The spinor representation of minimal surfaces in space

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A $\mathbb{C}$-linear holomorphic bundle map $\mathcal{T}(M)\xrightarrow{\omega}\mathcal{T}^*(\overline{\mathbb{C}})$ determines (and is determined by) a minimal surface $\tilde{M}$ in R^3 covering M. The geometric meaning of ω is the following:

i) M is a Riemann surface, $\tilde{M}$ a certain Abelian covering space, $\mathcal{T}M$ the holomorphic tangent line bundle of M, $\mathcal{T}^*(\overline{\mathbb{C}})$ the holomorphic cotangent line bundle of the Riemann sphere $\overline{\mathbb{C}}$, and $\mathcal{T}(M)\xrightarrow{\omega}\mathcal{T}^*(\overline{\mathbb{C}})$ determines the minimal surface (see v)).

ii) The induced map in the base $M\xrightarrow{g}\overline{\mathbb{C}}$ is the classical Gauss map (made holomorphic instead of anti-holomorphic by composing with the antipodal map of the sphere).

iii) The quadratic differential q defined by $q(v) = \langle\omega(v), dg(v)\rangle$ for $v \in \mathcal{T}(M)$ is the holomorphic quadratic differential associated with constant mean curvature whose real foliations are the lines of curvature and whose measure is $\sqrt{-K}\,dm$, K the Gaussian curvature and dm defined by the Riemannian metric. Note that ω is determined by g and q, and these are linked by a common zeroes condition.

iv) The Riemannian metric on $\tilde{M}$ as a minimal surface in space is the pull back of the Hermitian metric on $\mathcal{T}^*(\overline{\mathbb{C}})$. So $\tilde{M}$ is immersed precisely when ω is a bundle isomorphism.

v) *Classical Weierstrass representation of minimal surfaces*

Let us recall how a minimal surface is constructed from ω. Think of $\mathcal{T}^*(\overline{\mathbb{C}})$ as the tautological line bundle over $\mathbb{C}P^2$ (= lines in $\mathbb{C}^3$) restricted to the quadric $\{z_1^2 + z_2^2 + z_3^2 = 0\}$. Then if $v \in \mathcal{T}(M)$ let $\varphi_1(v), \varphi_2(v), \varphi_2(v)$ be the coordinates of $\omega(v)$ in $\mathbb{C}^3 - \{0\}$. We obtain three holomorphic 1-forms on M satisfying $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$. The coordinates of the minimal surface immersion are defined by integrating these forms and taking real parts. Thus there are period conditions which insure the coordinates are well defined on M. In general, the coordinate functions $x = (x_1, x_2, x_3)$ are defined on an Abelian cover $\tilde{M}$ of M, they are harmonic, and the algebraic equation satisfied by $\varphi_1, \varphi_2, \varphi_3$ is equivalent to the assertion that $\tilde{M} \xrightarrow{x} R^3$ is conformal.

Conversely, an Abelian cover $\tilde{M}$ of M, a "conformal immersion" of $\tilde{M}$ to R^3 with harmonic coordinates, equivariant with respect to a representation of $\pi_1 M$ into the translations of R^3 determines $\varphi_1, \varphi_2, \varphi_3$ by differentiating and then ω, using $\varphi_1(v), \varphi_2(v), \varphi_3(v)$ as coordinates of $\omega(v)$.

The classical Weierstrass representation also makes use of the ancient parametrization of solutions of $a^2 + b^2 = c^2$ by $(a, b, c) = (u^2 - v^2, 2uv, u^2 + v^2)$. We can reformulate this step in terms of holomorphic spinors, namely sections of the line bundles $\sqrt{T}$ and $\sqrt{T^*}$. The period conditions mentioned above have a neat expression in terms of these spinors.

vi) *The Spinor representation of minimal surfaces*

By topology there are in general many complex line bundles $\sqrt{T}$, namely solutions of $(\sqrt{T}) \otimes_{\mathbb{C}} (\sqrt{T}) \simeq T$ over a Riemann surface M. These "square roots" or "spin structures" are parametrized by quadratic functions $\varphi : H_1(M, Z/2) \to Z/2$, that is $\varphi(x+y) = \varphi(x) + \varphi(y) +$ intersection (x, y). Any immersed orientable surface in R^3 has such a function φ defined by $\varphi(x) =$ number of twists mod 2 in a band around x.

Alternatively, a spin structure on an immersed surface M can be defined by inducing the spin structure on the 2-sphere to M using the Gauss map.

Since $\pi_1(SO_3) = Z/2$ is generated represented by a path of frames where one vector stays fixed and the other 2 rotate the twisting band description of the spin structure and the Gauss map description can be correlated.

For our minimal surface $\tilde{M} \to R^3$ defined by $T(M) \xrightarrow{\omega} T^*(\mathbb{C})$ we can go a bit further. The unique spin structure on $T^*(\overline{\mathbb{C}})$, $\sqrt{T^*}$, is just the tautological line bundle L over $\overline{\mathbb{C}} = \mathbb{C}P^1 =$ lines in $\mathbb{C}^2$. The double covering $L \to T^*(\mathbb{C})$ can be induced using ω (in the immersed case) to a double covering $g^*L \to T(M)$. *Define* $\sqrt{T(M)}$ to be g^*L. Then we have a canonical bundle map $\sqrt{T(M)} \xrightarrow{\sqrt{\omega}} \sqrt{T^*}$. We call $\sqrt{\omega}$ the ***spinor representation*** of the minimal surface.

