DIFFERENTIABLE STRUCTURES ON FRACTAL LIKE SETS, DETERMINED BY INTRINSIC SCALING FUNCTIONS ON DUAL CANTOR SETS[1]

by Dennis Sullivan

There is an easy notion of differentiable structure on a topological space. In the case of an embedded Cantor set in the line the differentiable structure records the fine scale geometrical structure. We will discuss two examples from the theory of one dimensional smooth dynamical systems, namely Cantor sets dynamically defined by i) folding maps on the boundary of chaos, and by ii) smooth expanding maps.

In example i) there is a remarkable discovery due to M. Feigenbaum[1] and independently P. Coullet and C. Tresser[2] that there is a universality or rigidity in the fine geometric structure of the Cantor set attractor for folding maps on the boundary of chaos. Feigenbaum expressed this discovery in terms of a universal scaling function for the Cantor set. Both papers offer an explanation motivated by the renormalization group idea of physics. These discoveries were empirical, and even today after much theoretical work they are not well understood. For example, the fine structure is codified by a scaling function defined on a logically distinct perfect set – the dual Cantor set. The main unsolved mystery is why the renormalizations converge. We prove here the rigidity conjecture assuming renormalization converges[2] §5,6. We also prove a converse. The proofs use the theory of the second example and a study of non linearity based on the bounded geometry of the Cantor set.

In the example ii) the Cantor set is the opposite of an attractor. It is the maximal invariant set of a $C(1,\alpha)$ expanding mapping of a 1-dimensional manifold. Now the fine structure of the Cantor set is not rigid but depends on many parameters. A complete set of invariants is again a scaling function but now the scaling function is an arbitrary Holder continuous function on a perfect set. Here the theoretical discussion is complete, straightforward and easy §1,2,3.

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[1]Proceedings of the Herman Weyl Symposium, Duke University, to appear in the series of the Proceedings of Symposia in Pure Mathematics, Vol. 48.

[2]In earlier unpublished work with Feigenbaum we proved the rigidity differently assuming a definite rate of convergence. Recently, David Rand has also derived a rigidity result.

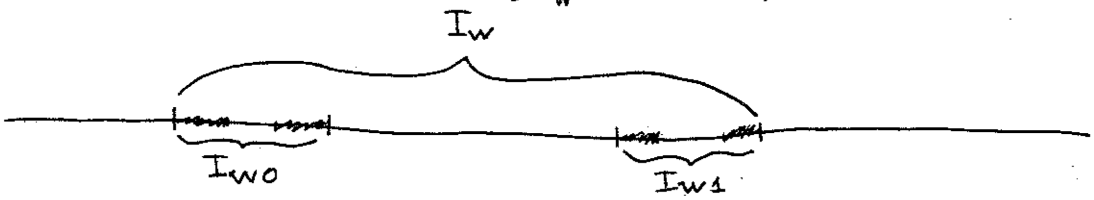
§1 *Differentiable structures on fractal sets*.

Let X be a topological space which is locally compact and can be locally embedded in R^n. If Q denotes some adjective like smooth, real analytic, complex analytic, etc. defining a pseudogroup of local isomorphisms of $\mathbb{R}^n$, we can define *a Q-structure on X of dimension n*. Say that a collection of local embeddings in R^n is *Q-coherent* if whenever i and j are two such embeddings defined near $x \in X$ there is a local Q isomorphism φ of R^n so that $\varphi \circ i = j$ near x. Then a Q-structure (of dimension n) on X is a maximal collection of Q-coherent local embeddings whose domains cover all of X.

§2 *Linear differentiable structures on Cantor sets*.

For concreteness let C denote the set of one sided infinite sequences of 0's and 1's with the product topology. Let $C(1,\alpha)$ denote the pseudogroup of smooth local diffeomorphisms of R with α-Holder continuous derivatives, for all α $0 < \alpha \leq 1$. We denote this pseudogroup $C(1,\alpha)$ (instead of the usual symbol) because α is not fixed.

