QUANTIZED CALCULUS ON S^1 AND QUASI FUSCHIAN GROUPS.

A. Connes and D. Sullivan

The theory of distributions works well for a number of problems involving non smooth functions such as these generated by the variational calculus. It is however notoriously incompatible with *products*, i.e. products of distributions only make sense in rare cases. The reason for that is simple since the notion of distribution on a manifold V is invariant under any continuous *linear* transformation of $C^\infty(V)$ while such linear transformations affect arbitrarily the *algebra* structure of $C^\infty(V)$. In the quantized calculus which we propose, the differential of a function f is an *operator* in Hilbert space, namely:

$$df = [F, f].$$

in particular this operator can undergo all the operations of the functional calculus such for instance as:

$$T \to |T|^p$$

where $|T|$ is the absolute value of the operator T and p a positive non integer real number.

This gives meaning to an expression such as $|df|^p$ while f is a non differentiable function. We shall show the power of this method by giving a formula for the hausdorff measure on the quasi Fuschian circles which appear in the theory of uniformization of pairs of Riemann surfaces with the same genus.

Our formula will invoke the operator

$$f(Z)|dZ|^p$$

where Z is a highly non differentiable function on the manifold S^1 and the Fredholm module $(\mathcal{H}, F)$ is given by $\mathcal{H} = L^2(S^1)$, on which functions on S^1 act by multiplication, and F the Hilbert transform $Fe_n = \mathrm{Sign}(n)e_n \ \ \forall n \in \mathbb{Z}$, $(e_n)_{n\in\mathbb{Z}}$ being the canonical orthonormal basis of $L^2(S^1)$, $e_n(t) = \exp(2\pi int) \ \ \forall t \in \mathbb{R}/\mathbb{Z}$, $n \in \mathbb{Z}$. We first need to explain how non smooth functions appear in the theory of Riemann surfaces. It is indeed, at first, somewhat surprising that a pair of compact Riemann surfaces $\sum_+, \sum_-$ of the same genus, or equivalently a pair of points in the moduli space $\mathcal{M}_g$, does generate a non smooth function on S^1. Let $\Gamma = \pi_1(\sum_+) = \pi_1(\sum_-)$ be the fundamental group of $\sum_\pm$. The point is the following result of L. Bers which provides a common uniformization for both $\sum_+$ and $\sum_-$.

Theorem 1. [B] *With the above notations there exists an isomorphism $h : \Gamma \to PSL(2, \mathbb{C})$ of Γ with a discrete subgroup of $PSL(2, \mathbb{C})$ whose action on $P_1(\mathbb{C}) = S^2$ has a Jordan curve C as limit set and is proper with quotient $\sum_\pm$ on the connected components $U_\pm$ of the complement of C.*

The discrete subgroup $h(\Gamma)$ is called a *quasi-Fuschian* group and its limit set C is called a *quasi circle*. It is a Jordan curve whose Hausdorff dimension is strictly bigger than one ([Bowen]). Let us choose a coordinate in $P_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ in such a way that $\infty \in \sum_-$ and use the Riemann mapping theorem to get a conformal equivalence:

$$Z : D \to \Sigma_+ \subset \mathbb{C}$$

where $D = \{z \in \mathbb{C}, |z| < 1\}$ is the unit disk. By the Caratheodory theorem ([C]) the holomorphic function Z extends continuously to $\overline{D} = D \cup S^1$ and yields a homeomorphism

$$Z : S^1 \to C.$$

the non differentiability of Z on S^1 is of course a consequence of the non smoothness of the Jordan curve C.

Since the range of the function Z on S^1 is equal to C, we see that the spectrum of the operator of multiplication by Z in $L^2(S^1)$ is also equal to C so that, for $p > 0$, the following operator:

$$f(Z)|dZ|^p$$

where f is a function on $\mathbb{C}$, and $dZ = [F, Z]$ as above, only involves the *restriction* of f to the subset C of $\mathbb{C}$, and depends of course linearly on f.

We shall prove the following formula for the Hausdorff measure μ on the compact set C:

Theorem 2. *There exists a smallest $p \in]1, 2[$ such that $dZ \in \mathcal{L}^{p,\infty}$, p is the Hausdorff dimension of C. The Hausdorff measure μ of C is given by the formula:*

$$\int_C f d\mu = \mathrm{Trace}_\omega(f(Z)|dZ|^p).$$

Here the two sided ideal $\mathcal{L}^{p,\infty}$ of compact operators are defined by the condition:

$$\mathcal{L}^{p,\infty}(\mathcal{H}) = \left\{ T \text{ compact}, \sum_0^N \mu_n(T) = O\left(\sum_1^N n^{-\frac{1}{p}}\right)\right\}$$

where $\mu_n(T)$ is the n^{th} eigenvalue of $|T|$ for a compact operator T. For $p > 1$ the condition is equivalent to

$$\mu_n(T) = O\left(n^{-\frac{1}{p}}\right) \qquad (n \to \infty).$$

We refer to the detailed discussion of these ideals and of real interpolation theory in Appendices B and C.