There are two holomorphic spinors $\{u, v\}$ associated to the spinor representation $\sqrt{\omega}$. If $s \in \sqrt{T(M)}$, let $u(s)$ and $v(s)$ be defined as the coordinates of $\sqrt{\omega}(s)$ in $\mathbb{C}^2 - \{0\} = \{$ nonzero vectors of $L\}$.

Thus we have shown the three holomorphic 1-forms $\varphi_1, \varphi_2, \varphi_3$ satisfying $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$ in the classical Weierstrass representation can be written $u^2 - v^2$, $2uv$, $i(u^2 + v^2)$ where u and v are holomorphic sections of a naturally defined spin reduction $\sqrt{T^*M} \equiv \text{Hom}\left(\sqrt{TM}, \mathbb{C}\right)$, namely u and v are two holomorphic spinors.

Conversely, given two holomorphic spinors u, v on M not both zero we obtain an immersed minimal surface. Namely, define $\varphi_1, \varphi_2, \varphi_3$ by the above formulae.

vii) *The period conditions*

For the real parts of the integrals of $\varphi_1, \varphi_2, \varphi_3$ to be well defined on M we need to know these 1-forms have purely imaginary periods. Using the spinor representation $(\varphi_1, \varphi_2, \varphi_2) = \left(u^2 - v^2, 2uv, i(u^2 + v^2)\right)$ we see these period conditions are equivalent to the

$$\text{“spinor period relations”} = \begin{cases} \text{i)} & u^2 \text{ and } v^2 \text{ have conjugate periods} \\ \text{ii)} & uv \text{ has imaginary periods} \end{cases},$$

where u^2, v^2, uv are the holomorphic 1-forms associated to the spinors u, v.

viii) Note that $G\ell(2, \mathbb{C})$ acts on spinor representations of minimal surfaces

$$(u, v) \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} (u, v)$$

and the subgroup $R^* \times SU(2) = \left\{ \begin{pmatrix} \alpha & -\beta \\ \overline{\alpha} & \overline{\beta} \end{pmatrix} \right\}$ preserves the period conditions.

ix) *Complete minimal surfaces of finite conformal type*

Suppose that M is a compact surface N with finitely many punctures $\{a_1, a_2, \ldots, a_v\} \subset N$, that M is immersed in R^3 as a complete minimal surface with the geometry asymptotically planar near each puncture. As Osserman observed, the Gauss map extends over N in this case.

In terms of the Weierstrass representation $\mathcal{T}(M)\xrightarrow{\omega}\mathcal{T}^*\mathbb{C}$ we have

a) ω is a bundle isomorphism;

b) ω has quadratic poles at each puncture;

c) ω satisfies the period conditions.

In terms of the spinor representation $\sqrt{\mathcal{T}(M)}\xrightarrow{\sqrt{\omega}}\sqrt{\mathcal{T}^*\mathbb{C}}$ where $\sqrt{\omega}$ is defined by the holomorphic spinors (u,v) one has

a) u and v have no common zeroes on $M-\{a_1,\ldots,a_p\}$.

b) u and v have at most simple poles at each of the punctures a_r and both cannot be holomorphic near any puncture.

c) u and v satisfy the period conditions 1) u^2 and v^2 have conjugate periods on $N-\{a_1,\ldots,a_p\}$ and 2) uv has imaginary periods there.

x) Conversely, we can try to construct minimal surfaces this way. Holomorphic spinors may not exist on a compact surface N. However, by Riemann Roch $(\dim(D)-\dim(\mathcal{T}^*-D)=|D|-g+1)$ there is always at least a 1-dimensional subspace of holomorphic spinors on $N-\{a\}$ with at most a single order pole at a $(D=\sqrt{\mathcal{T}}+a,\ |D|=g)$. Thus even if N is without holomorphic spinors we have p-dimensional spaces of spinors as required in ix).

Varying N and $\{a_1,\ldots,a_p\}$ yields $3g-3+p$ $\mathbb{C}$-parameters. Varying (u,v) among 2-dimensional subspaces of spinors gives $2p-4$ more. That makes $3g+3p-7$ complex parameters plus 4 more real parameters for the choice of u,v in the 2-dimensional subspace (see viii)). Thus we have $6g+6p-10$ real parameters to put up against the $(2g+p-1)$ 3 real period conditions. *So we have more real parameters than conditions* $(3p-7)$ *when there are at least three punctures.*

xi) *Embedded minimal surfaces*

If we have a complete minimal surface of finite conformal type and planar ends as in ix) which is *embedded* in R^3, the quadratic function $H_1(M,Z/2)\xrightarrow{\varphi}Z/2$ associated to the spin structure has a special property. First of all φ is zero for the cycles around the punctures. Secondly, the *Arf invariant* of this quadratic function (now thought of as a quadratic function on the closed surface $H_1(N, Z/2) \xrightarrow{\varphi} Z/2$) *must be zero*. This is so abstractly because one knows the Arf invariant represents the unique cobordism invariant* of compact immersed oriented surfaces in R^3, and clearly embedded surfaces bound.

More simply, the Arf invariant of φ is zero exactly when there is a good basis for the mod 2 intersection form on which φ vanishes. Such a basis for an embedded surface can be constructed by looking at the kernels of homology to each complementary component (after suitably filling in the punctures topologically).

* For cobordisms of unoriented immersed surfaces there is a complete $Z/8-$ Arf invariant associated to quadratic functions $H_1(M, Z/2) \to Z/4$.