BOUNDED STRUCTURE OF INFINITELY RENORMALIZABLE MAPPINGS

by

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The period doubling universality observed by Feigenbaum is one example of an apparently large theory of geometric rigidity of critical orbits of dynamical systems. Even in the folding mappings of the interval there are uncountably many different topological forms illustrating the following phenomenon :

The orbit of critical point of f up to closest return times picks out collections of intervals permuted by f. (see § 3). The intersection of these interval collections is a labeled Cantor set *whose geometry at fine scales is conjecturally determined only by the sequence of permutations and the critical exponent of the mapping* f.

Here we describe how to get geometric bounds on the Poincaré return maps to these intervals in the C^2, power law singularity case. There are two ideas. Koebe distortion §1 and combinatorics § 2, 3, 4. These bounds show that limits of these Poincare return maps or renormalizations always exist §5 and have a certain analytic form : their inverse branches have extensions to the upper half plane which are limits of composition of roots $\sqrt[t]{z}$ and linear mappings § 6, 7.

The bounds which only depend on the critical exponent asymptotically, show the rigidity conjecture implies strong convergence results for iterated renormalization § 7.

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renormalization of two mappings brings them together in every C^k *topology* $k \geq 2$, *that makes sense*. § 6. This statement works even when the combinatorics of the successive renormalizations is unbounded.

<u>Appendix 1</u> : The projective length of $J \subset I = [a,b] \subset [c,d]$ is $\int_a^b \left(\frac{dx}{x-c} + \frac{dx}{d-x}\right)$. The infinitesimal distortion of this length by a map f is $\left(\left(\frac{f'x}{fx-fc}\right) + \left(\frac{f'x}{fd-fx}\right)\right) / \left(\left(\frac{1}{x-c}\right) + \left(\frac{1}{d-x}\right)\right) =$ $f'x\left(\frac{fd-fc}{d-c}\right)\left(\frac{x-c}{fx-fc}\right)\left(\frac{d-x}{fd-fx}\right)$. Taking the logarithm and writing $\log\left(\frac{fx-fy}{x-y}\right) = \Delta(x,y)$, which becomes $\log f'x$ on the diagonal, we obtain a difference of differences of Δ over the vertical (or horizontal) sides of the square $[(c,d),(x,c),(d,x),(x,x)]$. Dividing by the appropriate side length of the square one obtains a difference of difference quotients of Δ. This quantity is controlled by the modulus of continuity of the gradient of $\Delta(x,y)$, which exists if the original map is C^2.

Thus the distortion by an f in C^2 of the projective length $[J \subset I]$ is estimated by $1 + o(|I|)$ where the "small o" comes from the modulus of continuity of the gradient of $\log\left(\frac{fx-fy}{x-y}\right)$. (cf [1]).

<u>Appendix 2</u> : (boundedness of total non linearity).

As we pull back $I_c = (k,2k)$ $(k-1)$ times its projective length in the interval with definite buffer zones on each side only *changes* a bounded amount by Koebe and § 5. On the other hand each passage near the critical value creates a compression, whereas the C^2 calculation (appendix 1) shows the total change at other moments is negligible. If we calculate the compression part and invoke the first statement above, we prove the

<u>Proposition</u>. Write $[j,k+j] = [a_j,b_j]$ with an origin at the critical value. Then $\sum_{j=2}^{k} |\log \frac{b_j}{a_j}|$ is bounded independent of k. This means the *total non linearity* of the diffeomorphic part of renormalizations is bounded universally.

<u>Appendix 3</u> : The figure 4 show a gap whose size is comparable to the size of I_C. For all other intervals the mirror image about the critical point of f of an interval is contained in a gap or it lies ouside the interval $[2,1]$. As we go deep into renormalization the ratio of the length of $(2,c)$ to the length of $(1,c)$ cannot go to zero, where c is the critical point, because the shape of the renormalized f is bounded and the combinatorics stays away from that of a period two critical point. These remarks provide sufficient reason for the assertion :

The total length of the intervals tends to zero exponentially fast in the depth of renormalization. In particular the Cantor set has measure zero and the contribution to Koebe distortion away from the critical point tends exponentially fast to zero in the depth if f'' *is Lipschitz.*

[1] W de Melo and S J Van Strien. Schwarzian derivative and beyond. Bull of Amer Math Soc 18, 1988, pp 159-162.

[2] H. Epstein and J. Lascoux. Analyticaly Properties of the Feigenbaum Function. Comm. in Math. Physics 81, 413-453 (1981).