# RENORMALIZATION, ZYGMUND SMOOTHNESS AND THE EPSTEIN CLASS

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To randomize a deck of n-cards one may turn over one of the split stacks before shuffling. The resulting permutation of order n if irreducible is called a *folding permutation* because it may be accomplished by a continuous mapping f of the real line to itself which folds the line once. The orbit of the turning point is finite and f restricted to this finite orbit is the folding permutation.

In fact there are two nontrivial theorems about this realization.

i) Any smooth unimodal shape for a graph can be ajusted vertically as a graph to realize any folding permutation (Sharkovski, 1959, Milnor-Thurston 1975 et al.)

ii) For the parabolic shape the vertical positioning is *unique* for each folding permutation. In particular, the geometry of the finite critical point orbit is determined uniquely up to affine equivalence by the combinatorical structure of the folding permutation. (Douady, Hubbard, Milnor, Sullivan, Thurston, ...).

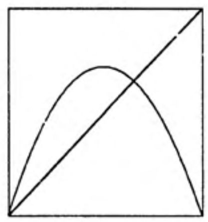
The first theorem uses the order structure of the real line. To the best of my knowledge, all proofs of the second statement use holomorphic dynamics and quasi conformal mappings.

Physicists working numerically discovered surprising and far reaching generalizations of this geometric rigidity in the context of dynamical systems defined by smooth folding mappings. Certain infinite limiting versions of geometric rigidity suggested by example ii) occur in the general smooth families of i). Geometric structures that should depend on infinitely many parameters - like a Cantor set critical orbit up to smooth change of coordinates - only depend on a few, such as the power law at the critical point and the combinatorics (the vertical parameter).

These physicists, Feigenbaum in the U.S. and Coullet and Tresser in France introduced the language and scenarios of statistical physics to describe these phenomena because they were reminiscient of the universal exponents and renormalization group of critical points and phase changes in statistical mechanics. [F1], [F2], [CT].

*Chaos, Order, and Patterns*, Edited by R. Artuso *et al.*
Plenum Press, New York, 1991

Let us first say what "renormalization" means. Start with a folding mapping preserving some interval I and suppose some power of f preserves some interval RIcontaining the original turning point figure 1a) . The graph of the inset box is that of a new folding mapping of the interval called a renormalization of f. If n and RI are taken to be minimal then we denote this renormalization Rf. Renormalization is a partially defined mapping from the space of all folding mappings of the interval, into itself.



folding mappings of I

If the original f is varied by a vertical parameter say, RI and Rf will be defined in between the two extreme cases, Figure 1b

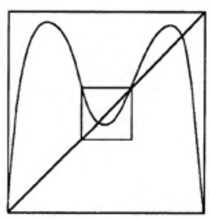


Figure 1a

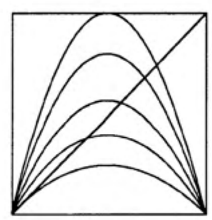


Figure 1b

As Rf varies between these two extremes all folding permutations occur . We may have that for $f_1 = Rf$ there are interior disjoint intervals in $I_1 = RI$ permuted by $f_1$ according to a second permutation $\sigma_1$. We may consider $f_2 = Rf_1$ etc. If the renormalization process is always possible we generate $f, f_1, f_2, \ldots, f_n$, and permutations $\sigma_0\,\sigma_1 \ldots \sigma_n \ldots$ and we say f is *infinitely renormalizable* with combinatorial type $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_2, \ldots)$.

Let $\tau$ denote the unique permutation of order 2. One finds an infinitely renormalizable f of combinatorial type $(\tau, \tau, \tau, \ldots)$ at the limit of the cascade of period doubling bifurcations. The orbit closure of the critical point for such an f is a binary cantor set which was discovered numerically to have a universal geometric structure characterized by a countable set of self similarity ratios. Some aspects and consequence of this geometry for example the celebrated $\delta = 4.6692...$ ratio for the cascade itself were measured in fluid experiments of Maurice Libchaber. The presence of this numerical value in a system of many dimensions is explained by the n-dimensional perturbation of the Feigenbaum renormalization picture (Collet, Eckman, Koch [CEK]).

