = 1, 2, \ldots$ is (uaa) when measured in terms of a finite coordinate cover from this structure. Compare [25].

A maximal covering of a solenoid by charts so that the overlap homeomorphisms are affine in the leaf direction is called a *transversally continuous affine structure* on the solenoid.

THEOREM. There are canonical one-to-one correspondences between

(a) complex structures on L (up to Teichmüller equivalence)
(b) transversally continuous affine structures on the solenoid $\tilde{S}$ so that $\tilde{d}$ is affine from leaf to leaf (up to equality)
(c) the set of (uaa) structures for d (up to equality).

PROOF. (b) $\to$ (a). Add (by the natural construction) a half space to each affine line of $\tilde{S}$. The $\tilde{d}$ extends to a complex affine map $\tilde{D}$ between these half spaces. Drop off the boundary and form the quotient lamination. The Ahlfors-Beurling extension [22] defines a (smooth) equivalence with L.

(a) $\rightarrow$ (b). Start with a complex structure, put on the transversally continuous hyperbolic structure [12], and form the cyclic cover of L. Observe each leaf has a preferred point at infinity so the leaves are naturally upper half spaces and the deck transformation complex affine from leaf to leaf.

Pass to the ideal boundary, which is $\tilde{S}$ with the deck transformation becoming $\tilde{d}$, to obtain the transversally continuous affine structures on leaves affinely permuted by $\tilde{d}$.

(b) $\rightarrow$ (c). Consider the charts on S^1 defined by the affine structures on leaves of $\tilde{S}$ and the natural projection $\tilde{S} \rightarrow S^1$. Since $\tilde{d}$ is affine from leaf to leaf and $\tilde{d}^{-1}$ contracts the leaves uniformly we obtain a (uaa) structure for d. Namely, apply $\tilde{d}^{-1}$ enough times so that the continuity of the affine structure takes over [27].

(c) $\rightarrow$ (b). The definition of a (uaa) structure for d has been devised for this step. If we apply an iterated branch of d^{-1} to a segment on S^1 it becomes small and picks up an approximate affine structure from a finite covering by charts. Further iteration introduces small affine distortion by definition.

COROLLARY. There is a canonical bijection between the real analytic expanding mappings of S^1 up to real analytic conjugacy and the dense subset $T(L)' \subset T(L)$ defined by transversally locally constant complex structures.

PROOF. (i) The usual distortion lemma for $C^{1+\alpha}$ expanding maps e shows the smooth structure (after transport by the conjugacy [24] between e and the standard degree 2 mapping d) determines a (uaa) structure for d.

(ii) Also for two such C^r systems, r not an integer, a conjugacy which is (uaa) or which is Lipschitz is automatically C^r. This follows from the well known blow down and blow up argument applied to a point where the conjugacy is asymptotically affine (aa). If eigenvalues at periodic points are equal, the Markov intervals have comparable sizes and the conjugacy is Lipschitz.

(iii) Consider the complex analytic extension of a real analytic mapping to a neighborhood of the circle. Drop off the circle and form the inverse limit. One obtains an open neighborhood of one end of the cyclic cover of L.

Putting (i), (ii) and (iii) together proves the result.

COROLLARY. Inside $T(L)$ we see the real analytic expanding systems on S^1 as the transversally locally constant theory and the $C^{1+\alpha}$ expanding systems on S^1 as the transversally Hölder continuous theory. For these dense subsets of $T(L)$ the moduli of the annuli, being in general the eigenvalues at the periodic points, form a complete set of invariants.

PROOF. The transverse Hölder structure comes from the depth structure of the transversal Cantor set. One checks the steps of the theorem are compatible with this depth *qua* Hölder structure [27].

Problem. Are the eigenvalues *qua* moduli of annuli a complete set of invariants for all the elements of $T(L)$?

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