Also Trace_ω is the Dixmier trace (cf. Chapter 6 section 2) which for a positive operator $T \in \mathcal{L}^{1+}$ is a suitable limit of the averages $\frac{1}{\log N} \sum\limits_0^N \mu_n(T)$ (cf. [D]).

We have used extensively the Dixmier trace as a replacement for the integration against the volume form in a Riemannian manifold, but Theorem 2 above shows that the domain of applicability of our integration, using the Dixmier trace, is wider.

α) Regularity of f and the size of $df = [F, f]$.

Here we shall just state the known results of function theory on S^1 which relate the regularity of f to the size of df, measured in terms of the Schatten ideals $\mathcal{L}^p$ (or more generally the real interpolation ideals $\mathcal{L}^{p,q}$).

Let $L^\infty(S^1)$ be the von Neumann algebra in $\mathcal{H}$ generated by the multiplication operators. We know, with F the Hilbert transform in $\mathcal{H} = L^2(S^1)$, that $(\mathcal{H}, F)$ is a Fredholm module over the C^* algebra $C(S^1)$ and thus the first natural question is to characterize these $f \in L^\infty(S^1)$ which satisfy:

$$[F, f] \in k \quad \text{(i.e. } [F, f] \text{ is a } \textit{compact} \text{ operator).}$$

The answer is known (cf; []) and involves the *mean oscillation* of the function f. Let us recall that given any interval I of S^1 one lets $I(f)$ be the mean: $\frac{1}{|I|}\int_I f\, dx$ of f on I and one defines for $a > 0$ the mean oscillation of f by:

$$M_a(f) = \sup_{|I| \leq a} \frac{1}{|I|} \int_I |f - I(f)|.$$

A function is said to have bounded mean oscillation (BMO) if the $M_a(f)$ are bounded independently of a. This is of course true if $f \in L^\infty(S^1)$. A function f is said to have *vanishing* mean oscillation (VMO) if $M_a(f) \to 0$ when $a \to 0$. Let us then recall (cf. [])

Theorem 3. ([S]) *Let $f \in L^\infty$, then*

$$[F,f] \in k \Leftrightarrow f \in VMO.$$

Moreover $L^\infty \cap VMO = (H^\infty + C(S^1)) \cap (\overline{H}^\infty + C(S^1))$ where $H^\infty = L^\infty(S^1) \cap H^2$ is the (non selfadjoint) subalgebra of $L^\infty(S^1)$ of boundary values of holomorphic functions inside D (cf. [S]). The algebra $L^\infty \cap VMO$ is *strictly larger* than $C(S^1)$, its elements are called *quasicontinuous* functions. It is a C^* algebra by construction.

The next question is to characterize the functions $f \in L^\infty$ for which

$$[F,f] \in \mathcal{L}^p$$

for a given real number $p \in [1,\infty[$.

This question has a remarkably nice answer due to V.V. Peller [P] in terms of the Besov spaces $B_p^{1/p}$ of measurable functions.

Definition 4. *Let $p \in [1,\infty[$. Then the Besov space $B_p^{1/p}$ is the space of measurable functions f on S^1 such that*

$$\int\int |f(x+t) - 2f(x) + f(x-t)|^p \, t^{-2} \, dx \, dt < \infty.$$

For $p > 1$ this condition is equivalent to:

$$\int\int |f(x+t) - f(x)|^p \, t^{-2} \, dx \, dt < \infty$$

and the corresponding norms are equivalent. For $p = 2$ one recovers the Sobolev space of Fourier series,

$$f(t) = \sum_{n \in \mathbb{Z}} a_n \, \exp(2\pi \, int) \, , \, \sum |n| \, |a_n|^2 < \infty.$$

The result of V.V. Peller is then the following:

Theorem 5. ([P]) *Let $f \in L^\infty(S^1)$, $p \in]1,\infty[$, then $[F,f] \in \mathcal{L}^p \Leftrightarrow f \in B_p^{1/p}$.*

For $p = 1$ and $f = f^+ + f^-$ where $f^+ \in H^2, \overline{f}^- \in H^2$ one has $[F,f] \in \mathcal{L}^1 \Leftrightarrow f^\pm \in B_1^1$ (cf. [P]).