We will consider those $C(1,\alpha)$ structures on the Cantor set C where if C_w = {sequences with initial n-segment = w} then there is a finite coordinate cover so that in a chart containing C_w we have the picture



where $I_{w'}$ denotes the smallest interval containing $C_{w'}$. In other words we want I_{w0} and I_{w1} to be disjoint.

We define in terms of the coordinate cover the *ratio geometry of w* to be the 3 ratios length I_{w0}/length I_w, length I_{w1}/length I_w, length w-gap/length I_w.

Definition We say the differential structure has bounded geometry if in addition to the above disjointness property these ratios are bounded away from zero (uniformly in w).

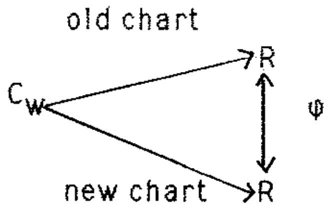
Lemma 1 If length I_w tends exponentially fast to zero in length w, the coordinate ratio function $w \rightarrow$ ratio geometry is determined exponentially fast in length w by the differentiable structure.

Proof: Changes of coordinates being $C(1,\alpha)$ have exponentially small non-linearity on intervals of exponentially small size.

Now for some cover of C given by a finite system of charts deform the embeddings into R. Namely, imagine changing the lengths of the I_w and the gaps without changing the local ordering of points of C.

Theorem 2 If C has bounded geometry and if the ratio functions of the deformed charts are only changed by an exponentially small error in length w, then the new charts belong to the original differentiable structure.

Proof: We fill in the diagram locally to construct φ



defined between the images of C. The difference quotient $\varphi(x) - \varphi(y)/x-y$ for $x, y \in C_w$ has the form $a_1+a_2+.../b_1+b_2+...$ where a_i and b_i are respective gap lengths in the two different charts and the sums are infinite. These are determined exponentially fast by the respective lengths of I_w and the ratio functions.

One sees the difference quotient for x,y in C_w, is Holder continuous with approximate value (new chart length I_w)/(old chart length I_w).

An elementary extension lemma shows φ has local $C(1,\alpha)$ extensions for some $0 < \alpha \leq 1$. QED

We say two ratio functions are exponentially equivalent if they differ by exponentially small quantities in length w.

Theorem 3 There is a one to one correspondece between $C(1,\alpha)$ differentiable structures on C of bounded geometry with given local order on the one hand and exponential equivalence classes of bounded from zero ratio functions, $\{w\} \to$ ratio geometry, on the other.

Proof: One way is Theorem 2. Conversely, if an abstract ratio function is bounded away from zero one builds the Cantor set C in R directly satisfying i) and ii).

§3 *Differentiable structures with smooth magnification and scaling functions.*

Now we ask the question: when is the shift map $(\varepsilon_0\varepsilon_1\varepsilon_2 \dots) \overset{J}{\mapsto} (\varepsilon_1\varepsilon_2\dots)$ of C locally a smooth diffeomorphism of class $C(1,\alpha)$ for some given differentiable structure on C.

There is a subtlety we will not deal with here. We will only characterize the situation when one of the two equivalent properties holds:

i) J is smooth and for some smooth metric $J' \geq \lambda > 1$ or,

ii) J is smooth and the structure has bounded geometry.

The basic fact for everything is that the non-linearity of the composition

$$I_{w_1} \xrightarrow{J} I_{w_2} \xrightarrow{J} I_{w_3} \xrightarrow{J} \ldots I_{w_n} \quad \text{where } w_{k+1} = {*}w_k, \quad \text{will be controlled by}$$

$\Sigma(\text{length } I_w)^\alpha$ which is part of a geometric series. (See Apendix 1.) This implies the ratio geometry of w stops changing exponentially fast in length w if we add arbitrary symbols to w on the left.

