On negative curvature, variable, pinched, and constant

by

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We will study certain geometric properties of Riemannian manifolds M^{n+1} in terms of dynamical properties of geodesic lines in the unit tangent sphere bundle S^{2n+1}. This study was prompted by the observation that a dynamical calculation related to quasi-linearity corresponds exactly to $\frac{1}{4}$ pinching of negative curvature. (See Theorem 2.)

Oriented geodesic lines may be grouped into asymptotic equivalence classes as time tends to plus infinity or as time tends to minus infinity. Thus ℓ_1 and ℓ_2 lie on the same "leaf" of $\mathcal{A}_+$ (respectively $\mathcal{A}_-$) if there is x_1 in ℓ_1 and x_2 in ℓ_2 so that distance $\big(g_t(x_1), g_t(x_2)\big) \to 0$ as time tends to plus infinity (respectively, minus infinity) where g_t is the geodesic flow on S^{2n+1}. When M^{n+1} has bounded negative curvature $-b^2 \leq k \leq -a^2$, each leaf L^{n+1} of $\mathcal{A}^+$ or $\mathcal{A}^-$ is a smooth $n+1$-manifold which is a covering space of M^{n+1} via the composition $L^{n+1} \subset S^{2n+1} \to M^{n+1}$.

We say two negatively curved manifolds are g-homeomorphic if there is a homeomorphism G between their unit sphere bundles preserving oriented geodesic lines, and we say G is a g-homeomorphism. Gromov and Mostow showed each isomorphism of fundamental groups in the compact case leads to a rather canonical g-homeomorphism. (The correspondence between geodesic lines is canonical.)

Theorem 1. *A closed $\frac{1}{4}$-pinched (i.e., $4a^2 < b^2$) negatively curved manifold M^{n+1} is g-homeomorphic to a constant negative curvature -1 manifold (a hyperbolic manifold) if the geodesic flow in the asymptotic leaves is transversally irreducible relative to Lebesgue measure.*

Remark. The meaning of the condition in Theorem 1 is the following. The tangent bundle to the leaves $\mathcal{A}_+$ modulo the subbundle tangent to geodesic lines defines a continuous n-dimensional bundle $\mathcal{C}$ over S^{2n+1} (we only need Borel measurable to make the definition). The condition *transversally irreducible relative to*

Lebesgue measure means there is no Borel measurable subbundle of $\mathcal{C}$ almost everywhere invariant under the geodesic flow, where a.e. means relative to Lebesgue measure on S^{2n+1}.

Examples. A compact hyperbolic manifold is *measurably irreducible.* A compact complex hyperbolic manifold is not (see below).

Recently, Gromov and Thurston [Pinching constants for hyperbolic manifolds, *Inventiones Math.* **1986**] produced arbitrarily pinched negative curved manifolds of dimension $4, 5, 6, \ldots$ which are not g-homeomorphic to hyperbolic manifolds. By the theorem all these examples must have *measurably reducible* geodesic flows.

If we sharpen the statement of Theorem 1, the condition of *measurable irreducibility* becomes necessary and sufficient for the conclusion. For this we need some more concepts.

We say the geodesic flow is *uniformly quasi-conformal* or uniformly quasi-similar if the action of g_t on the transversal bundle $\mathcal{C}$ to the geodesic lines in the asymptotic leaves of $\mathcal{A}^+$ (figure 1)

figure 1

is given by linear transformations which are a bounded distance from similarities, with a bound uniform in points of S^{2n+1} and uniform in time. Because of the time reversal involution on S^{2n+1}, this quasi-similarity implies the equivalent property in $\mathcal{A}_-$, the negative asymptotic foliation. For a hyperbolic manifold of constant negative sectional curvature the geodesic flow acts by precise similarities in these tangent spaces and so is *uniformly quasi-conformal.*

We say the geodesic flow is *uniformly quasi-linear* if the derivatives of g_t acting on the fibres of $\mathcal{C}$ are uniformly continuous in the asymptotic leaves of $\mathcal{A}_+$ with constants and moduli of continuity independent of the leaf in S^{2n+1} and the time. *Uniformly quasi-similar* implies *uniformly quasi-linear* (see Proposition 1).

A negatively curved manifold obtained from the ball in $\mathbb{C}^n$ by dividing out by a discrete group of holomorphic motions (a complex hyperbolic manifold) has a uniformly quasi-linear but not quasi-similar geodesic flow, in fact for $n = 2$ the action is by exact "homotheties" of the 3-dimensional Heisenberg group

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \lambda x & \lambda^2 z \\ 0 & 1 & \lambda y \\ 0 & 0 & 1 \end{pmatrix} .$$

These concepts are related by the following two theorems.

Theorem 2. *A complete $\frac{1}{4}$-pinched negatively curved manifold has a uniformly quasi-linear geodesic flow.*

Theorem 3. *Suppose the geodesic flow is ergodic and uniformly quasi-linear. Then the geodesic flow is uniformly quasi-similar if and only if it is measurably irreducible.*

We say that a *g-homeomorphism G is transversally quasi-conformal* if there is a constant K so that the maps induced by G in the local quotients of asymptotic leaves by geodesic lines (say using unit size flow boxes) are all K-quasi-conformal homeomorphisms.