As an immediate corollary of Theorem 5 one has:

Corollary 6. *Let $\alpha \in]0,1[$ and $f \in C^\alpha$ be Hölder continuous of exponent α, then $[F,f] \in \mathcal{L}^p \;\; \forall p > \frac{1}{\alpha}$.*

It follows in particular that the Fredholm module $(\mathcal{H}, F)$ is $p+1$ summable over the algebra C^α of Hölder continuous functions of exponent $\alpha > \frac{1}{p+1}$ and thus the $(2k+1)$ dimensional character of $(\mathcal{H}, F)$ makes sense on C^α for $\alpha > \frac{1}{2k+2}$. The formula for this character is the same as the one already given in Chapter 3, section 2 Proposition 3, except for the small change due to the replacement of $\mathbb{R} \cup \{\infty\}$ by $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Theorem 7. a) *Let $p = 2k+2$. The following equality defines a cyclic cocycle of dimension $2k+1$ on the algebra $L^\infty \cap B_p^{1/p}$*

$$\tau(f^0, f^1, \ldots, f^{2k+1}) = \int \prod_0^{2k+1} d\theta_i \left(\prod_{j=0}^{2k+1} \frac{f^j(\theta_j) - f^j(\theta_{j+1})}{tg\left(\frac{\theta_j - \theta_{j+1}}{2}\right)} \right) \frac{1}{tg\left(\frac{\theta_{2k+1} - \theta_0}{2}\right)}.$$

b) *Let* $u \in L^\infty \cap B_p^{1/p}$ *be such that* $u^{-1} \in L^\infty$, *then* $u^{-1} \in L^\infty \cap B_p^{1/p}$ *and* $\tau(u, u^{-1}, \dots, u, u^{-1})$ *is an integer equal to the Fredholm index of the Toeplitz operator* PuP.

It follows immediately from Proposition 8 of section 1 and the above Theorem 5. In particular if $u \in C^\alpha$, $\alpha \geq (2k+2)^{-1}$ then the winding number of u is given by the above formula. The latter formula would involve the difference quotients: $\frac{f^j(t_j)-f^j(t_{j+1})}{t_j - t_{j+1}}$ if we had worked with $\mathbb{R}$ instead of S^1.

By Proposition 6 of section 1 the restriction of the above cyclic cocycle τ_k to $C^\infty(S^1)$ is cohomologous to $\lambda_k S^k \tau_1$, with $\tau_1(f^0, f^1) = \int_{S^1} f^0 df^1$. More generally the restriction of τ_k to $L^\infty \cap B_p^{1/p}$, $p \leq 2k$ is cohomologous to $S\tau_{k-1}$.

We shall end this section by giving known reformulations of the Besov spaces $A_p^{1/p} = \{f \in B_p^{1/p} \ , \ \widehat{f}(n) = 0$ for $n < 0\}$. Given $f \in A_p^{1/p}$ we denote also by f the holomorphic function inside the unit disk D with f as boundary values.

Proposition 8. ([S]) a) *For* $1 \leq p < \infty$ *one has* $f \in A_p^{1/p}$ *iff*

$$\int_D |f''|^p \ (1-|z|)^{2p-2} \ dz \ d\overline{z} < \infty.$$

b) *For* $1 < p < \infty$ *one has* $f \in A_p^{1/p}$ *iff*

$$\int_D |f'|^p \ (1-|z|)^{p-2} \ dz \ d\overline{z} < \infty.$$

One can also reformulate the condition $f \in A_p^{1/p}$ using the L^p norm of the truncations of the Fourier series of f, $\sum \widehat{f}(k) \ e^{ik\theta}$, between $k = 2^n$ and $k = 2^{n+1}$. More precisely ([P]) one lets for each $n \in \mathbb{N}$, w_n be the trigonometric polynomial

$$w_n = \sum_{2^{n-1}}^{2^{n+1}} c_k \ e^{ik\theta}$$

where $c_k = (k - 2^{n-1})/2^{n-1}$ for $2^{n-1} \leq k \leq 2^n$ and $c_k = (2^{n+1} - k)/2^n$ for $2^n \leq k \leq 2^{n+1}$.
Then the operator $f \rightarrow f * w_n$ of convolution by w_n is the same as the multiplication of the Fourier coefficients $\widehat{f}(k)$ by c_k, these operators add up to the identity, and one has:

Proposition 9. ([P]) *The space* $A_p^{1/p}$ *is the space of boundary values of holomorphic functors inside* D *such that:*

$$\sum 2^n \ \| w_n * f \|_p^p < \infty.$$

Using $w_{-n} = \overline{w}_n$ for $n < 0$ one can then check that the following conditions on $f \in L^\infty(S^1)$ are equivalent, for any $p \in [1, \infty[$:

$$[F, f] \in \mathcal{L}^p \quad , \quad \sum_{n \in \mathbb{Z}} 2^{|n|} \ \| w_n * f \|_p^p < \infty.$$

β) **The class of** df **in** $\mathcal{L}^{p,\infty}/\mathcal{L}_0^{p,\infty}$.