Thus there is a limiting ratio geometry $\sigma(\ldots\varepsilon_2\varepsilon_1\varepsilon_0)$ attached to each left infinite word. These limit ratios are called the scaling function of the differentiable structure. This proves

Theorem 4 If the shift map on the Cantor set of right infinite words is smooth $(C(1,\alpha))$ in a structure of bounded geometry, the coordinate dependent ratio function $w \rightarrow$ ratio geometry defines a limiting scaling function which is coordinate cover independent and attached intrinsically to the differentiable structure. The scaling function assigns to each left infinite word a triple of positive ratios adding up to one.

Remark: The proof shows this scaling function is exp-continuous on $(\ldots\varepsilon_2\varepsilon_1\varepsilon_0)$, namely there is exponentially fast determination of the value of σ by knowledge of initial n-segment of $(\ldots\varepsilon_2\varepsilon_1\varepsilon_0)$. We call this property Holder continuity of the scaling function σ.

Theorem 5: Conversely, if there is a Holder continuous limiting scaling function for the differentiable structure (as in the remark) the shift is a smooth $C(1,\alpha)$ expanding map (in some smooth metric).

Remark: All Holder continuous scaling functions on $\{\ldots\varepsilon_2\varepsilon_1\varepsilon_0\}$ occur in this discussion.

The proof of theorem 5 involves exactly the same consideration as that of theorem 2. One sees the relevant difference quotient is Holder using the scaling function. A standard argument shows the shift is expanding in some smooth metric because the bounded geometry implies all the derivatives at period points are greater than unity.

Summary Differentiable structures on C where the shift is a C(1,α) expanding map are precisely those structures which have bounded geometry and whose asymptotic ratio geometry is described by a scaling function.

$$\{\ldots\varepsilon_2\varepsilon_1\varepsilon_0\} \xrightarrow{\sigma} \text{ratio geometry.}$$

All Holder continuous σ occur in this discussion. *There is a one to one correspondence between these C(1,α) structures and exponentially continuous scaling functions Theorems 3,4,5.*

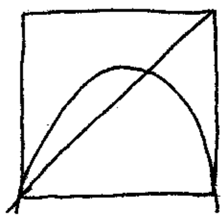
Furthermore if the structure admits a C(k,α) refinement so that the shift is C(k,α), this structure is also determined uniquely by the same scaling function k=0,1,... ; k= ∞, or k = ω . In fact, a shift commuting homeomorphism between structures which has a non zero derivative at one point, already is the restriction of a C(k,α) equivalence. (Appendix, part ii) of corollary)

An unsolved problem here is to determine what further properties of the scaling function σ allows higher smoothness. From earlier work we also know that if the structure is at least C^2 and for any smooth metric the second derivative of the shift is non zero at some point of C, the scaling function itself is determined by the thermodynamics of C which we know to be determined by the underlying Lipschitz structure. By thermodynamics we mean a certain mathematical discussion whose input is the sizes of the I_w, the set of numbers obtained by taking k-fold products of σ over k-fold shifts of k-periodic sequences.

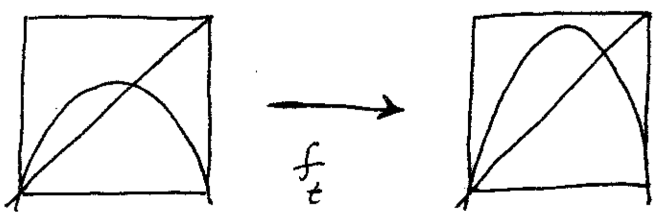
§4 *The period doubling attractor* (Informal discussion)

Let us consider the simplest class of maps which allows a transition from very simple dynamics to complicated dynamics with exponential effects.

These are the folding maps of an interval $I \xrightarrow{f} I$ which have a turning point c in I so that f is increasing before c and decreasing after c.