The physicists'scenario was as follows. For the $(\tau, \tau, ...)$ combinatorics, iteration of renormalization leads one to a special mapping $F(\tau, \tau, ...)$ which is fixed by renormalization. The geometry of any f of type $(\tau, \tau, ...)$ at deep levels becomes that of $F(\tau, \tau, ..., \tau ...)$ since $R_\tau ... R_\tau R_\tau f \to F(\tau, \tau, ...)$. The $\delta$ above measured the rate of expansion of $R_\tau$ in the space of folding mappings transverse to the codimension one manifold of maps of type $(\tau, \tau, ...)$. This picture works beautifully in the numerical experiments of Feigenbaum. This numerics was proved rigourously in a computer assisted discussion in a open neighborhood of some $F(\tau, \tau, ...)$ in Lanford [L1].

The existence of an $F(\tau, \tau, ...)$ was verified mathematically by Epstein [E] working in the space of folding mappings $h{\cdot}q$ = (diffeomorphism) $\cdot$ (quadratic polynomial) where $h^{-1}$ has an injective complex analytic extension to $\mathbb{C}$-(x real but not in an open neighborhood of this dynamical interval}. We say these mappings have the Epstein form and write $f \in E(J)$ where J is the open neighborhood of the dynamical interval. Epstein also treated the other critical exponents in the $(\tau, \tau, ...)$ case.

In the rest of this paper we discuss why this Epstein class E(J) is important for renormalization of dynamical systems. Namely the theorem of § 3 states any limit of renormalization starting from sufficient finite smoothness must lie in Epstein class E(J) for some interval strictly larger than the dynamical interval. We also characterize in terms of the technique used (cross ratio distortion) exactly what smoothness class is required. It is : if f = hQ = diffeomorphism . quadratic polynomial, then log h' satisfies the Zygmund condition § 1.

In the complete paper [S1] we prove generalizations of the scenario described above. In particular the Cantor set of any mapping f of type $(\sigma_0, \sigma_1, ...)$ where degree $\sigma_i$ is uniformly bounded has universal asymptotic ratios. These ratios can be computed from canonical analytic functions $F( ... \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, ...)$ which are precisely the limits of infinite renormalization starting from the Zygmund class of smoothness.

The theorem presented here about the Epstein class is the first step in this program. For the complete discussion see [S1].

## 1 Poincaré length distortion and smoothness class one plus Zygmund

We want to study the smoothness required for a diffeomorphism h to only distort cross ratios of small standard 4-tuples by an amount commensurable to the size of the 4-tuple.

One cross ratio [a, b, c, d] can be computed by

$$-\log\,[a, b, c, d] = \iint_S \frac{dx\,dy}{(x - y)^2}\ , \quad a < b < c < d\ ,$$

where S is the square $\{(x,y) \mid a \le x \le b\ ,\ c \le y \le d\}$ .

Thus the distortion by h , given by

$$\log \frac{[ha, hb, hc, hd]}{[a, b, c, d]}\text{, equals } \int_S \mu - (h \times h)^* \mu$$

where $\mu$ is the measure, $\dfrac{dx\,dy}{(x - y)^2} = \mu$ .

Calculating the integral we get $\left(\dfrac{1}{(x - y)^2} - \dfrac{h'x\,h'y}{(hx - hy)^2}\right)$, or

$$\frac{1}{(x - y)^2}\left[1 - \frac{h'x\,h'y}{[h']^2_{xy}}\right]$$

where $[h']_{xy} = \underset{[x,y]}{\text{average}}\ h'$ is the average of h' over the interval [x,y] .

Because we are assuming b - a = c - b = d - c for every point (x,y) in the square S the factor $\dfrac{1}{(x - y)^2}$ is commensurable to 1/area S . Thus a small bound $\varepsilon$ on $\log \dfrac{h'x\,h'y}{[h']^2_{xy}}$ yields the bound $\varepsilon$ on the distortion of cross ratio, $\log \dfrac{[ha, hb, hc, hd]}{[a, b, c, d]}$ .

Calculating this log we get
$\log h'x + \log h'y - 2 \log [h']_{xy}$ .

Let us replace the last term with the average taken after the log to obtain a) where

a) is $(\log h'x + \log h'y - 2\,[\log h']_{xy})$ with an error of twice b), where

b) is $(\log \underset{[x,y]}{\text{average}}\,(h') - \underset{[x,y]}{\text{average}}\,(\log h'))$ .