Let T be a compact operator in a Hilbert space $\mathcal{H}$, we let $\mu_0(T) \geq \mu_1(T) \geq \cdots$ be the list of eigenvalues of $|T| = (T^*T)^{1/2}$ and recall (appendix B) that for each N the following defines a norm on the ideal k of compact operators:

$$\sigma_N(T) = \sum_0^{N-1} \mu_n(T).$$

We let (appendix B) $\mathcal{L}^{p,\infty}$ be the interpolation ideal given by the condition:

$$T \in \mathcal{L}^{p,\infty} \Leftrightarrow \sigma_N(T) = O\left(\sum_1^N n^{-1/p}\right).$$

For $p = 1$ this means that $\sigma_N(T) = O(\log N)$ while for $p > 1$ the conditions $\sigma_N(T) = O\left(N^{1-\frac{1}{p}}\right)$ and $\mu_N(T) = O(N^{-1/p})$ are equivalent to $T \in \mathcal{L}^{p,\infty}$.

We let $\mathcal{L}_0^{p,\infty} \subset \mathcal{L}^{p,\infty}$ be the closure in the Banach space $\mathcal{L}^{p,\infty}$ of the ideal $\mathcal{R}$ of finite rank operators. One has

$$T \in \mathcal{L}_0^{p,\infty} \Leftrightarrow \sigma_N(T) = o\left(\sum_1^N n^{-1/p}\right).$$

For $p > 1$ this is equivalent to $\mu_n(T) = o(n^{-1/p})$ but for $p = 1$ the condition $\mu_n(T) = o\left(\frac{1}{n}\right)$ is stronger than $T \in \mathcal{L}_0^{1,\infty}$. Let then $p > 1$ and $f \in C(S^1)$ be such that its quantum differential $df = [F, f]$ belongs to $\mathcal{L}^{p,\infty}$. The main result of this section is that if we work modulo $\mathcal{L}_0^{p,\infty}$ then the following rules of calculus are valid:

a) $(df)g = g\, df \quad \forall g \in C(S^1)$

b) $d(\varphi(f)) = \varphi'(f)df \quad \forall \varphi \in C^\infty$ (Spectrum (f))

c) $|d(\varphi(f))|^p = |\varphi'(f)|^p\ |df|^p$.

In a) and b) the equalities mean that the following operators belong to the ideal $\mathcal{L}_0^{p,\infty}$:

$$[F, f]g - g[F, f] \quad , \quad [F, \varphi(f)] - \varphi'(f)[F, f].$$

In c) the equality holds modulo $\mathcal{L}_0^{1,\infty}$:

$$|[F, \varphi(f)]^p - |\varphi'(f)|^p\ |[F, f]|^p \in \mathcal{L}_0^{1,\infty}.$$

In fact we shall prove that the characteristic values of the latter operator are $o\left(\frac{1}{n}\right)$ which is a stronger result.

The above rules a) b) c) are *classical* rules of calculus but they are applied to a non differentiable function f, for which the distributional derivative f' cannot undergo the operation $x \to |x|^p$ as does the quantum differential.

We shall thus prove: (with $p \in [1, \infty[$ for a) b)).

Theorem 10. a) *Let $f \in L^\infty(S^1)$ be such that $[F, f] \in \mathcal{L}^{p,\infty}$ and let $g \in C(S^1)$. Then $[F, f]g - g[F, f] \in \mathcal{L}_0^{p,\infty}$.*

b) *Let $X_1, \ldots, X_n \in C(S^1)$, $X_j = X_j^*$, be such that $[F, X_j] \in \mathcal{L}^{p,\infty}$ and $\varphi \in C^\infty(K)$ be a smooth function on the joint spectrum $K \subset \mathbb{R}^n$ of the X_j's (i.e. $K = X(S^1)$). Then:*

$$[F, \varphi(X_1, \ldots, X_n)] - \sum \partial_j\ \varphi(X_1, \ldots, X_n)[F, X_j] \in \mathcal{L}_0^{p,\infty}.$$

c) *Let $p > 1$, $Z \in C(S^1)$ be such that $[F, Z] \in \mathcal{L}^{p,\infty}$ and φ be a holomorphic function on $K =$ Spectrum $Z = Z(S^1)$. Then:*

$$|[F, \varphi(Z)]|^p - |\varphi'(Z)|^p\ |[F, Z]|^p \in \mathcal{L}_0^{p,\infty}.$$

In fact as we already mentioned we shall prove the stronger result that $\mu_n(T) = o\left(\frac{1}{n}\right)$ for the operator T appearing in c).