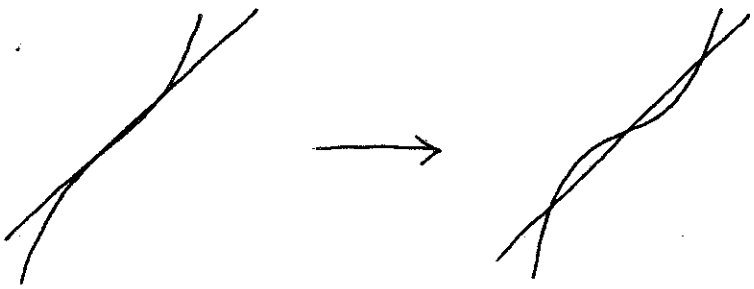
If there is a parameter t in the formula for f which raises the graph enough



and the family has appropriate smoothness there will be a parameter value a where f_a has an attractive Cantor set (all but a countable sequence of points are asymptotic to C).

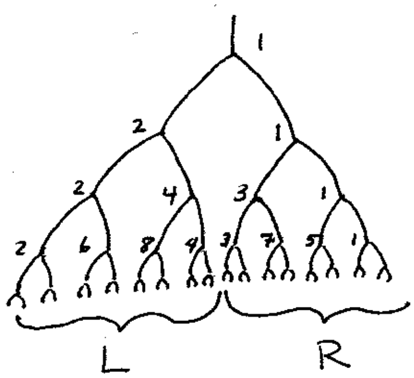
This Cantor set is the closure of the forward orbit of the turning point and is created by an infinite sequence of period doublings bifurcations of known form:

Figure 2



The forward orbit of the critical point denoted {1,2,3,...} increases its complexity as t increases to a. At a sequence of values a_1, a_2, a_3, ... tending to a, the critical point has 2^n – periodicity (figure 3).

Figure 3



The numbers in L at stage n are just those at stage (n-1) doubled and reversed in order. The numbers in R at stage n are obtained from those in L at stage n by subtracting 1 and reversing the order.

In the limiting map f_a (at the "boundary of chaos") there is a 2-adic Cantor type structure on which f acts by "adding 1" . The more precise statement is that the closure of the orbit of the critical point is a Cantor set on which f is equivalent to $(\varepsilon_0\varepsilon_1 ...) \mapsto g(\varepsilon_0\varepsilon_1 ...)$ where $g(\varepsilon_0\varepsilon_1 ...)$ is, change the first zero to a one and all previous ones to zeroes (adding "1" in the 2-adic integers.)

The identification can be chosen so that the critical point is 111··· and the critical value is 000··· .

Feigenbaum made the remarkable discovery that for many examples deep ratios in the Cantor set have asymptotic limits independent of the family f_t. His calculations involved smooth functions with quadratic turning points. The only way to change the fine scaling in practice was to change the nature of the critical point or to introduce other critical points.

§5 *The Feigenbaum Rigidity Conjecture*

Let us formulate a precise rigidity statement corresponding to the Feigenbaum discovery. We assume f is a folding map $f: I \to I$ satisfying

i) f has a Lipschitz first derivative i.e. $f \in C(1,1)$.

ii) f has exactly one critical point c, namely $f'c = 0$, and $f'x \neq 0$ for $x \neq c$.

iii) f in some $C(1,1)$ coordinate system near c is just $(x-c)^2 + f(c)$.

iv) fc, f^2c, f^3c, .. is deployed in the interval in terms of order as described in §4.

Rigidity Conjecture: The closure of the forward orbit of the critical point $\{fc, f^2c, \ldots\}$ is the 2-adic Cantor set C of one sided sequences of 0's and 1's with f acting on C by adding "1". The $C(1,\alpha)$ differentiable structure on C induced from its embedding in R is unique and described by a universal scaling function (§3).