Let us say h satisfies the *local Koebe condition* if for |x - y| sufficiently small one of the equivalent conditions hold

1) $\left[1 - \dfrac{h'x\,h'y}{[h']^2_{xy}}\right] = 0(|x - y|)$ (here 0(s) means a term at most k.s as $s \to 0$.)

2) $\log \dfrac{h'x\,h'y}{[h']^2_{xy}} = 0(|x - y|)$

**Note.** if both a) and b) are 0(|x - y|) , then by the above 1) and 2) hold.

**Proposition.** *If* h *satisfies the local Koebe condition then the* h *distortion of cross ratios of small standard 4-tuples is commensurable to the size of the 4-tuple.*

**Proof.** The above calculation.

Expression a) suggests the Zygmund condition on continuous functions

$$Z: \quad \varphi(x) + \varphi(y) - 2\,\varphi(\tfrac{x+y}{2}) = 0(|x - y|)\,.$$

**Proposition.** *If* $\varphi$ *satisfies* Z *on an interval* J *then average* $\varphi$ *over* J *is the value of* $\varphi$ *at the midpoint with an error* 0(*length* J ).

**Proof.** Think of the uniform measure on J as two dirac masses moving out uniformly from the center. Use the Z condition to replace the average of $\varphi$ at the moving points by the value at the center. Q.E.D.

**Corollary.** *If* log h' *is Zygmund then expression* a) *is* $0(|x - y|)$ .

**Proof.** Use the proposition, then the definition of Z again.

There is a converse to the corollary.

Say $\varphi$ satisfies the *average property* if $\underset{[x,y]}{\text{average}}\ \varphi = \frac{1}{2}(\varphi x + \varphi(y)) + 0(|x - y|)$ .

**Proposition.** *If* $\varphi$ *satisfies the average property for all intervals* $J \subset I$ , *then* $\varphi$ *satisfies the Zygmund property for all pairs* x,y *in* I .

**Proof.** Apply the average property to the intervals $\left[x, \frac{x+y}{2}\right]$, $\left[\frac{x+y}{2}, y\right]$, and $[x,y]$ and combine averages of averages to get the Z-property for x,y .

**Corollary.** *The Zygmund property is equivalent to the average property.*

**Proof.** Propositions above.

**Conclusion A.** *expression a) is* $0(|x - y|)$ *iff* log h' *is Zygmund.*

Now we consider when expression b) is $0(|x - y|)$ . We are concerned with small intervals J and we assume h' is continuous. Then h' varies only a little from one of its values $h'(x_0) = a$ . The expression b) is unchanged if we multiply h'x by 1/a . Write 1/a h' on J as $1 + \varepsilon$ where $\varepsilon$ is a small function. Expand the two terms of b)

$$\log \frac{1}{|J|} \int_J 1 + \varepsilon \;-\; \frac{1}{|J|} \int_J \log 1 + \varepsilon$$

$$= \left( \frac{1}{|J|} \int_J \varepsilon - \frac{1}{2}\left( \frac{1}{|J|} \int_J \varepsilon \right)^2 \cdots \right) - \left( \frac{1}{|J|} \int_J \varepsilon - \frac{\varepsilon^2}{2} \cdots \right)$$

$$= -\frac{1}{2}\left( \frac{1}{|J|} \int_J \varepsilon \right)^2 + \frac{1}{|J|} \int_J \varepsilon^2/2 \cdots$$

Now the first term could be zero so there would be no cancellation. Thus we are forced to estimate each brutally with absolute values. Assume $\varepsilon$ is Holder of order 1/2 on J ,

$|\varepsilon(x) - \varepsilon(y)|^2 \le C_J\, |x - y|$ . Since $\varepsilon$ is zero at $x_0$ we get the estimate $C_J \cdot$length J for the sum of the absolute values. Also if $C_J \cdot$length J is sufficiently small the higher order terms can be ignored.

**Conclusion B.** expression b) is $0(|x - y|)$ if h' is Holder of order 1/2 . The coefficient for $|x - y| < \varepsilon$ is estimated by the *normalized* $\frac{1}{2}$ *Holder norm* : take the sup over all intervals J of length $\le \varepsilon$ of $C_J$ above where $1 + \varepsilon = h'(x) / h'(x_0)$ for convenient $x_0$ in J and we assume $C_J$ |length J | is sufficiently small.