Proof of a) and b)

a) The map from $C(S^1)$ to $\mathcal{L}^{p,\infty}$ given by:

$$g \mapsto [F, f]g - g[F, f] = T(g)$$

is norm continuous. Thus it is enough to show that for $g \in C^\infty(S^1)$ the image $T(g)$ belongs to $\mathcal{L}_0^{p,\infty}$. In fact one has $T(g) \in \mathcal{L}^1$ since g commutes with f while $[F, g] \in \mathcal{L}^1$.

b) The map from $C^\infty(K)$ to $\mathcal{L}^{p,\infty}$ given by:

$$\varphi \mapsto [F, \varphi(X_1, \ldots, X_n)]$$

is continuous. Thus it is enough to check that the statement is true for polynomials, which easily follows from a).

The proof of c) involves general estimates on the map $A \to |A|^p$ with respect to the norms σ_N. We first recall that by [D] [K] the map $A \to |A|$ is a Lipschitz map from $\mathcal{L}^{p,\infty}$ to itself provided that $p > 1$.

We shall need the following lemma:

Lemma 11. *Let $\alpha \in]0, 1[$. There exists $C_\alpha < \infty$ such that for any compact operators A, B in $\mathcal{H}$ one has:*

$$\frac{1}{N} \sigma_N (|A|^\alpha - |B|^\alpha) \leq C_\alpha \left(\frac{1}{N} \sigma_N(A - B) \right)^\alpha .$$

Proof. Let $r = \frac{\alpha}{2}$ so that $0 < r < \frac{1}{2}$. One has

$$|A|^\alpha = (A^*A)^r = \lambda_r \int_0^\infty \frac{A^*A}{t + A^*A} \, t^{r-1} \, dt$$

where $\lambda_r \neq 0$.

One has

$$\frac{A^*A}{t + A^*A} - \frac{B^*B}{t + B^*B} = t(t + B^*B)^{-1} \, (A^*A - B^*B)(t + A^*A)^{-1}$$
$$= t(t + B^*B)^{-1} \, (A^* - B^*)A(t + A^*A)^{-1} + (t + B^*B)^{-1} \, B^*(A - B)(t + A^*A)^{-1} \, t.$$

Let p_N be the norm $\frac{1}{N} \sigma_N$, then we get from the inequality $p_N(XYZ) \leq \| X \| \; p_N(Y) \| Z \|$ that:

$$p_N \left(\frac{A^*A}{t + A^*A} - \frac{B^*B}{t + B^*B} \right) \leq p_N((A^* - B^*)) \, \frac{1}{2\sqrt{t}} + p_N(A - B) \, \frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{t}} \, p_N(A - B).$$

We used the inequalities: $\| A(t + A^*A)^{-1} \| \leq \frac{1}{2\sqrt{t}}$, $\| (t + B^*B)^{-1} \, B^* \| \leq \frac{1}{2\sqrt{t}}$. Moreover since the operator norm of $\frac{A^*A}{t+A^*A} - \frac{B^*B}{t+B^*B}$ is less than 1 we get:

$$p_N \left(\frac{A^*A}{t + A^*A} - \frac{B^*B}{t + B^*B} \right) \leq \mathrm{Inf}(1, t^{-1/2} \, p_N(A - B)).$$

We then use as in [K] the inequality:

$$\int_0^\infty \mathrm{Inf}(1, t^{-1/2} \, a) \, t^{r-1} \, dt \leq \int_0^\infty \frac{2a}{t^{1/2} + a} \, t^{r-1} \, dt = 4\lambda_{2r}^{-1} \, a^{2r}$$

so that $p_N (|A|^\alpha - |B|^\alpha) \leq 4 \frac{\lambda_r}{\lambda_{2r}} \, p_N(A - B)^\alpha$.

As an immediate corollary of this lemma we get:

Proposition 12. *Let $p > 1$.*

1) *Let A, B be bounded operators such that $A - B \in \mathcal{L}_0^{p,\infty}$ then $|A|^\alpha - |B|^\alpha \in \mathcal{L}_0^{p/\alpha,\infty}$ for any $\alpha < 1$.*

2) *If $A, B \in \mathcal{L}^{p,\infty}$ and $A - B \in \mathcal{L}_0^{p,\infty}$ then for any $\alpha \leq p$ one has $|A|^\alpha - |B|^\alpha \in \mathcal{L}_0^{p/\alpha,\infty}$.*

3) *For A, B as in 2) and $\alpha = p$ one has $\mu_N(|A|^p - |B|^p) = o\left(\frac{1}{N}\right)$.*

Proof. 1) One has $p_N(A - B) = o(N^{-1/p})$ by hypothesis, it then follows from the lemma that $p_N\left(|A|^\alpha - |B|^\alpha\right) = o(N^{-\alpha/p})$.