Theorem 4. *Suppose two compact g-homeomorphic negatively curved manifolds have uniformly quasi-similar geodesic flows. Then any g-homeomorphism between them has to be transversally quasi-conformal.*

Theorem 5. *A compact negatively curved manifold M is g-homeomorphic to a hyperbolic manifold by a G which is transversally quasi-conformal if and only if the geodesic flow of M is uniformly quasi-similar.*

Theorems 2,3,4, and 5 yield the more precise form of Theorem 1 promised above.

Corollary. *A compact $\frac{1}{4}$-pinched negatively curved manifold is quasi-conformally g-homeomorphic to a hyperbolic manifold if and only if its geodesic flow is measurably irreducible.*

Remark. An obvious gap in our study is a theorem describing how often measurable irreducibility or uniform quasi-similarity occurs among $\frac{1}{4}$-pinched metrics. For example is it an open condition on negative metrics?

Section 1. Quasi-linearity

First we calculate in R^n. If φ is a local diffeomorphism, let $J\varphi$ denote the field of Jacobian matrices of partial derivatives and let $\eta(\varphi)$ denote the matrix of one-forms recording the nonlinearity of φ, namely $\eta(\varphi) = (J\varphi)^{-1}d(J\varphi)$. If η is any matrix of 1-forms defined in the range of a diffeomorphism φ define η^φ by $\eta^\varphi = J\varphi^{-1}\cdot(\varphi^*\eta)\cdot J\varphi$. Differentiating the chain rule for J, $J(\varphi_2\varphi_1) = \varphi_1^*(J\varphi_2)\cdot J\varphi_1$ yields the *cocycle identity for the nonlinearity*

$$\eta(\varphi_1\varphi_2) = \eta(\varphi_1) + \big(\eta(\varphi_1)\big)^{\varphi_1} \ .$$

Iterating the cocycle identity yields

$$\eta(\varphi_n\varphi_{n-1}\cdots\varphi_1) = \eta(\varphi_1) + \eta(\varphi_2)^{\varphi_1} + \cdots + \eta(\varphi_n)^{\varphi_{n+1}\cdots\varphi_1} \ .$$

Now suppose for each k, $\eta(\varphi_k) = J\varphi_k^{-1}dJ\varphi_k$ is bounded by N, and each φ_k contracts each tangent vector by a factor no larger than $1/\lambda < 1$. Then only one obstruction remains to derive a bound on the nonlinearity of the composition $\varphi_n\varphi_{n-1}\cdots\varphi_1$. In the sum consider the k^{th} term

$$\eta(\varphi_k)^{\varphi_{k-1}\cdots\varphi_1} = J(\varphi_{k-1}\cdots\varphi_1)^{-1}(\varphi_{k-1}\cdots\varphi_1)^*\eta(\varphi_k)J(\varphi_{k-1}\cdots\varphi_1) \ .$$

The inner factor is bounded by $N \cdot (1/\lambda)^{k-1}$. If the conjugation has a controlled effect the series would be geometric and a bound results. This works in several cases:

i) in dimension one conjugation is trivial and we arrive at a bound called the "basic distortion lemma". Namely $\eta(\varphi_n \cdots \varphi_1) \leq N\left(\frac{\lambda}{\lambda-1}\right)$, a bound independent of n.

ii) In complex dimension one, the φ_k complex analytic, we arrive at a complex analytic version of the basic distortion lemma.

iii) If the φ_k are uniformly K quasi-conformal then conjugation has a bounded effect and the sum is bounded by $K \cdot \left(\frac{\lambda}{\lambda-1}\right) \cdot N$.

iv) If the quasi-conformality of $\varphi_{k-1} \cdot \varphi_1$ grows at a slow enough exponential rate, namely $\leq C\mu^{k-1}$ where $\mu < \lambda$, then we again have a bound

$$\eta(\varphi_n \cdots \varphi_1) \leq C \cdot N \cdot \left(\frac{1}{1-\mu/\lambda}\right) .$$

Note: 1) There will be a bound as required in iv) if for each contraction φ_k the biggest and the smallest eigenvalues λ_M and λ_m of $(J\varphi_k \;\; J\varphi_k^t)^{1/2}$ satisfy λ_M/λ_m the quasi-conformality is strictly less than the contraction $1/\lambda_M$. In other words $\lambda_M^2 \leq c\lambda_m$ where $c < 1$ is valid for all points in domain φ_k and all k.

2)

3) Bounds on the nonlinearity of a composition $\eta(\varphi_n\varphi_{n-1} \cdots \varphi_1)$ implies

$$J(\varphi_n\varphi_{n-1} \cdots \varphi_1)$$

has a modulus of continuity independent of n using the following:

Lemma. *If $g(x)$ is a smooth field of matrices in an open set U in R^n so that $\|g^{-1}dg\| \leq M$ then*

$$\|g(x)g(y)^{-1}\| \leq \exp M\ell$$

where ℓ is the length of any path in U connecting x and y, and the norm n matrix satisfies $\|A \cdot B\| \leq \|A\| \cdot \|B\|$.