Let us note that Zygmund functions are Holder $\alpha$ for all $\alpha < 1$ . However, the $\alpha$-Hölder constants are not determined by the Z-norm. Let us also note the normalized $\frac{1}{2}$ Holder norm of h' can be estimated by the usual $\frac{1}{2}$ Holder norm of log h' - the best C such that

$$|\log h'x - \log h'y|^2 \le C\, |x - y| .$$

Now we can summarize the above by the

**Theorem.** a) *If* log h' *is Zygmund then* h *satisfies the local Koebe distortion condition. The coefficient is controlled by the Zygmund norm of* log h' *and the* $\frac{1}{2}$ *Holder norm of log* h' . *Conversely,*

b) *if* log h' *is* $\frac{1}{2}$ *Holder, then the local Koebe condition for* h *implies* log h' *is Zygmund.*

**Proof.** The above discussion has been a proof of a). For part b) recall from above the local Koebe inequality implies expression a) plus expression b) is $0(|x - y|)$ . The $\frac{1}{2}$ Holder implies expression b) is $0(|x - y|)$ . Thus expression a) is $0(|x - y|)$ . But this implies log h' is Zygmund by the third little proposition. Q.E.D.

**Problem.** Derive necessary and sufficient conditions for the integral distortion to be commensurable to the linear scale. (In the above discussion we have estimated the integral by the integrand.) For the sketch of the solution see [S1].

## 2 The Koebe distortion argument of Denjoy, de Melo-Van Strien, Swiatek, Yoccoz, et al and Zygmund smoothness

Consider a composition g of many diffeomorphisms $f_i$ between tiny intervals $J_i$ all lying disjointly in some big interval I , $f_i : J_i \to J_{i+1}$ .

The *classical Denjoy argument* estimates $\log |g'x / g'y|$ $x,y \in$ domain g in terms of the $\sum_i$ total variation $|\log f_i'|$ . This will be finite say if $f_i = f / I_i$ and $\log f'$ has bounded variation on I . The proof is the chain rule.

The new argument called the *Koebe principle* for one dimensional real dynamics treats the case when the factors can be divided into two groups so that relative to some coordinate system on I

i) for one group a Denjoy type argument can be used at least to study cross ratios.

ii) the factors in the other group decrease Poincaré length (a type of cross ratio) (because of a positive Schwarzian condition) even though $\log f_i'$ has unbounded variation.

Here if L, M, R is a partition of an interval T into 3 consecutive subintervals (the left, the middle, and the right) the *Poincaré length of* M *in* T is $\log(1 + \frac{MT}{LR})$. It is the length of M in the Riemannian metric on $T = [a,b]$ corresponding to the form $|dx| / x - a + |dx| / b - x$.

The additive change of P-length along a composition is additive over the factors. *In a decomposition such as* i) ii) *above, the increase in* P*-length is controlled by the factors of type* i) *because there is a decrease for the factors of type* ii). *This is the first idea cf. Swiatek* [SW].

The *second idea is the four intervals argument.* Let J, L, M, R be contiguous equal length intervals and let h be a homeomorphism of the union into the real line so that one of hL and hM is *much smaller* than other. Discard from the original 4 intervals the outer interval next to the one of L, M called s made smaller. Let T denote the remaining three L, M, X and let $\ell \subset T$ be the one of L or M made larger. The P-length of $\ell \subset T$ is $\log 4$. The P-length of $h(\ell) \subset hT$ is very large because $h(\ell)$ is much larger than h(s) and h(T) is of course greater than hX. Thus one has the analogue of complex Koebe distortion :

**Real Koebe distortion.** *If a homeomorphism* $h : I \to$ reals *does not increase unit* P*-lengths too much the asymmetric distortion for interior symmetric triples is controlled.*

More precisely if $x,y \in I$ satisfy $|x - y|$ is as small as the distance to $\partial I$ and $z = x + y / 2$, then $\frac{1}{M} \le (h(x) - h(z)) / (hy - hz) \le M$ where M can be calculated from the bound B on the additive increase of Poincaré length of unit Poincaré length subintervals $J \subset T$ where $T \subset I$, i.e. the B defined by

$(\text{P-length of } hJ \subset hT \ - \ \text{P-length } J \subset T)_+ \le B$

for all $J \subset T$ so that P-length $J \subset T = 1$.