2) Let $\alpha < 1$ be such that p/α is an integer k $(k > 1)$. Then by 1) one has $|A|^\alpha - |B|^\alpha \in \mathcal{L}_0^{k,\infty}$, while $|A|^\alpha, |B|^\alpha \in \mathcal{L}^{k,\infty}$.

Let $S = |A|^\alpha$, $T = |B|^*$. One has $\mu_N(S) = O(N^{-1/k})$, $\mu_N(T) = O(N^{-1/k})$, $\mu_N(S - T) = 0(N^{-1/k})$. Thus using the inequality $\mu_{n_1+n_2+n_3}(XYZ) \leq \mu_{n_1}(X)\, \mu_{N_2}(Y)\, \mu_{N_3}(Z)$ (cf. appendix B) as well as the equality

$$S^k - T^k = \sum S^j(S - T)T^{k-j-1}$$

we get that $\mu_N(S^k - T^k) = o\left(\frac{1}{N}\right)$.

The proof of 2) is the same.

Let us apply this proposition for the proof of theorem 10 c). First, as in b) we have:

$$[F, \varphi(Z)] - \varphi'(Z)[F, Z] \in \mathcal{L}_0^{p,\infty}$$

so that by proposition 12 b) we get

$$|[F, \varphi(Z)]|^p - |\varphi'(Z)[F, Z]|^p \in \mathcal{L}_0^{1,\infty}.$$

Thus we just need to show that:

$$|\varphi'(Z)[F, Z]|^p - |\varphi'(Z)|^p\ |[F, Z]|^p \in \mathcal{L}_0^{p,\infty}.$$

One can replace $\varphi'(Z)$ by $f = |\varphi'(Z)|$ and replace $[F, Z]$ by $T = |[F, Z]|$ since $f[F, Z] - [F, Z]f \in \mathcal{L}_0^{p,\infty}$. Thus it is enough to use the following lemma:

Lemma 13. *Let $p > 1$, $T \in \mathcal{L}^{p,\infty}$, $T \geq 0$, let f bounded, $f \geq 0$ such that $fT - Tf \in \mathcal{L}_0^{p,\infty}$. Then*

$$f^{p/2}\, T^p\, f^{p/2} - \left(f^{1/2}\, T\, f^{1/2}\right)^p \in \mathcal{L}_0^{1,\infty}$$

γ) The Dixmier trace of $f(Z)|dZ|^p$.

Let us take the notations of theorem 1, so that Γ is a quasi-Fuschian subgroup of $PSL(2, \mathbb{C})$ and $C \subset P_1(\mathbb{C})$ its limit set in $P_1(\mathbb{C})$. Also we let Z be a holomorphic function in the unit disk $D = \{z \in \mathbb{C}\ ;\ |z| < 1\}$, continuous on $\overline{D} = D \cup S^1$ and which is a conformal equivalence of D with the bounded connected component $\sum^+$ of the complement of C in $P_1(\mathbb{C})$. By construction there is an isomorphism $g \to g_+$ of Γ with a Fuschian subgroup Γ_+ of $PSL(2, \mathbb{R})$ such that:

$$(*) \qquad g \circ Z = Z \circ g_+ \qquad \forall g \in \Gamma$$

where we consider $PSL(2, \mathbb{R}) = SU(1, 1)$ as the group of automorphisms of D.

Let us first use the equality $(*)$ to reexpress the condition

$$[F, Z] \in \mathcal{L}^q$$

in simpler terms.

Lemma 14. *Let $q > 1$, then $[F, Z] \in \mathcal{L}^q$ iff the following Poincaré series is convergent for some (and equivalently all) point $z \in \sum_+$:*

$$\sigma(q) = \sum_{g \in \Gamma} |g'(z)|^q < \infty.$$

Moreover there are constants c_q, C_q bounded away from 0 and ∞ for $q \geq q_0 > 1$ such that

$$c_q\ \sigma(q) \leq \text{Trace}|[F, Z]|^q \leq C_q\ \sigma(q).$$

Proof. By construction the function $Z \in C(S^1)$ extends to a holomorphic function in D so that we can apply the criterion given by proposition 8 b), which also gives an estimate on the $\mathcal{L}^p$ norm of $[F, Z]$ by the following expression:

$$\int_D |Z'(z)|^q\ (1 - |z|)^{q-2}\ dz\ d\bar{z}.$$

For z in D, $1 - |z|$ and $1 - |z|^2$ are comparable, so that we may as well consider the following expression:

$$\int_D |Z'(z)|^q\ (1 - |z|^2)^q\ (1 - |z|^2)^{-2}\ dz\ d\bar{z}.$$

If we endow D with its canonical hyperbolic Riemannian metric of curvature -1, the last expression is equivalent to

$$\int_D \| \nabla Z \|^q\ dv$$

where ∇Z is the gradient of the function Z whose norm is evaluated with respect to the Riemannian metric, and where dv is the volume form on the Riemannian manifold D. Let then $g \in PSL(2, \mathbb{R}) = SU(1, 1)$, since it acts as an isometry on D one has:

$$\| \nabla(Z \circ g) \| (p) = \| \nabla(Z) \| (gp) \quad \forall p \in D.$$

For $g_+ \in \Gamma_+$ one has $Z \circ g_+ = g \circ Z$ so that

$$\| \nabla(g \circ Z) \| (p) = \| \nabla(Z) \| (g_+ p) \quad \forall p \in D.$$

The left hand side is equal to $|g'(Z(p))|\ \ \| \nabla Z \| (p)$, so that:

$$\| \nabla(Z) \| (g_+ p) = |g'(Z(p))|\ \ \| \nabla Z \| (p) \quad \forall p \in D\ ,\ g \in \Gamma_+.$$

Let then $D_1 \subset D$ be a compact fundamental domain for the Fuschian group Γ_+, we have the equality:

$$\int_D \| \nabla Z \|^q\ dv = \int_{D_1} \sum_{\Gamma} |g'(Z(p))|^q\ \ \| \nabla Z \| (p).$$

The compactness of D_1 then gives the required uniformity in $p \in D_1$ so that the conclusion follows.

Let now p be the Hausdorff dimension of the limit set C. One has $p > 1$ ([B]) and by [S] it follows that the Poincaré series $\sigma(q)$ is convergent for any $q > p$, and diverges for $q = p$. Thus we get so far:

Proposition 15. *One has $[F,Z] \in \mathcal{L}^q$ iff $q > p =$ Hausdorff dimension of C.*

But in fact we need to know that $[F,Z] \in \mathcal{L}^{p,\infty}$ and that $\mathrm{Trace}_\omega(|[F,Z]|^p) > 0$.

Lemma 16. *One has $[F,Z] \in \mathcal{L}^{p,\infty}$.*

Proof. From each interpolation theory (appendix B) and the above criterion: $\| \nabla Z \| \in L^q(D,dv) \Leftrightarrow [F,Z] \in \mathcal{L}^q$ we just need to show that:

$$\| \nabla Z \| \in L^{p,\infty}(D,dv)$$

where the Lorentz space $L^{p,\infty} = L^p_{\text{weak}}$ is the space of functions h on D such that for some constant $c < \infty$:

$$v(\{z \in D\ ;\ |h(z)| > \alpha\}) \leq c\, \alpha^{-p}.$$

Thus the proof of lemma 14 shows that all we need to prove is that (uniformly for $a \in D_1$) the sequences $|g'Z(a)|$; $g \in \Gamma$ belong to $\ell^{p,\infty}(\Gamma)$, i.e.:

$$\mathrm{Card}\{g \in \Gamma\ ;\ |g'(Z(a))| > \alpha\} = O(\alpha^{-p}).$$

this follows from [S] corollary 10.

Next, the pole like behaviour of $\int_D \| \nabla Z \|^s\ dv$ for $s \to p_+$, which follows from [S], and the fact that the residue at $s = p$ is *not zero* ([S]) imply a similar behaviour for $\mathrm{Trace}(|[F,Z]|^s)$, so that the characteristic values

$$\mu_n = \mu_n(|F,Z]|)$$

satisfy the following conditions:

α) $\mu_n = O(n^{-1/p})$ (by lemma 16)

β) $(s-p)\sum \mu_n^s \geq c > 0$ for $s \in\ ]p, p+\varepsilon]$.

One can then use the following Tauberian lemma:

Lemma 17. ([HL][W]) *Let μ_n be a decreasing sequence of positive real numbers satisfying α) β) then*

$$\liminf \frac{1}{\log N} \sum_0^N \mu_n^p \geq c.$$

Let us sketch a proof for the convenience of the reader. We may assume that $p = 1$, replacing μ_n by μ_n^p. Then an easy calculation shows that if we define for $x > 0$,

$$\alpha(x) = \sum_{\mu_n \geq e^{-x}} \mu_n$$

then $\alpha(x) = O(x)$ for $x \to +\infty$, and the hypothesis β) means that there exists $y_0 \in \mathbb{R}$ such that for $y \geq y_0$,