A corollary of the mere existence of the scaling function for C is that the shift map J of the Cantor set is a $C(1,\alpha)$ expanding map (§3). In the 2-adic notation $x = \varepsilon_0 + \ldots + \varepsilon_i 2^i + \ldots$, $f(x) = x+1$ on C, and the shift map J is $x \to$ greatest integer in $1/2x = [1/2x]$. The calculation $1/2[x+1+1] = [1/2x] + 1$ shows $\boxed{Jff = fJ}$. This is the topological form of celebrated Cvitanović–Feigenbaum functional equation[4].

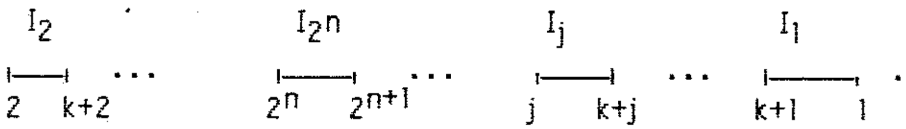
This equation can be iterated to obtain $J^n f^{2^n} = fJ^n$. Now J^n provides 2^n diffeomeorphisms between I_C, the interval subtending C, and the 2^n I_w's, the intervals subtending the Cw's, length $w = n$. The I_w are each invariant by f^{2^n} and the branches of $J^{\pm n}$=n provides smooth conjugacies between f^{2^n}/I_w and f/I_C.

For example, let $I_{w(n)}$, where $w(n)$ has with n 1's, converge down to the critical point $111111\cdots$ fixed by J. If $\alpha = J'(11111\cdots)$, then by calculus $\alpha^n J^{-n}_{w(n)}$ has a limit. Thus if $J^{-n}_n = J_{w(n)}$ and $f_n = f^{2^n}/i_{w(n)}$, then $\alpha^n J_n f J_n^{-1} \alpha^{-n} = \alpha^n f_n \alpha^{-n}$ has a limit. (These equations only hold on the Cantor set, a detail we will clarify in the next paper. The limit g satisfies the Cvitanović–Feigenbaum functional equation $\boxed{\alpha g^2 \alpha^{-1} = g}$, where $g = \lim_{n \to \infty} \alpha^n f^{2^n} \alpha^{-n}$, $\alpha = J'(111\cdots)$. This proves the first part of Theorem 6. The rest is explained by the argument of the next section.

Theorem 6 If the period doubling Cantor set has a scaling function, than the n^{th} renormalization of f, $\alpha^n f^{2^n} \alpha^{-n}$ converges to g, a solution of the CF functional equation $\alpha g^2 \alpha^{-1} = g$. The limit g only depends on the scaling function.

§6 *The Rigidity Conjecture and Renormalization*

The folding maps f we are considering satisfy (by hypothesis) that $\{1,2,3,...,2k\}$ denoting the first 2k forward iterates of the critical point $k=2^n$ are deployed



Consequently, $f^{2^n} = f^k$ preserves each of the indicated intervals and is a folding map of the same form. Each of these is called an n^{th} renormalization of f.

As we observed in §5 one of these renormalizations after linear rescaling by powers of $J'(111\cdots) = \sigma(\cdots 111)$ converge assuming the part of the conjecture about the existence of the scaling function σ.

There is a converse.

Theorem 7 If the renormalization of f about the critical point converges (in the C^0 topology to a folding map with a quadratic critical point), then the Cantor set of f has a scaling function only dependent on this renormalization limit.

<u>Corollary</u>: If two folding maps have the same renormalization limits there is a $C(1,\alpha)$ diffeomorphism between their Cantor sets conjugating the dynamics on the Cantor sets.

<u>Proof of Corollary</u>: Theorem 7 and Summary of §3.

We make the proof of Theorem 7 assuming the Cantor sets have bounded geometry. We will expose our general result (valid for all maps described at the beginning of the section) on bounded geometry and general a priori estimates on the non-linearity of renormalization in the next paper.

Now consider the measure $|dx/x|$ restricted to all the intervals at the n^{th} level except I_{2^n} containing the critical point. Here x=0 is the critical point. By induction on n we prove two properties:

i) the density of the measure is quasi-constant on each interval

ii) the total mass is controlled independent of n.