**Remark.** The point here as in Koebe distortion for schlicht mappings is we go from one analytic condition (in that case holomorphic ; in this case positive Schwarzian or controlled P-length increase) to interior control on the non-linearity.

We describe the dynamical Koebe distortion principle for a rather general class of dynamical systems. Let M be a compact one-manifold provided with a differentiable structure where overlap homeomorphisms $h_{\alpha\beta}$ are continuously differentiable and the $\log h_{\alpha\beta}'$ have bounded Zygmund norm (see § ).

Suppose $f : M \to M$ is a smooth mapping with finitely many critical points where $f' = 0$. At a non singular point assume $\log f'$ is Zygmund. At a singular point $c_i$ suppose there are coordinate systems in the (1 + Zygmund) structure so that f takes the form $x \to |x|^{r_i} + v_i$ or $x \to (\text{sign } x)(|x|^{r_i}) + v_i$ where $r_i > 1$.

Assume we have a long composition g of diffeomorphisms $f_i : J_i \to J_{i+1}$ where the $J_i$ are disjoint in M and $f_i^{-1} = f$ restricted to $J_{i+1}$.

**Theorem.** *For the composition* g *the increase in Poincaré length and therefore the interior non linearity of* g *in domain* g *is controlled by constants of the coordinate systems and local models of* f *and are independent of the length of the composition* g.

**Proof.** We first need a lemma.

**Lemma.** If h is a diffeomorphism of the unit interval I and log h' is Zygmund, $T \subset I$ is a tiny interval $J \subset T$ has unit Poincaré length, then the Poincaré length of $hJ \subset hT$ is $1 + 0(\text{length } T)$. The coefficient is controlled by the Zygmund norm of log h' and the Holder $\frac{1}{2}$ norm of log h' squared.

**Proof of the lemma.** We have proved this in § 1 when J sits in the middle of T. In general J may be tiny and near one end of T. We have to calculate the integral of §1 over the rectangle R of figure

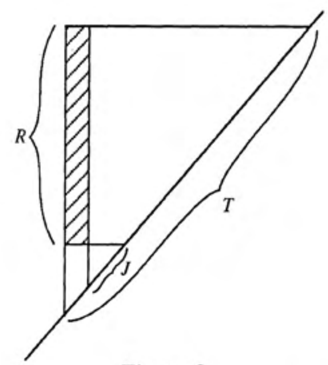



Figure 2

Using the local Koebe condition, and the fact that for a point in R the distance to the diagonal and the vertical distance to the diagonal are equivalent, the integral takes the form

$$a \cdot \int_a^b \frac{1}{t^2} 0(t)\, dt$$

where a~length J~distance $(J, \partial T)$, b~length T. This yields a log b/a which has order b when a and b are commensurable. This is the case already discussed. Otherwise if $a << b$, a log b/a is much smaller than b. This proves the lemma.

**Proof of Theorem.** i) As we go along the composition a Poincaré length is decreased if we are entirely within one of the coordinate systems for the singular point models because $f^{-1}$ has positive schwarzian there and maps of positive schwarzian decrease Poincaré length (de Melo and Van Strien) [MW]. ii) There are finitely many possible transitional cases for long intervals which don't fit inside one model or the other. We won't discuss these further. They are finite. iii) Finally we have the factors where the lemma applies. We view the lemma as saying intervals $J \subset T$ of any P-length $\geq 1$ cannot increase by more than the multiplicative *factor* $1 + 0(\text{length } T)$. By disjointness of the orbit of T this effect is controlled by the total length of M. Q.E.D.

## 3 Renormalization limits and schlicht mappings

Write unimodal mappings as Qh where Q is a quadratic polynomial and h is diffeomorphism with log h' bounded in $\frac{1}{2}$ Hölder, Zygmund sense of §1.

**Theorem.** *For the sequence of renormalizations* $R^n f = h_n Q$ *, the Zygmund* $\frac{1}{2}$ *Hölder size of* $\log h'_n$ *is bounded. Thus* $R_n f$ *is precompact for the topology of uniform convergence on* I *. Any* $C^0$ *limit* $g_\infty$ *of* $\{R_n f\}$ *belongs to the Epstein class* E(J) *for some interval* J *containing* I *plus definite space on either side.*

For the proof see [S1]

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