$$\int_{-\infty}^{\infty} K(y-x)\ \beta(x)\ dx \geq c$$

where $\beta(x) = \frac{\alpha(e^x)}{e^x}$ for $x \in \mathbb{R}$ and where

$$K(u) = e^{2u}\ e^{-e^u}\quad ,\quad u \in \mathbb{R}.$$

By construction β is positive and bounded, thus since $K \in L^1(\mathbb{R})$ with Fourier transform $\widehat{K}$ non vanishing, it follows from [W] that for any $g \in L^1(\mathbb{R})$ with $\int g(u)\ du = 1$ one has

$$\liminf_{y\to\infty} (g * \beta)(y) \geq c.$$

As α is an increasing function one easily concludes that

$$\liminf \beta(y) \geq c.$$

Thus $\lim \frac{\alpha(x)}{x} \geq c$. Now since $\mu_n \leq c'\, n^{-1}$ one has

$$\sum_{n\leq c'e^u} \mu_n \geq \sum_{\mu_n \geq e^{-u}} \mu_n = \alpha(u)$$

$\frac{1}{\log(c'e^u)} \sum_{n\leq c'e^u} \mu_n \geq \frac{\alpha(u)}{\log(c'e^u)}$, and the result follows.

We are now ready to prove the following theorem. In the statement we fix a Dixmier trace Tr_ω once and for all.

Theorem 18. *Let $\Gamma \subset PSL(2,\mathbb{C})$ be a quasi-Fuschian group, $C \subset P_1(\mathbb{C})$ its limit set and $Z \in C(S^1)$ the boundary values of a conformal equivalence of the disk D with the bounded component of the complement of C. Then let p be the lower bound of the set $\{q; [F,Z] \in \mathcal{L}^q\}$, one has $p =$ Hausdorff dim C and $[F,Z] \in \mathcal{L}^{p,\infty}$. Moreover there exists a non zero finite real number λ such that, with μ the p-dim Hausdorff measure on C,*

$$\int_C f\, d\mu = \lambda\, Tr_\omega(f(Z)|[F,Z]|^p) \quad \forall f \in C_0(\mathbb{C}).$$

Proof. The first part follows from proposition 15, lemma 16 and lemma 17. It also follows from this lemma that $Tr_\omega(|[F,Z]|^p) > 0$ so that we can consider the measure ν on C determined by the equality:

$$\nu(f) = Tr_\omega(f(Z)\, |[F,Z]|^p) \quad \forall f \in C(C).$$

We claim that this measure has conformal weight p, i.e. that for any $g \in \Gamma$ one has the equality:

$$\int f \circ g\, d\nu = \int |g'|^p\, f\, d\nu.$$

To prove this let $g_+ \in PSL(2,\mathbb{R}) = SU(1,1)$ be the corresponding element of Γ_+, its action on $L^2(S^1)$ (viewed as the space of $\frac{1}{2}$ densities on S^1) is a *unitary* operator W which commutes with the Hilbert transform F. Moreover

$$WZW^* = Z \circ g_+ = g \circ Z.$$

Thus we get the following equality:

$$W[F,Z]W^* = [F, g\circ Z].$$

It implies that $W|[F,Z]|^p\, W^* = |[F, g\circ Z]|^p$, thus:

$$W(f\circ g^{-1}(Z))\, |[F,Z]|^p)W^* = f(Z)\, |[F, g\circ Z]|^p.$$

Since the Dixmier trace Tr_ω is a trace we thus have:

$$Tr_\omega(f\circ g^{-1}(Z)\, |[F,Z]|^p) = Tr_\omega(f(Z)\, |[F,g\circ Z]|^p)$$

and by theorem 10 c) we have:

$$Tr_\omega(f(Z)\, |[F,g\circ Z]|^p) = Tr_\omega(f(Z)|g'(Z)|^p\, |[F,Z]|^p)$$

so that:

$$\int f \circ g^{-1}\, d\nu = \int f|g'|^p\, d\nu.$$

It then follows by [S] that ν is proportional to the p-dimensional Hausdorff measure on C.

The constant λ in theorem 18 should be independent of the choice of ω and given by:

$$\lambda = \frac{p}{2}\,\Gamma\left(\frac{p}{2}\right)(4\pi)^{p/2}.$$

We conjecture that this is true. It implies the following estimate on the best rational approximation of the function Z. Indeed, let μ_N be given by the distance (in the norm of $C(S^1)$) of Z with the restrictions to S^1 of rational fractions with at most n poles outside the unit disk. Then:

$$\mu_n \sim \lambda^{-1/p}\,|C|^{1/p}\,n^{-1/p}$$

where $\lambda = \frac{p}{2}\,\Gamma\left(\frac{p}{2}\right)(4\pi)^{p/2}$ and $|C|$ is the Hausdorff p dimensional Hausdorff mass of the limit set C.