Passing from level n to n+1 we cut away a middle piece from each interval which by i) and the bounded geometry reduces the total mass by a definite factor (and keeps property i). We also add a new interval near the critical point. This only adds a new term of bounded mass and quasi constant density because the interval is nicely situated with respect to the critical point by the bounded geometry assumption on C. This completes the induction.

Now the non linearity of f (the measure $(f''/f')dx$) is controlled by a bounded density measure away from the critical point and the measure $|dx/x|$ near the critical point. Thus $|f''/f'||dx|$ is controlled by a measure satisfying i) and ii) above since $|dx/x|$ and bounded measures satisfy i) and ii).

Now consider the ratio geometry associated a long word w (length r say). Fix j. We can keep the j-segment on the right fixed and change the other digits to 1's by applying f no more than 2^{r-j} times. The ratio geometry of w is that of something at level j inside some interval at depth r−j. We transform this over to the critical point interval by applying f no more than 2^{r-j} times.

These iterates all have bounded non linearity by ii) above (applied to level r−j). We care about the distortion of an object j levels deeper. This object is exponentially smaller in j, relatively (by bounded geometry). The non linearity measure (by i) at level (r−j)) we see is exponentially smaller in j. Thus the distortion of the appropriate iterate of f restricted to the smaller object is exponentially small in j. Thus the ratio geometry of w of length r is the same as that of the word beginning with r−j ones and ending with the same last j segment as w with an exponentially small in j error. This much follows from the just bounded geometry assumption on the Cantor set.

Now let r increase still keeping the final j segment of w fixed. The structure of the ratio geometry of 1111111 ... 111(final j segment of w) only depends on the 2^j forward orbit of the renormalized map which is converging in C^0 to a folding map f_∞. Thus we can define the scaling function at arguments ...1111111 (word of length j) in terms of the first 2^j iterates of f_∞. Since j was unrestricted in the argument we have defined the scaling function at all left infinite words which are all 1's eventually. Moreover, by the first part of the argument these values are determined exponentially fast by the initial segments on the right. This proves theorem 7 assuming bounded geometry of C.

Appendix (Composition of Contractions in $C(k,\alpha)$).

Consider a composition g of diffemorphisms $I_1 \xrightarrow{f_1} I_2 \xrightarrow{f_2} \ldots \xrightarrow{f_n} I_{n+1}$ where $|f_i'| \le \lambda < 1$.

If x_1 and y_1 are 2 points in I_1 let $(x_{i+1}, y_{+i}) = (f_i(x_i),\ f_i(y_i))$.

Note $|x_i - y_i| \le \lambda^{i-1} |x_1 - y_1|$. If φ_i is a function on I_i satisfying $|\varphi_i(x) - \varphi_i y| \le C\,|x_i - y_i|^{\alpha}$ $0<\alpha\le 1$, then $\varphi_1(x_1) + \varphi_2(x_2) + \ldots$.is also Holder continuous with constant $C(1/1-\lambda^{\alpha})$ and same exponent α.

We apply this to $\varphi_i = D^k \log f_i'$ $k = 0,1,2,\ldots$ to see that if the f_i's satisfy $|D^k \log f_i'(x) - D^k \log f_i(y)| \le C|x - y|^{\alpha}$ then so does $D^k \log g'(x)$ for the same α and the new C as above.

Corollary 1) A compositon of uniforms contractions which are individually bounded in $C(k,\alpha)$ (as diffeomorphisms) is also in $C(k,\alpha)$. (same α and new constant).

2) If a sequence of such compositons is renormalized by post composition with linear maps to obtain mappings between unit intervals the sequence is precompact in $C(k,\alpha)$.

References

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2. P. Coullet, J. Tresser, *Iterations d'endomorphisms et groupe de renormalisation*, C.R. Acad.Sc., Paris **287**(1978), 577; Journal de Physique **C5**(1978) 25.